EXCEPTIONAL REPRESENTATIONS OF A DOUBLE QUIVER OF TYPE A, AND RICHARDSON ELEMENTS IN SEAWEED LIE ALGEBRAS

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Abstract. In this paper, we study the set of Δ-filtered modules of quasi-hereditary algebras arising from quotients of the double of quivers of type A. Our main result is that for any fixed Δ-dimension vector, there is a unique (up to isomorphism) exceptional Δ-filtered module. We then apply this result to show that there is always an open adjoint orbit in the nilpotent radical of a seaweed Lie algebra in $\mathrm{gl}_n(k)$, thus answering positively in this $\mathrm{gl}_n(k)$ case to a question raised independently by Michel Duflo and Dmitri Panyushev. An example of a seaweed Lie algebra in a simple Lie algebra of type E_8 not admitting an open orbit in its nilpotent radical is given.

1. Introduction

Let $\mathfrak{g}$ be a reductive Lie algebra over an algebraically closed field k of characteristic zero. A Lie subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ is called a seaweed Lie algebra if there exists a pair of parabolic subalgebras $(\mathfrak{p}, \mathfrak{p}')$ of $\mathfrak{g}$ such that $\mathfrak{q} = \mathfrak{p} \cap \mathfrak{p}'$ and $\mathfrak{p} + \mathfrak{p}' = \mathfrak{g}$ (we call such a pair of parabolic subalgebras weakly opposite). For example, take the pair consisting of a Borel subalgebra and its opposite.

Seaweed Lie algebras are introduced by Vladimir Dergachev and Alexandre Kirillov [7] in the case $\mathfrak{g} = \mathrm{gl}_n(k)$, and in the above generality by Dmitri Panyushev [12]. The set of seaweed Lie algebras in $\mathfrak{g}$ contains clearly all parabolic subalgebras of $\mathfrak{g}$ and their Levi factors. In particular, they provide new examples of index zero Lie algebras (or Frobenius Lie algebras) [7, 12, 15].

A general formula for the index of a seaweed Lie algebra conjectured in [15] was recently proved by Anthony Joseph [10]. This is an unexpected and pleasant surprise. Naturally, we would like to know to what extent certain classical results can be generalized to this large class of Lie subalgebras of $\mathfrak{g}$.

Let $\mathbf{G}$ be a connected reductive algebraic group whose Lie algebra is $\mathfrak{g}$. Let $(\mathfrak{p}, \mathfrak{p}')$ be a pair of weakly opposite parabolic subalgebras of $\mathfrak{g}$, and $\mathfrak{q} = \mathfrak{p} \cap \mathfrak{p}'$ the corresponding seaweed Lie algebra. Denote by $\mathbf{P}$ and $\mathbf{P}'$ the parabolic subgroups of $\mathbf{G}$ corresponding to $\mathfrak{p}$ and $\mathfrak{p}'$. Set $\mathbf{Q} = \mathbf{P} \cap \mathbf{P}'$.

We are interested in the following question raised by Michel Duflo and Dmitri Panyushev independently :

Question 1.1. *Is there an open $\mathbf{Q}$-orbit in the nilpotent radical $\mathfrak{n}$ of $\mathfrak{q}$?*

Equivalently, we may ask if there is an element $x \in \mathfrak{n}$ verifying $[x, \mathfrak{q}] = \mathfrak{n}$.

In the case where $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$, then the answer is yes, and this is a result that is commonly known as Richardson's Dense Orbit Theorem [14]. If there is an open $\mathbf{Q}$-orbit in the nilpotent radical of $\mathfrak{q}$, then an element in the open $\mathbf{Q}$-orbit is called a Richardson element of $\mathfrak{q}$.

Using some computations with the program Gap4, we are able to check that Richardson elements exist in any seaweed Lie algebra in a simple Lie algebra of rank ≤ 7. However, in type E_8, we found a seaweed Lie algebra whose nilpotent radical does not contain an open orbit.

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Our present task is to study the case of seaweed Lie algebras in $\mathrm{gl}_n(k)$. We prove that :

Theorem 1.2. *Any seaweed Lie algebra in* $\mathrm{gl}_n(k)$ *has a Richardson element.*

In this particular case, seaweed Lie algebras can be viewed as the stabiliser of a pair of weakly opposite flags. This provides a nice description of seaweed Lie algebras, and allows us to transfer the problem to a quiver representations setting, extending the one for parabolic subalgebras considered by Thomas Brüstle, Lutz Hille, Claus Ringel and Gerhard Röhrle [2].

More precisely, we associate to $\mathfrak{q}$ a certain double quiver $\tilde{Q}$ of type A together with a dimension vector $\mathbf{d}$. Then the quotient of the path algebra of $\tilde{Q}$ by certain relations $\mathcal{I}$ has a structure of a quasi-hereditary algebra with respect to some partial order on the vertices. See Section 2 for definitions and basic properties for quasi-hereditary algebras.

A family of quasi-hereditary algebras constructed as quotients of the double of quivers has appeared in different contexts, see [4, 5, 9, 11]. These quasi-hereditary algebras are sometimes called the twisted double incidence algebras of posets, see for example [5].

The existence of an open $\mathbf{Q}$-orbit in the nilpotent radical of $\mathfrak{q}$ corresponds then to the existence of an open $\mathrm{GL}(\mathbf{d})$-orbit in the set $\mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$ of Δ-filtered modules of dimension vector $\mathbf{d}$ verifying $\mathcal{I}$, where $\mathbf{d}$ is a dimension vector completely determined by the pair of weakly opposite flags associated to $\mathfrak{q}$.

The main theorem we prove is:

Theorem 1.3. *There exists a unique (up to isomorphism) exceptional* Δ*-filtered module for any given* Δ*-dimension vector.*

The idea of the proof consists firstly of constructing exceptional Δ-filtered modules with linear Δ-support, and then of constructing an exceptional Δ-filtered module by gluing together in a specific way these exceptional Δ-filtered modules with linear Δ-support. This explicit construction has the advantage of providing an explicit Richardson element. These constructions extend those used in [2].

The paper is organized as follows. In Sections 2 and 3, we give some generalities of quasi-hereditary algebras arising from quotients of the double of quivers. In particular, we show that any $\mathrm{GL}(\mathbf{d})$-orbit in $\mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$ meets a certain vector space R^α. In Section 4, we prove some properties of the gluing process. We generalize in Section 5 certain results in [2] to exceptional Δ-filtered modules of linear Δ-support. We prove our main theorem on the existence of an open $\mathrm{GL}(\mathbf{d})$-orbit in $\mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$ in Section 6. After reviewing some basic properties of seaweed Lie algebras in Section 7, we examine the case of seaweed Lie algebras in $\mathrm{gl}_n(k)$ in Section 8 where we explain how to associate to a seaweed Lie algebra the double of a quiver of type A, and the correspondence between orbits. Section 9 is devoted to seaweed Lie algebras in other simple Lie algebras where we give an example of a seaweed Lie algebra not admitting a Richardson element.

2. Quasi-hereditary algebras arising from quotients of the double of quivers

In this section, we first introduce necessary notation on representations of quivers. Then we recall the construction of a class of quasi-hereditary algebras as quotients of the double of quivers. We also prove some preliminary results on this class of quasi-hereditary algebras. Note that one of the important ingredients for quasi-hereditary algebras is Δ-filtered modules. At the end of this section, we describe varieties of Δ-filtered modules of the quasi-hereditary algebras.

2.1. **Representations of quivers.** Let $Q = (Q_0, Q_1, s, t)$ be a quiver, where Q_0 is the set of vertices, Q_1 is the set of arrows and s and t are two maps from Q_1 to Q_0 given by sending each arrow in Q_1 to its starting vertex and its terminating vertex, respectively. We assume that both Q_0 and Q_1 are finite sets. A vertex $i \in Q_0$ is called a sink vertex if there are no arrows in Q_1 starting from i and a source vertex if there are no arrows in Q_1 terminating at i. We say that i is an admissible vertex if it is either a source or a sink vertex. We denote by $M = (\{M_i\}_{i\in Q_0}, \{M_\alpha\}_{\alpha\in Q_1})$ a representation of Q, where each M_i is a k-vector space and M_α is a k-linear map from $M_{s(\alpha)}$ to $M_{t(\alpha)}$. We view each vertex as a trivial path in Q. A non-trivial path in Q is a sequence of arrows $\rho_1 \cdots \rho_l$ satisfying $t(\rho_i) = s(\rho_{i-1})$ for any $i \geq 2$. We denote by kQ the path algebra of Q. The path algebra kQ is a vector space with all the paths of Q as basis and the multiplication of any two paths ρ and γ given by

$$\rho \cdot \gamma = \begin{cases} \rho\gamma & \text{if } t(\gamma) = s(\rho); \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{J}$ be an ideal of the path algebra kQ. A representation $M = (\{M_i\}_{i\in Q_0}, \{M_\alpha\}_{\alpha\in Q_1})$ of Q is a representation of $(Q,\mathcal{J})$ if the maps M_α satisfy the relations in $\mathcal{J}$, that is, $\sum_{i_1,\cdots,i_l} c_{i_1,\cdots,i_l} M_{\alpha_{i_1}} \cdots M_{\alpha_{i_l}} = 0$ if $\sum_{i_1,\cdots,i_l} c_{i_1,\cdots,i_l}\alpha_{i_1}\cdots\alpha_{i_l} \in \mathcal{J}$. A morphism of two representations N and M of $(Q,\mathcal{J})$ is a family of linear maps $(f_i : N_i \longrightarrow M_i)_{i\in Q_0}$ such that $M_\alpha f_{s(\alpha)} = f_{t(\alpha)} N_\alpha$ for any $\alpha \in Q_1$. It is well-known that the category of representations of $(Q,\mathcal{J})$ is equivalent to the category of left modules of $kQ/\mathcal{J}$. We do not distinguish a representation of $(Q,\mathcal{J})$ from the corresponding $kQ/\mathcal{J}$-module.

Another way to understand representations of a quiver is through the representation variety $\mathrm{Rep}(Q,\mathcal{J},\mathbf{d})$ with dimension vector $\mathbf{d} = (d_i)_{i\in Q_0}$ defined as follows.

$$\mathrm{Rep}(Q,\mathcal{J},\mathbf{d}) = \left\{ (M_\alpha)_{\alpha\in Q_1} \;\middle|\; \begin{array}{l} \text{each entry } M_\alpha \in \mathrm{Hom}(k^{d_{s(\alpha)}}, k^{d_{t(\alpha)}}) \\ \text{and } (M_\alpha)_{\alpha\in Q_1} \text{ satisfies relations in } \mathcal{J} \end{array} \right\}.$$

Note that $\mathrm{Rep}(Q,\mathcal{J},\mathbf{d})$ is a closed subvariety of the affine space $\prod_{\alpha\in Q_1} \mathrm{Hom}(k^{d_{s(\alpha)}}, k^{d_{t(\alpha)}})$. In the case where $\mathcal{J} = 0$, we have $\mathrm{Rep}(Q,\mathcal{J},\mathbf{d}) = \prod_{\alpha\in Q_1} \mathrm{Hom}(k^{d_{s(\alpha)}}, k^{d_{t(\alpha)}})$. The group $\mathrm{GL}(\mathbf{d}) = \prod_i \mathrm{GL}_{d_i}(k)$ acts on $\mathrm{Rep}(Q,\mathcal{J},\mathbf{d})$ by conjugation and there is a one-to-one correspondence between the $\mathrm{GL}(\mathbf{d})$-orbits in $\mathrm{Rep}(Q,\mathcal{J},\mathbf{d})$ and isomorphism classes of $kQ/\mathcal{J}$-modules with dimension vector $\mathbf{d}$.

2.2. **Quasi-hereditary algebras and the double of quivers.** We shall recall the definition of quasi-hereditary algebras, see [3, 11].

Definition 2.1. *Let D be a finite-dimensional k-algebra with a set of representatives of simple D-modules $\{L(i)\}_{i\in I}$. Then D, together with a partial order $(I,\succeq)$, and a set of modules $\{\Delta(i)\}_{i\in I}$, is called a quasi-hereditary algebra if,*

1. *each $\Delta(i)$ has simple top $L(i)$,*
2. *we have $j \preceq i$ for any composition factor $L(j)$ of the radical of $\Delta(i)$,*
3. *the kernel of the projective cover $P(i) \longrightarrow \Delta(i)$ is filtered by the $\Delta(j)$ with $j \succeq i$.*

The modules in $\Delta = \{\Delta(i)\}_{i\in I}$ are called Verma modules. The subcategory of D-modules which are filtered by Verma modules is denoted by $\mathcal{F}(\Delta)$. We say that M is Δ-filtered if $M \in \mathcal{F}(\Delta)$.

