Solution to Exercise 4, Sheet 4

Exercise. For σ, $\tau \in \mathfrak{S}_N$, show that the segment $\mathcal{S}_{X^\sigma, X^\tau}$ (X^σ is the permutation matrix associated to σ) connecting X^σ and X^τ is an edge of the Birkhoff polytope B_N if and only if $\sigma^{-1}\tau$ is a cycle.

Proof. We first show that it suffices to consider the case where $\sigma = e$, the identity element. That is to say, S_{X^σ, X^τ} is an edge of B_N if and only if $S_{X^e, X^{\sigma^{-1}\tau}}$ is an edge of B_N.

Assume that S_{X^σ, X^τ} is an edge, then there exists a hyperplane $\mathcal{H}_{\alpha,b}$ such that

- $\mathcal{H}_{\alpha,b} \cap B_N = S_{X^\sigma, X^\tau}$;
- $B_N \subset \mathcal{H}^-_{\alpha,b}$.

We show that $S_{X^e, X^{\sigma^{-1}\tau}}$ is an edge. Indeed, we consider $\alpha' := \alpha \cdot X^\sigma$, and claim that

1. $\mathcal{H}_{\alpha',b} \cap B_N = S_{X^e, X^{\sigma^{-1}\tau}}$;
2. $B_N \subset \mathcal{H}^-_{\alpha',b}$.

We show the first part.

$\subset$: if $\mathbf{x} \in \mathcal{H}_{\alpha',b} \cap B_N$ then $b = \alpha'(\mathbf{x}) = \alpha(X^\sigma \mathbf{x})$, that is to say, $X^\sigma \mathbf{x} \in \mathcal{H}_{\alpha,b}$. Writing $\mathbf{x}$ into a convex combination of the permutation matrices shows that $X^\sigma \mathbf{x} \in B_N$, hence it is contained in S_{X^σ, X^τ}: there exists $0 \leq \lambda \leq 1$ such that $X^\sigma \mathbf{x} = \lambda X^\sigma + (1-\lambda) X^\tau$, hence $\mathbf{x} = \lambda X^e + (1-\lambda) X^{\sigma^{-1}\tau}$.

$\supset$: if $\mathbf{x} = \lambda X^e + (1-\lambda) X^{\sigma^{-1}\tau}$ for some $0 \leq \lambda \leq 1$, then $\mathbf{x} \in B_N$ and

$$\alpha'(\mathbf{x}) = \lambda\alpha(X^\sigma X^e + (1-\lambda) X^\sigma X^{\sigma^{-1}\tau}) = \alpha(\lambda X^\sigma + (1-\lambda) X^\tau) = b.$$

The second part: Let $\mathbf{x} \in B_N$. Write it as a convex combination of permutation matrices:

$$\mathbf{x} = \sum_{\tau \in \mathfrak{S}_N} \lambda_\tau X^\tau.$$

Then

$$\alpha'(\mathbf{x}) = \sum_{\tau \in \mathfrak{S}_N} \lambda_\tau \alpha(X^\sigma X^\tau) \leq b \sum_{\tau \in \mathfrak{S}_N} \lambda_\tau = b.$$

We proved that $S_{X^e, X^{\sigma^{-1}\tau}}$ is an edge. The same argument shows the other implication.

Now it suffices to show that S_{X^e, X^τ} is an edge if and only if τ is a cycle.

Assume that τ is not a cycle, that is to say, $\tau = c_1 c_2$ where c_1 and c_2 are multiplications of at least one cycle. We claim that

$$X^e + X^\tau = X^{c_1} + X^{c_2},$$

which means that S_{X^e, X^τ} is not an edge, as the middle point of the segment connecting X^e and X^τ can be expressed as a convex combination of X^{c_1} and X^{c_2}.

To show this claim, it suffices to compare the (i,j)-entry of both sides, which is straightforward.

Now we assume that $\tau = (i_1, \cdots, i_k)$ is a cycle and want to show that S_{X^e, X^τ} is an edge. That is to say, we want to find $\mathcal{H}_{\alpha,b}$ such that

- $\mathcal{H}_{\alpha,b} \cap B_N = S_{X^e,X^\tau}$;
- $B_N \subset \mathcal{H}_{\alpha,b}^-$.

For $1 \leq i,j \leq N$ we denote $\varepsilon_{i,j}$ be the linear function on the vector space $\mathcal{M}_N(\mathbb{R})$ mapping a matrix to its (i,j)-entry. We consider

$$\alpha := \sum_{s=1}^{k-1} \varepsilon_{i_s,i_{s+1}} + \varepsilon_{i_k,i_1} + \sum_{i=1}^{n} \varepsilon_{i,i}$$

and $b = N$. Let X be a double stochastic matrix in B_N. As its row and column sum are all 1, we know that $B_N \subset \mathcal{H}_{\alpha,b}$. Moreover, if $\alpha(X) = N$, then X must be supported on the set

$$\{(i_s, i_{s+1}), (i_k, i_1), (i,i) \mid s = 1, \cdots, k-1;\ i = 1, \cdots, N\},$$

which means that if the index set of a entry is not in this set, then the entry is zero. Again by the double stochastic property, for $j \in J$, $X_{j,j} = 1$ and

$$X_{i_1,i_2} = X_{i_2,i_3} = \cdots = X_{i_{k-1},i_k} = X_{i_k,i_1},$$

which implies that

$$X_{i_1,i_1} = \cdots = X_{i_k,i_k} = 1 - X_{i_1,i_2}.$$

Let $\lambda = X_{i_1,i_1}$ then $0 \leq \lambda \leq 1$ and

$$X = \lambda X^e + (1-\lambda) X^\tau.$$

This prove that $\mathcal{H}_{\alpha,b} \cap B_N \subset S_{X^e,X^\tau}$. The other inclusion is clear, as $\alpha(X^e) = N$ and $\alpha(X^\tau) = N$ implies that the segment $S_{X^e,X^\tau} \subset \mathcal{H}_{\alpha,b}$.