Solution to Exercise 5, Sheet 5

Exercise. Show that if P and Q are d-polytopes, and the face poset of P is a sub-poset of the face poset of Q, then P and Q are combinatorially equivalent.

Proof. First, what $\mathcal{L}(P)$ is a subposet of $\mathcal{L}(Q)$ means that there exists an injective poset homomorphism $f : \mathcal{L}(P) \to \mathcal{L}(Q)$ such that $f(\mathcal{L}(P))$ is a subposet of $\mathcal{L}(Q)$.

We prove by induction on the dimension d of the polytopes. The case $d = 1$ is clear as both of them can only by segments.

As $f(\mathcal{L}(P))$ is a subposet of $\mathcal{L}(Q)$, it suffices to show that f is surjective.

As any face is an intersection of facets, it suffices to show that P and Q have the same number of facets. We prove it by contradiction: assume that there exists a facet G of Q which is not in the image of f.

We first claim that there exists a facet G' in $f(\mathcal{L}(P))$ such that $G \cap G' \neq \emptyset$. Indeed, this is not hard to see by considering the following polar dual statement: one can not decompose the vertices of Q into two non-empty disjoint subsets V_1 and V_2 such that $V(Q) = V_1 \cup V_2$ and there is no edge between any vertex in V_1 and any vertex in V_2.

Let $H := G \cap G'$. Then H is a codimension 2 face. Let F' be a facet in $\mathcal{L}(P)$ such that $f(F') = G'$. By induction hypothesis, there exists a codimension 2 face K in $\mathcal{L}(P)$ such that $f(K) = H$. Then there exists a facet F'' in $\mathcal{L}(P)$ such that $F' \cap F'' = K$. Let G'' be the facet in $\mathcal{L}(Q)$ given by $f(F'')$. Then $H = G \cap G' \cap G''$ and G, G' and G'' are different facets of Q. This is not possible, as the polar dual statement of this fact says that an edge would have three vertices. This is the desired contradiction.