Definition 2.2. *A Δ-filtration of a Δ-filtered module M is defined to be a descending chain of submodules, $M = M(1) \supset M(2) \supset \cdots \supset M(r) = 0$, such that for any i, the module $M(i)/M(i+1)$ is isomorphic to $\Delta(j)$ for some j. Each $\Delta(l)$ appearing as a direct summand of $M(i)/M(i+1)$ for some i is called a Δ-composition factor of M. The Δ-length of M is defined to be the number of Δ-composition factors of M.*

From now on, we assume that the quiver Q has no oriented cycles. We denote by $\tilde{Q}$ the double of Q, that is $\tilde{Q}_0 = Q_0$ and $\tilde{Q}_1 = Q_1 \cup \{\alpha^* | \alpha^* \text{ is a reverse arrow of } \alpha \in Q_1\}$. Denote by $k\tilde{Q}$ the path algebra of $\tilde{Q}$. Let $\mathcal{I}$ be the ideal of $k\tilde{Q}$ generated by the following elements

1. $\alpha^*\alpha - \sum_{\gamma\in Q_1, s(\alpha)=t(\gamma)} \gamma\gamma^*$ for any $\alpha \in Q_1$.
2. $\beta^*\alpha$, where $\beta \neq \alpha \in Q_1$ with $t(\alpha) = t(\beta)$.

Note that the inclusion of quivers $Q \subseteq \tilde{Q}$, makes kQ a subalgebra of $k\tilde{Q}/\mathcal{I}$. Also, mapping each $\alpha^* \in \tilde{Q}_1 \backslash Q_1$ to zero gives us a surjective map of algebras $k\tilde{Q}/\mathcal{I} \longrightarrow kQ$. We view any $k\tilde{Q}/\mathcal{I}$-module as an kQ-module and any kQ-module as a $k\tilde{Q}/\mathcal{I}$-module via these two algebra homomorphisms.

Let $\Delta(i)$ be an indecomposable projective kQ-module with simple top $L(i)$. We define a binary relation $\succeq$ on the vertex set Q_0 by $i \succeq j$ if $\Delta(j) \subseteq \Delta(i)$ as kQ-modules. Since kQ is finite-dimensional, we see that $\succeq$ is a partial order on Q_0. We let $P(i)$ be a projective $k\tilde{Q}/\mathcal{I}$-module with simple top $L(i)$.

Proposition 2.3. [5] *The modules $\Delta(i)$ and the partial order $\succeq$ give $k\tilde{Q}/\mathcal{I}$ the structure of a quasi-hereditary algebra. Moreover gldim $k\tilde{Q}/\mathcal{I} \leq 2$.*

The following result is easy and well-known to experts.

Proposition 2.4. *The following are equivalent for a representation M of $(\tilde{Q}, \mathcal{I})$.*
- (i) *The representation M is Δ-filtered.*
- (ii) *The projective dimension of M is at most one.*
- (iii) *The representation M is projective as a kQ-module.*
- (iv) *For any arrow $\alpha \in Q_1$, the k-linear map M_α is injective and*

$$\mathrm{Im}(M_\alpha) \bigcap \sum_{\substack{\beta \in Q_1 \backslash \{\alpha\} \\ t(\beta)=t(\alpha)}} \mathrm{Im}(M_\beta) = 0.$$

A proof of the equivalence of (i)-(iii) of Proposition 2.4 can be found in [9] and the equivalence of (iii) and (iv) is clear for any quiver.

Definition 2.5. *Let M be a Δ-filtered module.*
1. *The Δ-dimension vector of M, denoted by $\underline{\dim}_\Delta(M)$, is the dimension vector with its i-th entry $(\underline{\dim}_\Delta(M))_i$ the multiplicity of $\Delta(i)$ as a Δ-composition factor in a Δ-filtration of M.*
2. *The Δ-support of M, denote by $\mathrm{supp}_\Delta(M)$, is the the full subgraph of Q with the set of vertices $\{i \in Q_0 | (\underline{\dim}_\Delta(M))_i > 0\}$.*

Since any Δ-filtered module M is projective as a kQ-module, there is a unique decomposition, up to isomorphism, of M as a kQ-module into a direct sum of projective kQ-modules. So the Δ-dimension vector of a Δ-filtered module is well-defined. We will also need the usual support of M, denoted by $\mathrm{supp}(M)$, defined via the ordinary dimension vector $\underline{\dim}(M)$. We have $\mathrm{supp}_\Delta(M) \subseteq \mathrm{supp}(M)$ with strict inclusion in general. Given a non-zero vector $\mathbf{d} \in \mathbb{N}^{Q_0}$, we denote by $\mathrm{supp}(\mathbf{d})$ the support of $\mathbf{d}$, which is the full subquiver of Q with the set of vertices $\mathrm{supp}(\mathbf{d})_0 = \{i | d_i > 0\}$.

We need some properties of the Δ-filtered modules. By the definition of $\mathcal{I}$, it is not difficult to see that the $k\tilde{Q}/\mathcal{I}$-module $\Delta(i)$ is projective for a source i, that is $\mathrm{Ext}^1_{k\tilde{Q}/\mathcal{I}}(\Delta(i), -) = 0$. A similar, but weaker, property holds for sinks.

Lemma 2.6. *If M is Δ-filtered and i is a sink in Q, then $\mathrm{Ext}^1_{k\tilde{Q}/\mathcal{I}}(M, \Delta(i)) = 0$.*

Proof. Observe that $\mathrm{Hom}_{k\tilde{Q}/\mathcal{I}}(\Omega\Delta(j), \Delta(i)) = 0$ for all j, where $\Omega\Delta(j)$ denotes the syzygy of $\Delta(j)$ as a $k\tilde{Q}/\mathcal{I}$-module. Hence $\mathrm{Ext}^1_{k\tilde{Q}/\mathcal{I}}(\Delta(j), \Delta(i)) = 0$. The lemma follows by using long exact sequences in homology and induction on the length of a Δ-filtration of M. □

Lemma 2.7. *Let i be a source in Q and let M be Δ-filtered with $d_i = (\underline{\dim}_\Delta(M))_i > 0$, then there exists a unique submodule $\Delta(i)^{d_i} \subseteq M$. Moreover the quotient module $M/\Delta(i)^{d_i}$ is Δ-filtered.*

Proof. We use induction on the length of a Δ-filtration

$$0 \subseteq M_1 \subseteq \cdots \subseteq M_t = M$$

of M. If $t = 1$, then $M = \Delta(i)$, and we are done. If $t > 1$, we have a short exact sequence

$$0 \longrightarrow M_{t-1} \longrightarrow M \longrightarrow M/M_{t-1} \longrightarrow 0$$

If $M/M_{t-1} \cong \Delta(i)$, then $M \cong M_{t-1} \oplus \Delta(i)$, since $\Delta(i)$ is projective, and by induction we are done. Otherwise, by induction we have the pushout Y in the following diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Delta(i)^{d_i} & = & \Delta(i)^{d_i} & & & & \\
 & & \downarrow{\scriptstyle g} & & \downarrow & & & & \\
0 & \longrightarrow & M_{t-1} & \longrightarrow & M & \longrightarrow & M/M_{t-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{Cok}(g) & \longrightarrow & Y & \longrightarrow & M/M_{t-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Here $\mathrm{Cok}(g)$ is Δ-filtered. This shows that Y is Δ-filtered. This proves the lemma. □

Lemma 2.8. *Let i be a sink in Q and let M be Δ-filtered with $d_i = (\underline{\dim}_\Delta(M))_i > 0$, then there exists a unique quotient module which is isomorphic to $\Delta(i)^{d_i}$. Moreover the submodule Y, satisfying $M/Y \cong \Delta(i)^{d_i}$, is Δ-filtered.*

Proof. We use induction on the length of a Δ-filtration

$$0 \subseteq M_1 \subseteq \cdots \subseteq M_t = M$$

of M. If $t = 1$, then $M = \Delta(i)$ and we are done. If $t > 1$, then we have a short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M/M_1 \longrightarrow 0$$

If $M_1 = \Delta(i)$, then $M \cong M/M_1 \oplus \Delta(i)$ by Lemma 2.6. If not, then by induction we get the pullback Y in the following diagram.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & Y & \longrightarrow & \mathrm{Ker}(g) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M & \longrightarrow & M/M_1 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow{\scriptstyle g} & & \\
 & & & & \Delta(i)^{d_i} & = & \Delta(i)^{d_i} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Thus Y is Δ-filtered, since M_1 and $\mathrm{Ker}(g)$ are Δ-filtered. This proves the lemma. □

Given a Δ-filtered module M with Δ-dimension vector $\mathbf{d} = (d_i)_i$, by iteration of Lemma 2.7 or Lemma 2.8 we obtain a nice Δ-filtration $M = M(0) \supset M(1) \supset M(2) \supset \cdots$ such that $M(i)/M(i+1) \cong \Delta(l_i)$. Moreover, for any i either $l_{i+1} \succeq l_i$ or they are non-comparable.

2.3. **Representation variety of Δ-filtered modules.** Let $\mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$ be the subset of $\mathrm{Rep}(\tilde{Q}, \mathcal{I}, \mathbf{d})$, containing the points which correspond to Δ-filtered modules, where $\mathcal{I}$ is the ideal defined in Section 2.2. Suppose that $\mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$ is not empty. Then by Proposition 2.4, for any vertex $i_0 \in Q_0$, we have $d_{i_0} \geq \sum_{\alpha \in Q_1, t(\alpha)=i_0} d_{s(\alpha)}$. We let $d'_{i_0} = d_{i_0} - \sum_{\alpha \in Q_1, t(\alpha)=i_0} d_{s(\alpha)}$ and decompose

$$k^{d_{i_0}} = \left(\bigoplus_{\alpha \in Q_1,\ t(\alpha)=i_0} k^{d_{s(\alpha)}} \right) \oplus k^{d'_{i_0}}.$$

Now define

$$R^\alpha = \{M \in \mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d}) | \text{ each map } M_\alpha \text{ is the standard embedding } k^{d_{s(\alpha)}} \subseteq k^{d_{i_0}}\}.$$

Proposition 2.9. *Suppose that the set* $\mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$ *is non-empty.*

(1) *The subset* R^α *is an affine space.*
(2) *The subset* $\mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$ *is open and irreducible in* $\mathrm{Rep}(\tilde{Q}, \mathcal{I}, \mathbf{d})$.

Proof. By the definition of $\mathcal{I}$, we see that any element in $\mathcal{I}$ is a linear combination of arrows in $\tilde{Q}_1 \backslash Q_1$ when we view arrows in Q_1 as constants. Thus R^α is the solution space of a linear system, and so it is an affine space. The openness of $\mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$ follows from Proposition 2.4 (iv). By Proposition 2.4 (iii), we know that the $\mathrm{GL}(\mathbf{d})$-orbit of any Δ-filtered representation M meets with R^α. So we have a surjective map $\mathrm{GL}(\mathbf{d}) \times R^\alpha \to \mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$, and thus $\mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$ is irreducible. □

Remark 2.10. *Note that Δ-filtered modules are characterized by vanishing of some extension groups (see* [6]*). Therefore, in general, for any quasi-hereditary algebra Λ, we have that the module variety* $\mathrm{mod}_\Delta(\Lambda, \mathbf{d})$ *of Δ-filtered modules with dimension vector* $\mathbf{d}$ *is open in the whole module variety* $\mathrm{mod}(\Lambda, \mathbf{d})$.

3. The double of a quiver of type A_n

From now on, we shall concentrate on the study of $k\tilde{Q}/\mathcal{I}$, where Q is a quiver of type A_n, and we denote by $\mathbf{D}$ the algebra $k\tilde{Q}/\mathcal{I}$. In this section, we first fix some notation concerning Q of type A_n. Then we give an example of $\mathbf{D}$ with Q of type A_5, illustrating the structure of $\mathbf{D}$.

Let $Q_0 = \{1, \dots, n\}$ be the set of vertices of Q. For $1 \leq i \leq n-1$, there is a unique arrow connecting vertices i and $i+1$ and we denote it by α_i. We denote by β_i the reverse arrow of α_i in $\tilde{Q}_1$, that is, $s(\beta_i) = t(\alpha_i)$ and $t(\beta_i) = s(\alpha_i)$. For the sake of convenience, we set $\alpha_0, \beta_0, \alpha_n$ and β_n to be zero arrows. Now the relations in $\mathcal{I}$ defined in Section 2.2 are as follows.

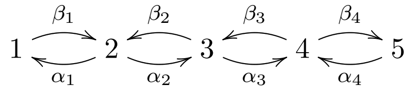
(1) $\beta_{j-1}\alpha_{j-1}$ and $\beta_j\alpha_j$ for a source j;
(2) $\beta_j\alpha_{j-1}$ and $\beta_{j-1}\alpha_j$ for a sink j;
(3) $\beta_{j-1}\alpha_{j-1} - \alpha_j\beta_j$ for j non-admissible and $t(\alpha_j) = j$;
(4) $\alpha_{j-1}\beta_{j-1} - \beta_j\alpha_j$ for j non-admissible and $s(\alpha_j) = j$.

We say that j is a successor of i, if $i \succ j$ and $i \succeq l \succeq j$ implies $j = l$ or $l = i$. In this case, i is called a predecessor of j. Note that if i is non-admissible then it has a unique successor and a unique predecessor. If i is sink it has no successor, and at most two predecessors, and exactly two if and only if i is an interior vertex of Q. Similarly, if i is a source, then it has no predecessor, and at most two successors, and exactly two if and only if i is an interior vertex of Q. Another description of $\succeq$ is given by $i \succeq j$ if and only if there is a path in Q from i to j.

Example 3.1. *Let Q be the quiver:*

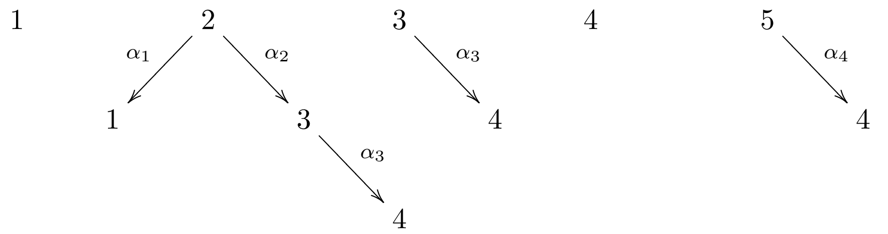
$$1 \xleftarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} 4 \xleftarrow{\alpha_4} 5$$

Then $\tilde{Q}$ is



and $\mathbf{D} = k\tilde{Q}/\mathcal{I}$ *with* $\mathcal{I}$ *generated by* $\{\beta_1\alpha_1, \beta_2\alpha_2, \alpha_2\beta_2 - \beta_3\alpha_3, \beta_3\alpha_4, \beta_4\alpha_3, \beta_4\alpha_4\}$. *We denote by* e_i *the trivial path in* $\tilde{Q}$ *corresponding to vertex* i.

(1) Each $\Delta(i)$ *is an indecomposable projective* kQ*-module and so it has a* k*-basis consisting of paths in* Q *which start from vertex* i. *So* $\Delta(1), \ldots, \Delta(5)$ *are, respectively, as follows.*

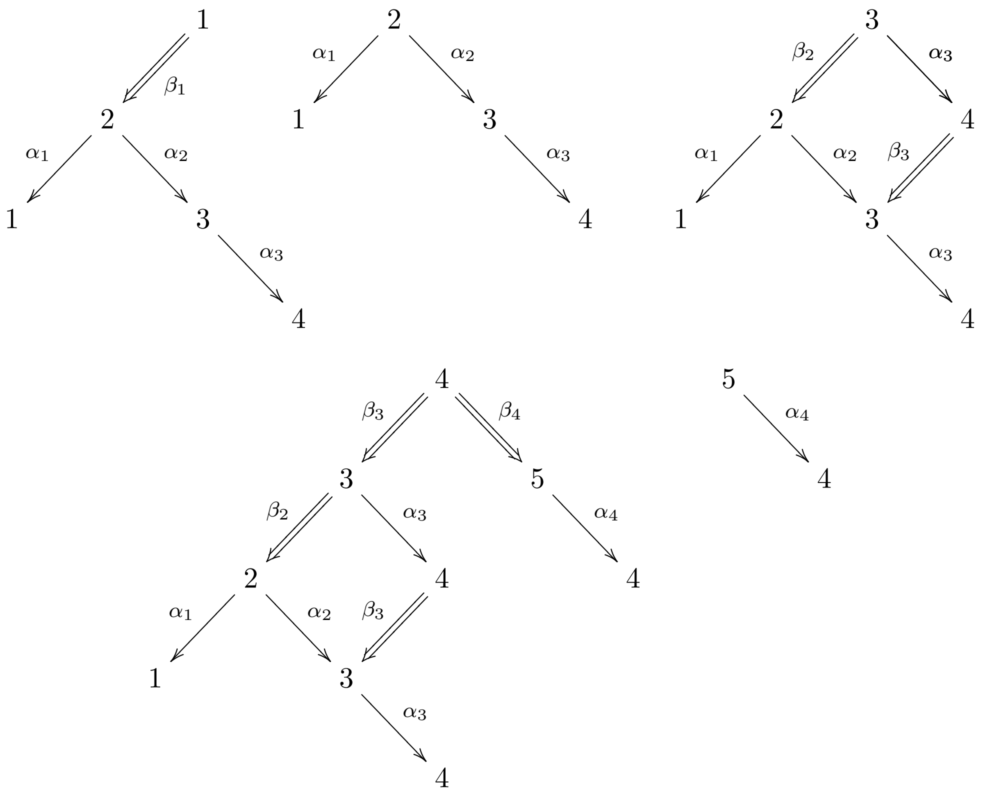


More precisely, the modules $\Delta(1), \ldots, \Delta(5)$ *have* k*-bases* $\{e_1\}$, $\{e_2, \alpha_1, \alpha_2, \alpha_3\alpha_2\}$, $\{e_3, \alpha_3\}$, $\{e_4\}$ *and* $\{e_5, \alpha_4\}$, *respectively.*

(2) Now the partial order defined in Section 2.2 is: $2 \succ 1$, $2 \succ 3 \succ 4$ *and* $5 \succ 4$.

(3) We now compute indecomposable projective $\mathbf{D}$*-modules* $P(1), \ldots, P(5)$. *Each of them has a* k*-basis consisting of residue classes of paths* $\overline{p}$, *such that* $p = ab$, *where* a *consists entirely of* α_j*'s and* b *consists entirely of* β_j*'s. More precisely, the modules* $P(1), \ldots, P(5)$ *have a* k*-basis consisting of* $\{e_1, \beta_1, \alpha_1\beta_1, \alpha_2\beta_1, \alpha_3\alpha_2\beta_1\}$, $\{e_2, \alpha_1, \alpha_2, \alpha_3\alpha_2\}$, $\{e_3, \beta_2, \alpha_3, \alpha_1\beta_2, \alpha_2\beta_2, \alpha_3\alpha_2\beta_2\}$, $\{e_4, \beta_3, \beta_2\beta_3, \alpha_3\beta_3, \alpha_1\beta_2\beta_3, \alpha_2\beta_2\beta_3, \alpha_3\alpha_2\beta_2\beta_3, \beta_4, \alpha_4\beta_4\}$ *and* $\{e_5, \alpha_4\}$, *respectively.*

(4) We illustrate the structure of modules $P(1), \ldots, P(5)$, *respectively, as follows, where the action by the* β_i*'s is represented by double arrows.*



(5) For any vertex i, the indecomposable projective $\mathbf{D}$-module $P(i)$ has a simple top $L(i)$ and has $\bigoplus_j P(j)$ as its maximal submodule, where j's are predecessors of i. That is, we have a short exact sequence

$$0 \longrightarrow \bigoplus_j P(j) \longrightarrow P(i) \longrightarrow L(i) \longrightarrow 0.$$

Remark 3.2. *The properties of $\mathbf{D}$ in Example 3.1 can be generalized to $k\tilde{Q}/\mathcal{I}$ for Q of other quivers without oriented cycles, see* [9] *for details.*

4. Gluing of Δ-filtered modules

In this section, we prove some further properties of Δ-filtered $\mathbf{D}$-modules. We first explain how to glue two Δ-filtered modules at an admissible vertex and obtain a new Δ-filtered module, which usually has higher dimension. We prove that any Δ-filtered module can be obtained by gluing Δ-filtered modules. Recall that a $\mathbf{D}$-module is said to be exceptional if $\mathrm{Ext}^1_{\mathbf{D}}(M, M) = 0$. Suppose that a Δ-filtered module M is exceptional and is glued from M' and M'' at an admissible vertex. Then we show that both M' and M'' are exceptional.

Let i be a sink or a source of Q. Let M' and M'' be two Δ-filtered modules with $\mathrm{supp}_\Delta(M') \subseteq \{1, \ldots, i\}$ and $\mathrm{supp}_\Delta(M'') \subseteq \{i, \ldots, n\}$, and $(\underline{\dim}_\Delta(M'))_i = (\underline{\dim}_\Delta(M''))_i = d_i > 0$.

Assume that i is a sink in Q. By Lemma 2.8, we have short exact sequences

$$0 \longrightarrow \mathrm{Ker}(f') \longrightarrow M' \xrightarrow{f'} \Delta(i)^{d_i} \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Ker}(f'') \longrightarrow M'' \xrightarrow{f''} \Delta(i)^{d_i} \longrightarrow 0.$$

Let M be given by the pullback of f' and f'', that is, we have a short exact sequence

$$0 \longrightarrow M \longrightarrow M' \oplus M'' \xrightarrow{(f' \; -f'')} \Delta(i)^{d_i} \longrightarrow 0.$$

Similarly, if i is a source, we have short exact sequences

$$0 \longrightarrow \Delta(i)^{d_i} \xrightarrow{f'} M' \longrightarrow \mathrm{Cok}(f') \longrightarrow 0$$

and

$$0 \longrightarrow \Delta(i)^{d_i} \xrightarrow{f''} M'' \longrightarrow \mathrm{Cok}(f'') \longrightarrow 0$$

Let M be the pushout of f' and f'', that is, we have a short exact sequence

$$0 \longrightarrow \Delta(i)^{d_i} \xrightarrow{\begin{pmatrix} f' \\ -f'' \end{pmatrix}} M' \oplus M'' \longrightarrow M \longrightarrow 0.$$

In both of these cases, we say that M is obtained by gluing M' and M'' at i.

Lemma 4.1. *Let M' and M'' be as above. If M is glued from M' and M'' at i, then M is Δ-filtered.*

Proof. If i is a sink, then we have an exact sequence

$$\mathrm{Ext}^2_{\mathbf{D}}(M' \oplus M'', -) \longrightarrow \mathrm{Ext}^2_{\mathbf{D}}(M, -) \longrightarrow \mathrm{Ext}^3_{\mathbf{D}}(\Delta(i)^{d_i}, -),$$

showing that M has projective dimension at most one. Therefore M is Δ-filtered by Proposition 2.4.

Now suppose i is a source. Then we have an exact sequence

$$\mathrm{Ext}^1_{\mathbf{D}}(\Delta(i)^{d_i}, -) \longrightarrow \mathrm{Ext}^2_{\mathbf{D}}(M, -) \longrightarrow \mathrm{Ext}^2_{\mathbf{D}}(M' \oplus M'', -).$$

This shows again that M is Δ-filtered. □

Proposition 4.2. *Let i be a sink or a source in Q. Then for any Δ-filtered module M with $(\underline{\dim}_\Delta(M))_i > 0$, there exists Δ-filtered modules M' and M'' with $(\underline{\dim}_\Delta(M'))_i = (\underline{\dim}_\Delta(M''))_i = (\underline{\dim}_\Delta(M))_i$ such that M is isomorphic to a module glued from M' and M'' at i.*

Proof. Let i be a source in Q. By Lemma 2.7, there is a short exact sequence

$$0 \longrightarrow \Delta(i)^{d_i} \xrightarrow{f} M \xrightarrow{g} \mathrm{Cok}(f) \longrightarrow 0,$$

where $\mathrm{Cok}(f)$ is Δ-filtered. Here $\Delta(i)^{d_i}$ is the submodule of M generated by M_i. Then $(\underline{\dim}(\mathrm{Cok}(f)))_i = 0$ and therefore $\mathrm{Cok}(f) = Y' \oplus Y''$, where $\mathrm{supp}(Y')$ is contained in $\{1, \dots, i-1\}$ and $\mathrm{supp}(Y'')$ is contained in $\{i+1, \dots, n\}$. We have short exact sequences

$$0 \longrightarrow \Delta(i)^{d_i} \longrightarrow M' \longrightarrow Y' \longrightarrow 0$$

and

$$0 \longrightarrow \Delta(i)^{d_i} \longrightarrow M'' \longrightarrow Y'' \longrightarrow 0,$$

where $M' = g^{-1}(Y')$ and $M'' = g^{-1}(Y'')$. By Proposition 2.4, we see that M' and M'' are Δ-filtered with their Δ-supports contained in $\{1, \dots, i\}$ and $\{i, \dots, n\}$, respectively. Now there is a short exact sequence

$$0 \longrightarrow X \longrightarrow M' \oplus M'' \xrightarrow{(g'\ g'')} M \longrightarrow 0,$$

where g' and g'' are the inclusion maps. We have $\underline{\dim}(X) = \underline{\dim}(\Delta(i)^{d_i})$ and X is projective as an kQ-module. Therefore $X \cong \Delta(i)^{d_i}$. Hence M is glued from M' and M'' at i.

Now suppose that i is a sink. We let N' and N'' be the submodules of M generated by $V'' = \bigoplus_{j=1}^{i-1} M_j$ and $V' = \bigoplus_{j=i+1}^{n} M_j$. By the relations in $\mathcal{I}$, we know that N'' is supported on the vertices $\{1, \dots, i\}$ and N' is supported on the vertices $\{i, \dots, n\}$. Now let $M' = M/N'$ and $M'' = M/N''$. We have $(N' \cap N'')_i = 0$, since M is Δ-filtered and so $\mathrm{Im}(M_{\alpha_{i-1}}) \cap \mathrm{Im}(M_{\alpha_{i+1}}) = 0$. Moreover, for all $j \neq i$ we have $N'_j = 0$ or $N''_j = 0$. Hence $N' \cap N'' = 0$. So we have an embedding

$$M \xrightarrow{\binom{f'}{f''}} M' \oplus M'',$$

where f' and f'' are the quotient maps. Now by comparing dimension vectors, we see that the cokernel of $M \longrightarrow M' \oplus M''$ is $L(i)^{d_i} \cong \Delta(i)^{d_i}$. Hence $M' \oplus M''$ is Δ-filtered and M is obtained by gluing M' and M'' at i. □

From the proof of Proposition 4.2, for a module M obtained by gluing two Δ-filtered modules M' and M'', we see that if i is a source, then M' and M'' can be viewed as submodules of M, and if i is a sink, then M' and M'' can be viewed as quotients of M.

Example 4.3. *We consider the quiver Q in Example 3.1. The projective module $P(4)$ has two submodules isomorphic to $P(3)$ and $P(5)$, respectively. We have a short exact sequence*

$$0 \longrightarrow P(4) \longrightarrow P(4)/P(3) \oplus P(4)/P(5) \longrightarrow \Delta(4) \longrightarrow 0,$$

that is, the module $P(4)$ can be glued from its quotient modules $P(4)/P(3)$ and $P(4)/P(5)$ at vertex 4.

Lemma 4.4. *Let $h : M \longrightarrow N$ be a morphism of two Δ-filtered modules which are obtained by gluing M', M'' and N', N'', respectively, at i. Then*

(1) *if i is a source, then f restricts to maps $h' : M' \longrightarrow N'$ and $h'' : M'' \longrightarrow N''$ of submodules.*
(2) *if i is a sink, then f induces maps $h' : M' \longrightarrow N'$ and $h'' : M'' \longrightarrow N''$ of quotient modules.*

Proof. Assume that i is a source. We have $h(M') \subseteq N'$ and $h(M'') \subseteq N''$, since $h(M_i) \subseteq N_i$. The lemma follows in this case.

Now suppose that i is a sink. We may assume that $M' = M/X$ and $M'' = M/Y$, where X and Y are the submodules of M generated by $\bigoplus_{j=i+1}^{n} M_j$ and $\bigoplus_{j=1}^{i-1} M_j$, respectively. We have $h(M_{i-1}) \subseteq N_{i-1}$ and $h(M_{i+1}) \subseteq N_{i+1}$. The lemma follows. □

Let M and N be modules glued from M', M'' and N', N'', respectively. If i is a sink, we construct a morphism $h : M \longrightarrow N$ from maps $h' : M' \longrightarrow N'$ and $h'' : M'' \longrightarrow N''$ provided there exists a map $w : \Delta(i)^{d_i} \longrightarrow \Delta(i)^{d_i}$, where $d_i = \underline{\dim}_\Delta(M)_i = \underline{\dim}_\Delta(N)_i$, such that the second square of the diagram of gluing sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & M' \oplus M'' & \longrightarrow & \Delta(i)^{d_i} & \longrightarrow & 0 \\ & & \vdots & & \downarrow{\scriptstyle h'\oplus h''} & & \downarrow{\scriptstyle w} & & \\ 0 & \longrightarrow & N & \longrightarrow & N' \oplus N'' & \longrightarrow & \Delta(i)^{d_i} & \longrightarrow & 0 \end{array}$$

commutes. Now h is a morphism $M \longrightarrow N$, which makes the first square of the above diagram commute. In this case, we say that h is obtained by gluing h' and h'' at i. We similarly define gluing of morphisms at a source i.

Lemma 4.5. *Let M be obtained by gluing M' and M'' at i. If M is exceptional, then M' and M'' are exceptional.*

Proof. Assume that M is exceptional. We first consider the case where i is a sink. We have a short exact sequence

$$0 \longrightarrow M \longrightarrow M' \oplus M'' \longrightarrow \Delta(i)^c \longrightarrow 0,$$

where $c = (\underline{\dim}_\Delta(M'))_i = (\underline{\dim}_\Delta(M''))_i$. By applying the functor $\mathrm{Hom}_{\mathbf{D}}(M, -)$ to this sequence and using Lemma 2.6, we see that $\mathrm{Ext}^1_{\mathbf{D}}(M, M' \oplus M'') = 0$. Then by applying $\mathrm{Hom}_{\mathbf{D}}(-, M')$, we get a surjection $\mathrm{Ext}^1_{\mathbf{D}}(\Delta(i)^c, M') \longrightarrow \mathrm{Ext}^1_{\mathbf{D}}(M' \oplus M'', M')$. From Lemma 2.8, we have a short exact sequence

$$0 \longrightarrow X \longrightarrow M'' \longrightarrow \Delta(i)^c \longrightarrow 0.$$

Note that $\mathrm{supp}(\mathrm{top}(X)) \cap \mathrm{supp}(M') = \emptyset$, where $\mathrm{top}(X)$ is the top of X. Thus $\mathrm{Hom}_{\mathbf{D}}(X, M') = 0$. We get an injection $\mathrm{Ext}^1_{\mathbf{D}}(\Delta(i)^c, M') \longrightarrow \mathrm{Ext}^1_{\mathbf{D}}(M'', M')$ by applying $\mathrm{Hom}_{\mathbf{D}}(-, M')$. Hence $\mathrm{Ext}^1_{\mathbf{D}}(M', M') = 0$. By symmetry, $\mathrm{Ext}^1_{\mathbf{D}}(M'', M'') = 0$.

Now assume that i is a source. We have a short exact sequence

$$0 \longrightarrow \Delta(i)^c \longrightarrow M' \oplus M'' \longrightarrow M \longrightarrow 0.$$

Applying $\mathrm{Hom}_{\mathbf{D}}(-, M)$, we get $\mathrm{Ext}^1_{\mathbf{D}}(M' \oplus M'', M) = 0$. Now by applying $\mathrm{Hom}_{\mathbf{D}}(M', -)$, we get a surjection $\mathrm{Ext}^1_{\mathbf{D}}(M', \Delta(i)^c) \longrightarrow \mathrm{Ext}^1_{\mathbf{D}}(M', M' \oplus M'')$. From Lemma 2.7, we have a short exact sequence

$$0 \longrightarrow \Delta(i)^c \longrightarrow M'' \longrightarrow Y \longrightarrow 0.$$

Since $\mathrm{supp}(M') \cap \mathrm{supp}(Y) = \emptyset$, we have $\mathrm{Hom}_{\mathbf{D}}(M', Y) = 0$. Using $\mathrm{Hom}_{\mathbf{D}}(M', -)$, we get an injection $\mathrm{Ext}^1_{\mathbf{D}}(M', \Delta(i)^c) \longrightarrow \mathrm{Ext}^1_{\mathbf{D}}(M', M'')$. Hence $\mathrm{Ext}^1_{\mathbf{D}}(M', M') = 0$. By symmetry, $\mathrm{Ext}^1_{\mathbf{D}}(M'', M'') = 0$. This finishes the proof. □

5. Exceptional Δ-filtered modules with a linear Δ-support

In this section, we first recall from [2] the construction of exceptional Δ-filtered modules with Δ-dimension vector a non-zero vector in $\mathbb{N}^n$ where Q is linearly oriented. We modify this construction to the case where the orientation of Q is arbitrary and the given dimension vector has a linear support, that is, as a subquiver of Q the support is linearly oriented. In particular, we say that M is a module with linear Δ-support if the subquiver $\mathrm{supp}_\Delta(M)$ is linearly oriented. We will define an order on Δ-filtered modules. We also prove some properties of the exceptional modules with a linear Δ-support.

5.1. **Exceptional Δ-filtered modules for Q of linear orientation.** Following [2], where Q is of linear orientation with vertex 1 a sink and vertex n a source, a module M is isomorphic to a nonzero submodule of $P(1)$ if and only if the socle of M is $L(1)$. Such a module M is indecomposable, exceptional and Δ-filtered. Moreover, the map sending a submodule M of $P(1)$ to the set of vertices of its Δ-support affords a bijection between the set of submodules of $P(1)$ and the set of the subsets of $\{1, \dots, n\}$. We denote by $\Delta(I)$ the submodule of $P(1)$ with I as the set of vertices of its Δ-support. Given $\mathbf{d}$ a non-zero vector in $\mathbb{N}^n$, define $I(\mathbf{d}) = \mathrm{supp}(\mathbf{d})_0$, the set of vertices of $\mathrm{supp}(\mathbf{d})$. We denote by $\mathbf{d}_{I(\mathbf{d})}$ the dimension vector with

$$(\mathbf{d}_{I(\mathbf{d})})_i = \begin{cases} 1 & \text{if } i \in I(\mathbf{d}), \\ 0 & \text{otherwise.} \end{cases}$$

Now define inductively a module $\Delta(\mathbf{d})$ as follows: $\Delta(\mathbf{d}) = \Delta(I(\mathbf{d})) \oplus \Delta(\mathbf{d} - \mathbf{d}_{I(\mathbf{d})})$. In this way we also obtain a descending sequence of subsets $I(\mathbf{d}) \supseteq I(\mathbf{d} - \mathbf{d}_{I(\mathbf{d})}) \supseteq \cdots$.

Theorem 5.1. [2] *Given $\mathbf{d}$ a non-zero vector in $\mathbb{N}^n$. The module $\Delta(\mathbf{d})$ constructed above is the unique (up to isomorphism) exceptional Δ-filtered module with Δ-dimension vector $\mathbf{d}$.*

5.2. **Exceptional Δ-filtered modules with a linear Δ-support.** Let $\mathbf{d}$ be a non-zero vector in $\mathbb{N}^n$ with a linear support. Suppose that the set of vertices of $\mathrm{supp}_\Delta(\mathbf{d})$ is $\{i, i+1, \cdots, j\}$, where any vertex in $\{i+1, \cdots, j-1\}$ is a non-admissible vertex of Q. We may suppose that i is a source in $\mathrm{supp}(\mathbf{d})$ and j is a sink in $\mathrm{supp}(\mathbf{d})$. We consider the following subquiver of Q,

$$l \longleftarrow \cdots \longleftarrow i' \longrightarrow \cdots \longrightarrow i \longrightarrow i+1 \longrightarrow \cdots \longrightarrow j \longrightarrow \cdots \longrightarrow j' \ ,$$

where i' is a source vertex, l and j' are sink vertices and all other vertices are non-admissible vertices of Q. We denote by Q' the subquiver with the set of vertices $Q'_0 = \{i', i'+1 \dots, j, \cdots, j'\}$. Let $\tilde{Q}'$ be the double of Q' and $\mathcal{I}' = k\tilde{Q}' \cap \mathcal{I}$. Then $k\tilde{Q}'/\mathcal{I}'$ is quasi-hereditary as well and its Verma modules are given by the indecomposable projective kQ'-modules. Note that Q' is linearly oriented of type A. By Theorem 5.1, there is a unique Δ-filtered $k\tilde{Q}'/\mathcal{I}'$-module with Δ-dimension vector $\mathbf{d}$. We denote this $k\tilde{Q}'/\mathcal{I}'$-module again by $\Delta(\mathbf{d})$. Note that we can consider $\Delta(\mathbf{d})$ as a $\mathbf{D}$-module which is not necessarily Δ-filtered. Note also that all indecomposable projective kQ'-modules, except at i' when i' is interior, coincide respectively to the indecomposable projective kQ-modules with the same simple top. We construct a $\mathbf{D}$-module $M(\mathbf{d})$ as follows. If i is not an interior source of Q, we let $M(\mathbf{d}) = \Delta(\mathbf{d})$. We now consider the case where i is an interior source of Q. We write

$$\Delta(\mathbf{d}) = \bigoplus_{s \geq 1} \Delta(I_s),$$

where $I(\mathbf{d}) = I_1 \supseteq I_2 = I(\mathbf{d} - \mathbf{d}_{I(\mathbf{d})}) \supseteq \cdots$ and each $\Delta(I_s)$ is an indecomposable exceptional Δ-filtered $k\tilde{Q}'/\mathcal{I}'$-module. For each I_s, define $M_{I_s} = \Delta(I_s)$ if $(\underline{\dim}_\Delta(\Delta(I_s)))_i = 0$; otherwise define M_{I_s} as follows,

1. $(M_{I_s})_r = (\Delta(I_s))_r$ for $r \in \{i, \cdots, j'\}$ and $(M_{I_s})_\gamma = (\Delta(I_s))_\gamma$ for $\gamma \in \{\alpha_s, \beta_s\}_{s=i}^{j'-1}$,
2. $(M_{I_s})_r = (\Delta(I_s))_i$ for $r \in \{l, \cdots, i-1\}$ and $(M_{I_s})_\gamma = 1$ for $\gamma \in \{\alpha_s\}_{s=l}^{i-1}$ and $(M_{I_s})_\gamma = 0$ for $\gamma \in \{\beta_s\}_{s=l}^{i-1}$,

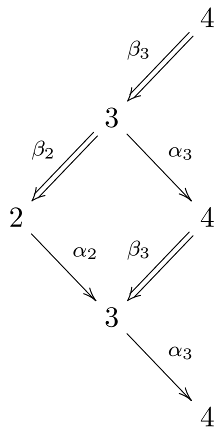
and zero elsewhere. By the construction, we see that M_{I_s} is Δ-filtered and the set of vertices of $\mathrm{supp}_\Delta(M_{I_s})$ is I_s. Now let

$$M(\mathbf{d}) = \bigoplus_{s \geq 1} M_{I_s}.$$

Example 5.2. *We consider the quivers and use the notation in Example 3.1. Let $\mathbf{d} = (0, 1, 1, 2, 0)$. Then*

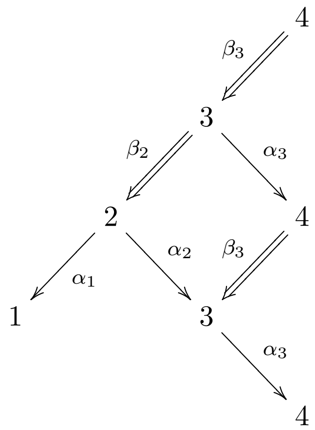
(1) $\mathrm{supp}(\mathbf{d}) = 2 \longrightarrow 3 \longrightarrow 4$ *, which is linearly oriented.*

(2) *Denote* $I(\mathbf{d})$ *by* I_1 *and so* $I_1 = \{2,3,4\}$ *and* $\mathbf{d}_{I_1} = (0,1,1,1,0)$. *Thus* $\mathbf{d} - \mathbf{d}_{I_1} = (0,0,0,1,0)$ *and* $I(\mathbf{d} - \mathbf{d}_{I_1}) = \{4\}$, *we denote it by* I_2. *Now* $\Delta(\mathbf{d}) = \Delta(I_1) \oplus \Delta(I_2)$ *with* $\Delta(I_2) = L(4)$ *and* $\Delta(I_1)$ *as follows.*



Note that $\Delta(I_1)$ *is a* Δ*-filtered* $k\tilde{Q}'/\mathcal{I}'$*-module, but not a* Δ*-filtered* $\mathbf{D}$*-module, where* $Q' = \mathrm{supp}(\mathbf{d})$ *and* $\mathcal{I}' = k\tilde{Q}' \cap \mathcal{I}$.

(3) *By our construction above we have* M_{I_1} *as follows.*



Since $(\underline{\dim}_\Delta(\Delta(I_2)))_2 = 0$, *we let* $M_{I_2} = \Delta(I_2)$. *Now* $M(\mathbf{d}) = M_{I_1} \oplus L(4)$.

Proposition 5.3. *Let* $\mathbf{d}$ *be a non-zero vector in* $\mathbb{N}^n$ *with a linear support as above. Then the module* $M(\mathbf{d})$ *constructed above is the unique (up to isomorphism) exceptional* Δ*-filtered module with* Δ*-dimension vector* $\mathbf{d}$.

Proof. In view of Proposition 2.9, we need only to show that the module $M(\mathbf{d})$ is exceptional. Suppose that L is a self-extension of $M(\mathbf{d})$. Then L is Δ-filtered. If i is not an interior source of Q, then this proposition follows from Theorem 5.1. We consider the case where $i = i'$ is interior. Since $(\underline{\dim}_\Delta M(\mathbf{d}))_i = \underline{\dim}(M(\mathbf{d}))_i = d_i$, we have $M(\mathbf{d})_{\alpha_r}$ is injective for $l \leq r \leq i-1$, and so is L_{α_r} for $l \leq r \leq i-1$. We have a short exact sequence

$$0 \longrightarrow M(\mathbf{d}) \xrightarrow{\lambda} L \xrightarrow{\mu} M(\mathbf{d}) \longrightarrow 0,$$

which induces another short exact sequence:

$$0 \longrightarrow \Delta(\mathbf{d}) \xrightarrow{\overline{\lambda}} L/\Delta(i-1)^{2d_i} \xrightarrow{\overline{\mu}} \Delta(\mathbf{d}) \longrightarrow 0.$$

Since $\Delta(\mathbf{d})$ is exceptional, there is a morphism $\overline{\eta} : L/\Delta(i-1)^{2d_i} \longrightarrow \Delta(\mathbf{d})$ such that $\overline{\eta}\overline{\lambda} = Id_{\Delta(\mathbf{d})}$. Now let $\eta : L \longrightarrow M(\mathbf{d})$ be defined by $\eta_r = (\overline{\eta}_r)$ for $i \leq r \leq j'$ and $\eta_r = M_{\alpha_r}\eta_{r+1}L_{\alpha_r}^{-1} : L_r \longrightarrow M(\mathbf{d})_r$ for $l \leq r \leq i-1$. We can check that η is a morphism from L to $M(\mathbf{d})$ and $\eta\lambda = Id_{M(\mathbf{d})}$. Thus L is a trivial self-extension of $M(\mathbf{d})$, and so $M(\mathbf{d})$ is exceptional. This finishes the proof. □

As a corollary of Proposition 5.3, we have the following result which is our version of Proposition 1 in [2] for an exceptional Δ-filtered module with linear Δ-support.

Corollary 5.4. *We use the same notation as above. Let I be a subset of Q_0 which is contained in a subquiver with linear orientation. Then for any $s \in I$, we have* $\operatorname{Ext}^1_{\mathbf{D}}(M_I, \Delta(s)) = 0 = \operatorname{Ext}^1_{\mathbf{D}}(\Delta(s), M_I)$.

By the construction of exceptional Δ-filtered modules with a linear Δ-support and Proposition 5.3, we have a bijection between the set of the indecomposable exceptional Δ-filtered modules with their Δ-supports contained in Q' and

$$\text{the set of the submodules of} \begin{cases} P(j')/P(j'+1) & \text{if } j' \text{ is interior,} \\ P(j') & \text{if } j' \text{ is not interior.} \end{cases}$$

Moreover, $M_J/S_J = \Delta(J)$, where $J \subseteq \{i', \dots, j'\}$ and S_J is the submodule of M_J, generated by $(M_J)_{i'-1}$. We also note that, as in [2], $J \mapsto M_J$ induces a bijection between non-empty subsets $J \subseteq \{i', \dots, j'\}$ and isomorphism classes of indecomposable exceptional Δ-filtered modules with their Δ-supports contained in Q'. Note that this bijection $J \mapsto M_J$ holds too, if Q' has an opposite orientation.

5.3. **An order on Δ-filtered modules.** In the next section, we will consider gluing exceptional modules with linear Δ-support to form exceptional Δ-filtered modules with arbitrary Δ-support. The construction will depend on an order on Δ-filtered modules. In this subsection, we define this order and prove some preliminary results.

Let M and N be two Δ-filtered modules with $(\underline{\dim}_\Delta(M))_i = 1 = (\underline{\dim}_\Delta(N))_i$ for an admissible vertex i. If i is a source, there is an inclusion $\Delta(i) \longrightarrow M$ by Lemma 2.7, and we define $M \geq_i N$ if the map $\operatorname{Hom}(M, N) \longrightarrow \operatorname{Hom}(\Delta(i), N)$ is surjective. That is, $M \geq_i N$ if for each map $f : \Delta(i) \longrightarrow N$ there is a map $h : M \longrightarrow N$ such that the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \Delta(i) & \longrightarrow & M \\ & & & \searrow^{f} & \downarrow h \\ & & & & N \end{array}$$

commutes. The order does not depend on any particular choice of inclusion $\Delta(i) \longrightarrow M$.

Similarly, if i is a sink, there is a quotient map $N \longrightarrow \Delta(i)$ by Lemma 2.8, and we define $M \geq_i N$, if $\operatorname{Hom}(M, N) \longrightarrow \operatorname{Hom}(M, \Delta(i))$ is surjective. By $M >_i N$ we mean that $M \geq_i N$ and $N \not\geq_i M$.

These orders are transitive and reflexive on the isomorphism classes of Δ-filtered modules M with $(\underline{\dim}_\Delta(M))_i = 1$. Here, transitivity is trivial and reflexivity follows from Lemma 2.7 and Lemma 2.8. We compute these orders for the indecomposable exceptional Δ-filtered modules M_J with J contained in the linearly oriented subquiver Q' as in Section 5.2. To simplify the notation we may assume that $i = i'$ and $j = j'$ are a source vertex and a sink vertex of Q, respectively. Similar to the proof of Proposition 5.3, we can prove the following using Lemma 4 in [2].

Lemma 5.5. *Let $J = \{j_1 \succ j_2 \succ \cdots \succ j_s\}$ and $J' = \{j'_1 \succ j'_2 \succ \cdots \succ j'_t\}$ be two subsets of $\{i, \dots, j\}$. Then the following conditions are equivalent:*

(i) *There is a monomorphism $M_J \longrightarrow M_{J'}$.*
(ii) *We have $s \leq t$ and $j_r \preceq j'_r$ for $r = 1, \dots, s$.*

Lemma 5.6. *Let $J = \{j_1 \succ j_2 > \cdots \succ j_s\}$ and $J' = \{j'_1 \succ j'_2 \succ \cdots \succ j'_t\}$ be two subsets of $\{i, \dots, j\}$, where $j_1 = i = j'_1$ is a source in Q. Then $M_J \geq_i M_{J'}$ if and only if $s \leq t$ and $j_r \preceq j'_r$ for $r = 1, \dots, s$.*

Proof. We have $M_J \geq_i M_{J'}$ if and only if there exists a monomorphism $M_J \longrightarrow M_{J'}$, since for any morphism f from M_J to $M_{J'}$, $\operatorname{Soc}(M_J) \subseteq \Delta(i)$ and $\operatorname{Ker}(f) \cap \operatorname{Soc}(M_J) \neq 0$ if $\operatorname{Ker}(f) \neq 0$. Now the lemma follows from our construction of M_J and $M_{J'}$ and Lemma 5.5. □

Lemma 5.7. *Let $J = \{j_1 \prec j_2 \prec \cdots \prec j_s\}$ and $J' = \{j'_1 \prec j'_2 \prec \cdots \prec j'_t\}$ be two subsets of $\{i, \ldots, j\}$, where $j_1 = j = j'_1$ is a sink in Q. Then $M_J \geq_j M_{J'}$ if and only if $s \geq t$ and $j_r \preceq j'_r$ for $r = 1, \ldots, t$.*

Proof. From Lemma 2.8, we have a surjection $M_J \longrightarrow \Delta(j)$. Assume that this surjection factors through a map $f : M_J \longrightarrow M_{J'}$. By Proposition 2.3, we have $\mathrm{Im}(f)$ is a Δ-filtered submodule of $M_{J'}$. Thus $\mathrm{Ker}(f)$ is Δ-filtered, following again from Proposition 2.3. Since $\mathrm{Ker}(f)$ is Δ-filtered, by Lemma 5.6, $\mathrm{Ker}(f) = M_{\{j_{s'+1},\ldots,j_s\}}$ for some $s' \in \{1, \ldots, s\}$. Hence $\mathrm{Im}(f) = M_{\{j_1,\ldots,j_{s'}\}}$. Using Lemma 5.5, we have $s' \leq t$ and $j_{s'-i} \preceq j'_{t-i}$ for $i = 0, \ldots, s' - 1$. But since f maps the top $\Delta(j)$ in M_J to the top $\Delta(j)$ in $M_{J'}$, we see that $s' = t$. Therefore $s \geq t$ and $j_i \preceq j'_i$ for $i = 1, \ldots, t$.

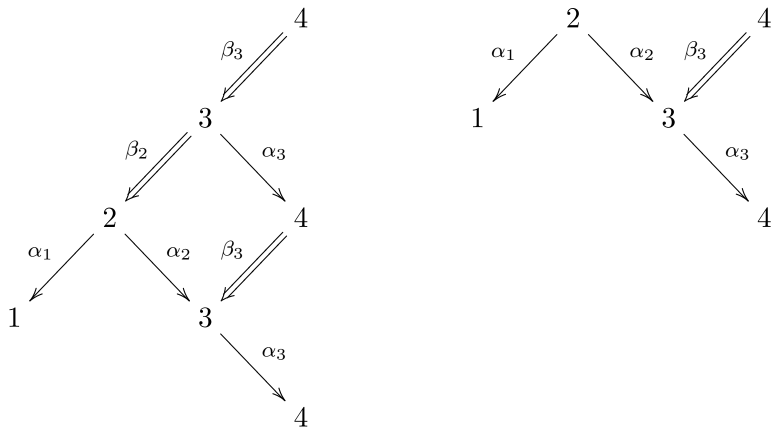
The converse follows from the construction of M_J and $M_{J'}$. □

As a corollary of Proposition 5.3 and Lemmas 5.6 and 5.7, we have the following:

Corollary 5.8. *Let $\mathbf{d}$ be a non-zero vector in $\mathbb{N}^n$ with $\mathrm{supp}(\mathbf{d})_0 \subseteq \{i, \cdots, j\}$. Then the indecomposable direct summands X of the exceptional Δ-filtered module $M(\mathbf{d})$ with $(\underline{\dim}_\Delta(X))_i = 1$ are totally ordered using $\geq_i$. Similarly, we have a total order using $\geq_j$.*

In the following we give some examples on the order defined above.

Example 5.9. *We consider the quiver and use the notation in Example 3.1. Let M and N, respectively, be the modules as follows.*



It is clear that we have an embedding N into M, which sends the submodule $\Delta(2)$ of N to the submodule $\Delta(2)$ of M. On the other hand, we have a morphism from M to N, which sends the top $\Delta(4)$ of M to the top $\Delta(4)$ of N. Therefore $M \geq_4 N$ and $N \geq_2 M$.

6. Exceptional Δ-filtered modules

This section is devoted to proving the following main result.

Theorem 6.1. *Given a non-zero vector $\mathbf{d} \in \mathbb{N}^n$, there exists a unique (up to isomorphism) exceptional Δ-filtered $\mathbf{D}$-module $M(\mathbf{d})$ with Δ-dimension vector $\mathbf{d}$.*

Let $\mathbf{d}$ be a non-zero vector in $\mathbb{N}^n$. We construct an exceptional Δ-filtered representation $M(\mathbf{d})$, with Δ-dimension vector $\mathbf{d}$. Following Lemma 4.5, we see that any indecomposable exceptional module is obtained by gluing exceptional modules with linear Δ-support. In the following we show how to glue the exceptional Δ-filtered modules with linear Δ-support to obtain exceptional Δ-filtered modules with arbitrary Δ-support.

Let $i_1 < i_2 \cdots < i_t$ be a complete list of interior admissible vertices in Q. Let $i_0 = 1$ and $i_{t+1} = n$ be the end vertices of Q. Let $\mathbf{d}^s$, for $s = 1, \ldots, t+1$, be the vector given by $(\mathbf{d}^s)_j = d_j$ if $j \in \{i_{s-1}, \ldots, i_s\}$ and zero elsewhere. Note that each support $\mathrm{supp}(\mathbf{d}^s)$ is a linearly oriented subquiver of Q.

Let $M(\mathbf{d}^s) = \bigoplus_l M_{J_l^s}$ be the exceptional Δ-filtered module from Proposition 5.3. We fix an ordering on the indecomposable direct summands of $M(\mathbf{d}^s)$ as follows. For $s = 1$, we assume that $(\underline{\dim}_\Delta(M_{J_l^1}))_{i_1} = 1$ for $l = 1, \ldots, d_{i_1}$ and that $M_{J_l^1} \geq_{i_1} M_{J_{l+1}^1}$ for $l = 1, \ldots, d_{i_1} - 1$. For each $s > 1$, we assume that $(\underline{\dim}_\Delta(M_{J_l^s}))_{i_{s-1}} = 1$ for $l = 1, \ldots, d_{i_{s-1}}$ and that $M_{J_l^s} \geq_{i_{s-1}} M_{J_{l+1}^s}$ for $l = 1, \ldots, d_{i_{s-1}}$. This is possible by Lemmas 5.6 and 5.7, and the fact that for each s the subsets J_l^s are totally ordered by inclusion. For $l > d_{i_s}$, $s \geq 1$, we fix an arbitrary order.

Let $\mathbf{c}^s$ be the Δ-dimension vector given by $(c^s)_j = d_j$ for $j \in \{1, \ldots, i_s\}$ and zero elsewhere. Here $\mathbf{c}^{t+1} = \mathbf{d}$ and $\mathbf{c}^1 = \mathbf{d}^1$. We will inductively construct an exceptional Δ-filtered module $M(\mathbf{c}^s)$ with Δ-dimension vector $\mathbf{c}^s$ for all $s = 1, \ldots, t+1$. If $t = 0$, then Q is linearly oriented, and we let $M(\mathbf{d}) = \Delta(\mathbf{d})$ as in [2]. Now suppose that $t > 0$. For $s = 1$, we let $M(\mathbf{c}^1) = M(\mathbf{d}^1)$. Suppose that we have $M(\mathbf{c}^s)$. We construct $M(\mathbf{c}^{s+1})$ by gluing $M(\mathbf{c}^s)$ and $M(\mathbf{d}^{s+1})$ at vertex i_s as follows.

First, we decompose $M(\mathbf{c}^s)$ into indecomposable direct summands

$$M(\mathbf{c}^s) = \bigoplus_l M_{K_l^s},$$

where $K_l^s \subseteq \{1, \ldots, i_s\}$ and $(\underline{\dim}_\Delta(M_{K_l^s}))_j = 1$ if $j \in K_l^s$ and zero elsewhere. We will show that $M_{K_l^s}$ is the unique, up to isomorphism, indecomposable exceptional Δ-filtered module with K_l^s as the set of vertices of its Δ-support. Moreover, we reorder the indecomposable direct summands of $M(\mathbf{c}^s)$ such that

(1) $(\underline{\dim}_\Delta(M_{K_l^s}))_{i_s} = 1$ for $l = 1, \ldots, d_{i_s}$;
(2) $M_{K_l^s} \geq_{i_s} M_{K_{l+1}^s}$ for $l = 1, \ldots, d_{i_s} - 1$.

Unlike the case of linear support, not all subsets of $\{1, \ldots, n\}$ will occur as the support of an indecomposable exceptional Δ-filtered module.

If $d_{i_s} = 0$, by induction we have $M(\mathbf{d}) = M(\mathbf{d}') \oplus M(\mathbf{d}'')$, where $(\mathbf{d}')_i = d_i$ and $(\mathbf{d}'')_j = d_j$ for $1 \leq i < i_s$, $i_s < j \leq n$ and zero elsewhere. Now suppose that $d_{i_s} \neq 0$ and define $M(\mathbf{c}^{s+1})$ as

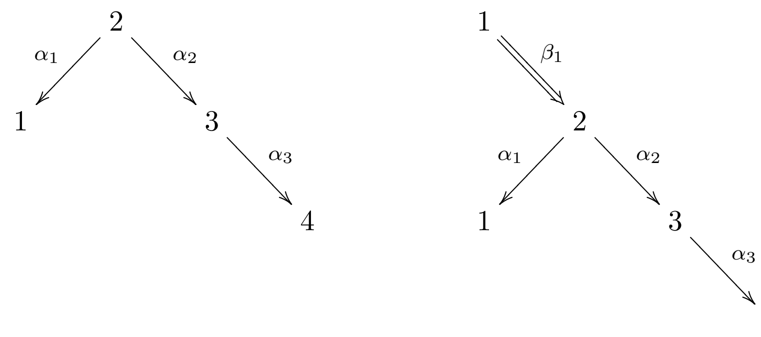
$$M(\mathbf{c}^{s+1}) = \bigoplus_l M_{K_l^{s+1}},$$

where $K_l^{s+1} = K_l^s \cup J_{d_{i_s}+1-l}^{s+1}$ and $M_{K_l^{s+1}}$ is obtained by gluing $M_{K_l^s}$ and $M_{J_{d_{i_s}+1-l}^{s+1}}$ at i_s for $l = 1, \ldots, d_{i_s}$, and $\bigoplus_{l > d_{i_s}} M_{K_l^{s+1}}$ is the direct sum of all the terms $M_{K_l^s}$ and $M_{J_l^{s+1}}$ with $l > d_{i_s}$. For $l > d_{i_s}$, we have that K_l^{s+1} is either $K_{l'}^s$ or $J_{l''}^{s+1}$ for some l', $l'' > d_{i_s}$. By the construction and the properties of gluing, we get the following proposition.

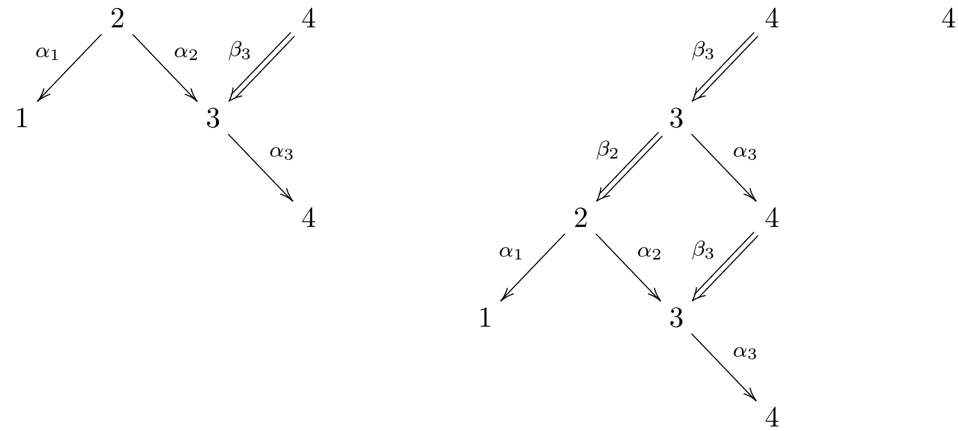
Proposition 6.2. *The representation $M(\mathbf{c}^s)$ constructed above is Δ-filtered and each of its direct summand $M_{K_l^s}$ is indecomposable.*

Example 6.3. *We consider the quiver and use the notation in Example 3.1 where $i_1 = 2$ and $i_2 = 4$. Let $\mathbf{d} = (1, 2, 1, 3, 2)$. We decompose $\mathbf{d}$ into its subvectors, $\mathbf{d}^1 = (1, 2, 0, 0, 0)$, $\mathbf{d}^2 = (0, 2, 1, 3, 0)$ and $\mathbf{d}^3 = (0, 0, 0, 3, 2)$. Each of these subvectors has a linear support.*

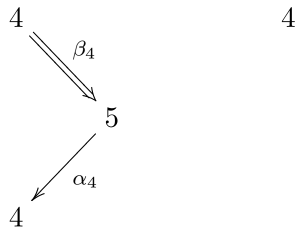
(1) Then we have $M(\mathbf{d}^1) = M_{J_1^1} \oplus M_{J_2^1}$*, where* $J_1^1 = \{2\} \subset J_2^1 = \{1,2\}$*. Thus* $M_{J_1^1} >_2 M_{J_2^1}$ *and they are as follows, respectively.*



The module $M(\mathbf{d}^2) = M_{J_1^2} \oplus M_{J_2^2} \oplus M_{J_3^2}$*, where* $J_1^2 = \{2,4\}$*,* $J_2^2 = \{2,3,4\}$ *and* $J_3^2 = \{4\}$*. Thus* $M_{J_1^2} >_2 M_{J_2^2}$ *and* $M_{J_1^2}$*,* $M_{J_2^2}$ *and* $M_{J_3^2}$ *are as follows, respectively.*

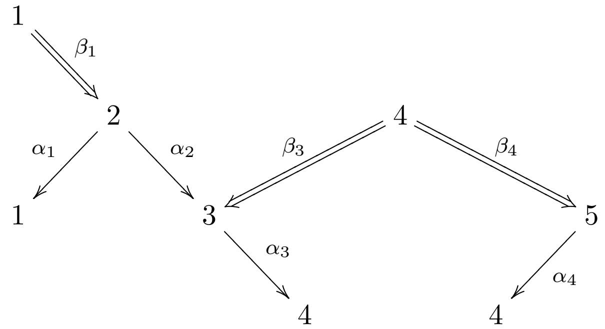


The module $M(\mathbf{d}^3) = M_{J_1^3} \oplus M_{J_2^3} \oplus M_{J_3^3}$*, where* $J_1^3 = J_2^3 = \{4,5\}$ *and* $J_3^3 = \{4\}$*. Thus* $M_{J_1^3} = M_{J_2^3} >_4 M_{J_3^3}$ *and they are as follows, respectively.*



(2) Glue $M(\mathbf{d}^1)$ *and* $M(\mathbf{d}^2)$ *at vertex* 2*. That is, glue* $M_{J_1^1}$ *and* $M_{J_2^2}$*, and glue* $M_{J_2^1}$ *and* $M_{J_1^2}$ *at vertex* 2*. Thus we obtain* $M(\mathbf{c}^2) = \bigoplus_{i=1}^3 M_{K_i^2}$*, where* $K_1^2 = J_1^1 \cup J_2^2 = J_2^2$*,* $K_2^2 = J_2^1 \cup J_1^2 = \{1,2,4\}$ *and* $K_3^2 = J_3^2$*. Moreover,* $M_{K_1^2} >_4 M_{K_2^2} >_4 M_{K_3^2}$*.*

(3) Finally we have $M(\mathbf{d}) = \bigoplus_{i=1}^3 M_{K_i^3}$*, where* $K_1^3 = K_1^2 \cup J_3^3 = \{2,3,4\}$ *and* $K_2^3 = K_2^2 \cup J_2^3 = \{1,2,4,5\}$ *and* $K_3^3 = K_3^2 \cup J_1^3 = J_1^3$*, and* $M_{K_1^3} = M_{J_2^2}$*,* $M_{K_3^3} = M_{J_1^3}$ *and* $M_{K_2^3}$ *is as follows.*



(4) These three modules $M_{K_1^3}$, $M_{K_2^3}$ and $M_{K_3^3}$ are indecomposable.

We need some lemmas for the inductive step. Let us fix some notation.

Let M and N be Δ-filtered modules obtained by successively gluing $M^a, \dots, M^u$ and $N^b, \dots, N^u$, respectively, where $1 \leq a \leq u \leq t+1$, $1 \leq b \leq u$, and for each i, there exist integers m_i and n_i such that

$$M^i = M_{J^i_{m_i}} \quad \text{and} \quad N^i = M_{J^i_{n_i}}.$$

For any l, we denote by $M^{\leq l}$ the module obtained by successively gluing M^j for $j = a, \dots, l$, and by $M^{\geq l}$ the module obtained by gluing M^j for $j = l, \dots, u$. We obtain M by gluing $M^{\leq l}$ and $M^{\geq l}$ at i_l. Similarly, for $N^{\leq l}$ and $N^{\geq l}$.

Lemma 6.4. *Assume that $J^j_{m_j} \subseteq J^j_{n_j}$ or $J^j_{n_j} \subseteq J^j_{m_j}$ for $\max\{a, b\} \leq j \leq u$. Then $M \geq_{i_u} N$ if and only if, either*

(1) *$M \cong N$, or*
(2) *there exists an l such that $M^l >_{i_l} N^l$ and $M^j \cong N^j$ for $j = l+1, \dots, u$.*

Proof. If $M \cong N$, then clearly $M \geq_{i_u} N$, so we may assume that $M \ncong N$. Then there exists an l such that $M^l \ncong N^l$ and $M^s \cong N^s$ for $s = l+1, \dots, u$. We need only to prove the lemma by showing that $M \geq_{i_u} N$ if and only if $M^l >_{i_l} N^l$.

First assume that $M \geq_{i_u} N$. Let $f : M \longrightarrow N$ be a morphism with the property that a surjective morphism $M \longrightarrow \Delta(i_u)$ factors through f, if i_u is a sink, and an injective morphism $\Delta(i_u) \longrightarrow N$ factors through f, if i_u is a source. Such a morphism f exists by the definition of $\geq_{i_u}$.

By Lemma 4.4, we have a map $g = f|_{M^u} : M^u \longrightarrow N^u$, the restriction of f to M^u, which shows that $M^u \geq_{i_u} N^u$. If $M^u \ncong N^u$, then $M^u >_{i_u} N^u$ and we are done. We will show that if $M^u \cong N^u$, then g is an isomorphism. If i_u is a source, then g is injective, and therefore an isomorphism. If i_u is a sink, then $\mathrm{Im}(g) \geq_{i_u} N^u$. So by Lemma 5.7, we see that the Δ-length of $\mathrm{Im}(g)$ is greater than or equal to the Δ-length of $N^u \cong M^u$. Hence g is surjective and therefore an isomorphism. Note any Δ-filtered module L with $(\underline{\dim}_\Delta(L))_i = 1$ and $(\underline{\dim}_\Delta(L))_j = 0$ for $j \in \{1, 2, \cdots, i-1\}$, where i is admissible, can always be viewed as a gluing of L and $\Delta(i)$ at vertex i. And so if we have $M^{\geq u-1} \cong N^{\geq u-1}$, we can assume either both N^{u-1} and M^{u-1} are zero or both are non-zero. Since g is an isomorphism, we have $M^{\leq u-1} \geq_{i_{u-1}} N^{\leq u-1}$. Using induction we get that $M^l >_{i_l} N^l$ and we are done.

For the converse, assume that $M^l >_{i_l} N^l$ and $M^j \cong N^j$ for $j = l+1, \dots, u$. If $l = u$, let $g : M^u \longrightarrow N^u$ be a morphism with the property that a surjective morphism $M^u \longrightarrow \Delta(i_u)$ factors through g, if i_u is a sink, and an injective morphism $\Delta(i_u) \longrightarrow N^u$ factors through g, if i_u is a source. Since $M^u \ncong N^u$ we see that $\Delta(i_{u-1})$ is in the kernel of g, if i_u is a sink, and that the top isomorphic to $\Delta(i_{u-1})$ is not in the image of g, if i_u is a source. So in both cases, we get a morphism $f : M \longrightarrow N$ by gluing g and the zero map $0 : M^{\leq u-1} \longrightarrow N^{\leq u-1}$ at i_{u-1}. This shows that $M \geq_{i_u} N$.

If $l < u$, then $M^{\leq u-1} \geq_{i_{u-1}} N^{\leq u-1}$ by induction and there exists a morphism $g : M^{\leq u-1} \longrightarrow N^{\leq u-1}$ such that a surjective morphism $M^{\leq u-1} \longrightarrow \Delta(i_{u-1})$ factors through g, if i_u is a source, and an injective morphism $\Delta(i_{u-1}) \longrightarrow N^{\leq u-1}$ factors through g, if i_u is a sink. In both cases, we may glue g with an isomorphism $M^u \longrightarrow N^u$ at i_{u-1}, and get a map $f : M \longrightarrow N$, which has the property that a surjective morphism $M \longrightarrow \Delta(i_u)$ factors through f, if i_u is a sink, and an injective morphism $\Delta(i_u) \longrightarrow N$ factors through f, if i_u is a source. This shows that $M \geq_{i_u} N$. □

Lemma 6.5. *The module $M \oplus N$ is exceptional if and only if $M^{\leq u-1} \oplus N^{\leq u-1}$ is exceptional and either*

(1) $M^{\leq u-1} \geq_{i_{u-1}} N^{\leq u-1}$ *and* $N^u \geq_{i_{u-1}} M^u$, *or*
(2) $N^{\leq u-1} \geq_{i_{u-1}} M^{\leq u-1}$ *and* $M^u \geq_{i_{u-1}} N^u$.

We need some preparation for the proof of this lemma. To simplify notation, we let $M' = M^{\leq u-1}$, $M'' = M^u$, $N' = N^{\leq u-1}$ and $N'' = N^u$.

Lemma 6.6. *We have* $\mathrm{Ext}^1_{\mathbf{D}}(M', \Delta(i_{u-1})) = 0 = \mathrm{Ext}^1_{\mathbf{D}}(\Delta(i_{u-1}), M')$, *and likewise for* N'.

Proof. We first consider the case where i_{u-1} is a source. Since $\Delta(i_{u-1})$ is a projective $\mathbf{D}$-module, and so $\mathrm{Ext}^1_{\mathbf{D}}(\Delta(i_{u-1}), M') = 0$. Thus we need only to prove $\mathrm{Ext}^1_{\mathbf{D}}(M', \Delta(i_{u-1})) = 0$. By the construction of M', we see that M' has a submodule Y with

$$(\underline{\dim}_\Delta(Y))_i = \begin{cases} (\underline{\dim}_\Delta(M'))_i & \text{if } 1 \leq i \leq i_{u-2} - 1, \\ 0 & \text{otherwise.} \end{cases}$$

We have an exact sequence $0 \longrightarrow Y \longrightarrow M' \longrightarrow X \longrightarrow 0$, where

$$(\underline{\dim}_\Delta(X))_i = \begin{cases} (\underline{\dim}_\Delta(M'))_i & \text{if } i_{u-2} \leq i \leq i_{u-1}, \\ 0 & \text{otherwise.} \end{cases}$$

That is, $\mathrm{supp}_\Delta(X)$ has a linear orientation. By applying $\mathrm{Hom}_{\mathbf{D}}(-, \Delta(i_{u-1}))$, we have an exact sequence

$$\mathrm{Ext}^1_{\mathbf{D}}(X, \Delta(i_{u-1})) \longrightarrow \mathrm{Ext}^1_{\mathbf{D}}(M', \Delta(i_{u-1})) \longrightarrow \mathrm{Ext}^1_{\mathbf{D}}(Y, \Delta(i_{u-1})).$$

By Corollary 5.4, we have $\mathrm{Ext}^1_{\mathbf{D}}(X, \Delta(i_{u-1})) = 0$. Note that the support of the top of the first syzygy $\Omega(Y)$ of Y is contained in $\{0, ..., i_{u-2} - 1\}$, and so $\mathrm{Hom}_{\mathbf{D}}(\Omega(Y), \Delta(i_{u-1})) = 0$. Thus $\mathrm{Ext}^1_{\mathbf{D}}(Y, \Delta(i_{u-1})) = 0$, and so $\mathrm{Ext}^1_{\mathbf{D}}(M', \Delta(i_{u-1})) = 0$.

Now suppose that i_{u-1} is a sink. By Lemma 2.6, we have $\mathrm{Ext}^1_{\mathbf{D}}(M', \Delta(i_{u-1})) = 0$. Thus we need only to prove that $\mathrm{Ext}^1_{\mathbf{D}}(\Delta(i_{u-1}), M') = 0$. By the construction of M', there exists a submodule Y of M' such that

$$(\underline{\dim}_\Delta(Y))_i = \begin{cases} (\underline{\dim}_\Delta(M'))_i & \text{if } i_{u-2} \leq i \leq i_{u-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that Y has linear Δ-support. We have $0 \longrightarrow Y \longrightarrow M' \longrightarrow X \longrightarrow 0$. By applying $\mathrm{Hom}_{\mathbf{D}}(\Delta(i_{u-1}), -)$, we get an exact sequence

$$\mathrm{Ext}^1_{\mathbf{D}}(\Delta(i_{u-1}), Y) \longrightarrow \mathrm{Ext}^1_{\mathbf{D}}(\Delta(i_{u-1}), M') \longrightarrow \mathrm{Ext}^1_{\mathbf{D}}(\Delta(i_{u-1}), X).$$

By Corollary 5.4, we have $\mathrm{Ext}^1_{\mathbf{D}}(\Delta(i_{u-1}), Y) = 0$. Note that $\Omega(\Delta(i_{u-1})) = P(i_{u-1} - 1) \oplus P(i_{u-1}+1)$ and so $\mathrm{Hom}_{\mathbf{D}}(\Omega(\Delta(i_{u-1})), X) = 0$, since $\mathrm{supp}(X)$ is contained in $\{1, ..., i_{u-2}-1\}$. Therefore $\mathrm{Ext}^1_{\mathbf{D}}(\Delta(i_{u-1}), X) = 0$ and so $\mathrm{Ext}^1_{\mathbf{D}}(\Delta(i_{u-1}), M') = 0$. This finishes the proof. □

Lemma 6.7. *We have* $\mathrm{Ext}^1_{\mathbf{D}}(M', N'') = 0 = \mathrm{Ext}^1_{\mathbf{D}}(M'', N')$.

Proof. We first consider the case where i_{u-1} is a source. We have an exact sequence

$$0 \longrightarrow \Delta(i_{u-1}) \longrightarrow N' \longrightarrow X \longrightarrow 0.$$

By applying $\mathrm{Hom}_{\mathbf{D}}(M'', -)$, we have an exact sequence

$$\mathrm{Ext}^1_{\mathbf{D}}(M'', \Delta(i_{u-1})) \longrightarrow \mathrm{Ext}^1_{\mathbf{D}}(M'', N') \longrightarrow \mathrm{Ext}^1_{\mathbf{D}}(M'', X).$$

By Corollary 5.4, we have $\mathrm{Ext}^1_{\mathbf{D}}(M'', \Delta(i_{u-1})) = 0$. Note that the vertex set $\mathrm{supp}(X)_0$ of the support of X is contained in $\{1, ..., i_{u-1} - 1\}$ and the support of the top of the syzygy of M'' is contained in $\{i_{u-1}, ..., i_u, i_u + 1\}$. Therefore $\mathrm{Ext}^1_{\mathbf{D}}(M'', X) = 0$. Thus $\mathrm{Ext}^1_{\mathbf{D}}(M'', N') = 0$.

By applying $\mathrm{Hom}_{\mathbf{D}}(M', -)$ to the exact sequence

$$0 \longrightarrow \Delta(i_{u-1}) \longrightarrow N'' \longrightarrow Y \longrightarrow 0,$$

we have an exact sequence

$$\mathrm{Ext}^1_{\mathbf{D}}(M', \Delta(i_{u-1})) \longrightarrow \mathrm{Ext}^1_{\mathbf{D}}(M', N'') \longrightarrow \mathrm{Ext}^1_{\mathbf{D}}(M', Y).$$

By Lemma 6.6, we have $\mathrm{Ext}^1_{\mathbf{D}}(M', \Delta(i_{u-1})) = 0$. By the construction of M', we have $\mathrm{Hom}_{\mathbf{D}}(\Omega(M'), Y) = 0$, and so $\mathrm{Ext}^1_{\mathbf{D}}(M', Y) = 0$. Hence $\mathrm{Ext}^1_{\mathbf{D}}(M', N'') = 0$.

Now suppse that i_{u-1} is a sink. By applying $\mathrm{Hom}_{\mathbf{D}}(-, N')$ and $\mathrm{Hom}_{\mathbf{D}}(-, N'')$, respectively, to the following exact sequences,

$$0 \longrightarrow Y \longrightarrow M'' \longrightarrow \Delta(i_{u-1}) \longrightarrow 0 \text{ and}$$
$$0 \longrightarrow X \longrightarrow M' \longrightarrow \Delta(i_{u-1}) \longrightarrow 0,$$

and by similar arguments as in the case where i_{u-1} is a source, we have $\mathrm{Ext}^1_{\mathbf{D}}(M'', N') = 0$ and $\mathrm{Ext}^1_{\mathbf{D}}(M', N'') = 0$. This finishes the proof. □

Lemma 6.8. *The following are equivalent:*

(i) $\mathrm{Ext}^1_{\mathbf{D}}(M, N) = 0$.

(ii) $\mathrm{Ext}^1_{\mathbf{D}}(M', N') = 0$, $\mathrm{Ext}^1_{\mathbf{D}}(M'', N'') = 0$, *and either* $M' \geq_{i_{u-1}} N'$ *or* $M'' \geq_{i_{u-1}} N''$.

Proof. We first consider the case where i_{u-1} is a source. By applying $\mathrm{Hom}_{\mathbf{D}}(-, N)$ to the gluing sequence of M at i_{u-1},

$$0 \longrightarrow \Delta(i_{u-1}) \longrightarrow M' \oplus M'' \longrightarrow M \longrightarrow 0,$$

we see that $\mathrm{Ext}^1_{\mathbf{D}}(M, N) = 0$ if and only if the following hold.

(1) $\mathrm{Hom}_{\mathbf{D}}(M' \oplus M'', N) \to \mathrm{Hom}_{\mathbf{D}}(\Delta(i_{u-1}), N)$ is surjective.

(2) $\mathrm{Ext}^1_{\mathbf{D}}(M' \oplus M'', N) = 0$.

We claim that (1) holds if and only if either $M' \geq_{i_{u-1}} N'$ or $M'' \geq_{i_{u-1}} N''$. Assume that (1) holds. Let $\lambda : \Delta(i_{u-1}) \to N$ be injective, and let $\mu = \binom{\mu_1}{\mu_2} : \Delta(i_{u-1}) \to M' \oplus M''$ be the embedding in the gluing sequence of M. By the assumption, there exists a map $f = (f_1, f_2) : M' \oplus M'' \to N$ such that $\lambda = f\mu = f_1\mu_1 + f_2\mu_2$. Since N has a unique submodule isomorphic to $\Delta(i_{u-1})$, we have that λ factors through either μ_1 or μ_2. If λ factors through μ_1, then the inclusion of $\Delta(i_{u-1})$ into N factors through f_1, since $\mathrm{Im}(\mu_1) \subseteq M'$. Hence $M' \geq_{i_{u-1}} N'$. Similarly, $M'' \geq_{i_{u-1}} N''$ if λ factors through μ_2. The converse is similar.

By Lemmas 6.6 and 6.7 and by applying $\mathrm{Hom}_{\mathbf{D}}(M' \oplus M'', -)$ to the gluing sequence of N, we get that $\mathrm{Ext}^1_{\mathbf{D}}(M' \oplus M'', N) = 0$ if and only if $\mathrm{Ext}^1_{\mathbf{D}}(M', N') = 0 = \mathrm{Ext}^1_{\mathbf{D}}(M'', N'')$. This proves the equivalence of (i) and (ii) in the case where i_{u-1} is a source.

Now suppose that i_{u-1} is a sink. By applying $\mathrm{Hom}_{\mathbf{D}}(M, -)$ to the gluing sequence of N we see that $\mathrm{Ext}^1_{\mathbf{D}}(M, N) = 0$ if and only if the following hold.

(1) $\mathrm{Hom}_{\mathbf{D}}(M, N' \oplus N'') \to \mathrm{Hom}_{\mathbf{D}}(M, \Delta(i_{u-1}))$ is surjective.

(2) $\mathrm{Ext}^1_{\mathbf{D}}(M, N' \oplus N'') = 0$.

Now similar to the case of a source, (1) holds if and only if either $M' \geq_{i_{u-1}} N'$ or $M'' \geq_{i_{u-1}} N''$ and (2) holds if and only if $\mathrm{Ext}^1_{\mathbf{D}}(M', N') = 0$ and $\mathrm{Ext}^1_{\mathbf{D}}(M'', N'') = 0$. Hence the equivalence of (i) and (ii) follows. □

We are now ready to prove Lemma 6.5.

Proof of Lemma 6.5. Suppose that $M \oplus N$ is exceptional, by Lemma 4.5, both $M' \oplus N'$ and $M'' \oplus N''$ are exceptional. By Corollary 5.8, either $M'' \geq_{i_{u-1}} N''$ or $N'' \geq_{i_{u-1}} M''$. Similarly, $M' \geq_{i_{u-1}} N'$ or $N' \geq_{i_{u-1}} M'$ by induction, using Lemma 6.4 and Proposition 5.3. If $M'' \not\geq_{i_{u-1}} N''$, then $M' \geq_{i_{u-1}} N'$ by Lemma 6.8. Thus (1) holds. If $N' \not\geq_{i_{u-1}} M'$, then again $N'' \geq_{i_{u-1}} M''$ by Lemma 6.8. Thus again (1) holds. Similarly, (2) holds if $N'' \not\geq_{i_{u-1}} M''$ or $M' \not\geq_{i_{u-1}} N'$.

We now prove the converse. By Lemma 5.6 and Lemma 5.7, we have either $\mathrm{supp}_\Delta(M'') \subseteq \mathrm{supp}_\Delta(N'')$ or $\mathrm{supp}_\Delta(N'') \subseteq \mathrm{supp}_\Delta(M'')$. Now by Corollary 5.4 and that both M'' and N'' are Δ-filtered, we have $\mathrm{Ext}^1_{\mathbf{D}}(M'', N'') = 0$ and $\mathrm{Ext}^1_{\mathbf{D}}(N'', M'') = 0$. Now the converse follows from Lemma 6.8 and the fact that $M' \oplus N'$ is exceptional. □

Proof of Theorem 6.1. We will show that $M(\mathbf{d})$ is exceptional, by induction on the number of interior admissible vertices of Q. If Q has no interior admissible vertices, then by Theorem 5.1 we know that $M(\mathbf{d})$ is exceptional. Note that $M(\mathbf{d})$ is always exceptional if $\mathrm{supp}(\mathbf{d})$ is linearly oriented, following from Proposition 5.3. Assume that $M(\mathbf{d})$ is exceptional if Q has less than t interior admissible vertices. Now suppose that Q has $t > 0$ interior admissible vertices. We shall show that $M(\mathbf{d})$ is exceptional.

We use the same notation as in the construction of $M(\mathbf{d})$. Recall that we have

$$M(\mathbf{c}^t) = \bigoplus_l M_{K_l^t}$$

where the $M_{K_l^t}$ are ordered such that

(1) $(\underline{\dim}_\Delta(M_{K_l^t}))_{i_t} = 1$ for $l = 1, \ldots, d_{i_t}$;
(2) $M_{K_l^t} \geq_{i_t} M_{K_{l+1}^t}$ for $l = 1, \ldots, d_{i_t} - 1$.

By the construction of $M(\mathbf{d})$, we may write

$$M(\mathbf{d}) = \bigoplus_{l=1}^{d_{i_t}} M_{K_l^{t+1}} \oplus \bigoplus_{l>d_{i_t}} M_{K_l^t} \oplus \bigoplus_{l>d_{i_t}} M_{J_l^{t+1}},$$

where $\bigoplus_{l>d_{i_t}} M_{K_l^t}$ is a direct sum of the indecomposable direct summands of $M(\mathbf{c}^t)$ whose Δ-support does not contain vertex i_t, $\bigoplus_{l>d_{i_t}} M_{J_l^{t+1}}$ a direct sum of the indecomposable direct summands of $M(\mathbf{d}^{t+1})$ whose Δ-support does not contain vertex i_t, and $M_{K_l^{t+1}}$ is obtained by gluing $M_{K_l^t}$ and $M_{J^{t+1}_{d_{i_t}+1-l}}$ at vertex i_t for $1 \leq l \leq d_{i_t}$. For convenience, we denote the representations $\bigoplus_{l=1}^{d_{i_t}} M_{K_l^{t+1}}$, $\bigoplus_{l>d_{i_t}} M_{K_l^t}$ and $\bigoplus_{l>d_{i_t}} M_{J_l^{t+1}}$ by X, Y and Z, respectively.

We have that Y is exceptional by induction, and following from Proposition 5.3, Z is exceptional. By our construction and Lemma 6.5 we know that X is exceptional. Comparing the support of Y and Z, we see that $\mathrm{Hom}_{\mathbf{D}}(\Omega(Z), Y) = 0 = \mathrm{Hom}_{\mathbf{D}}(\Omega(Y), Z)$, and so $\mathrm{Ext}^1_{\mathbf{D}}(Y, Z) = 0 = \mathrm{Ext}^1_{\mathbf{D}}(Z, Y)$. Hence $Y \oplus Z$ is exceptional.

We now prove that $X \oplus Y$ and $Y \oplus Z$ are exceptional. Assume that i_t is a source. Let T be the submodule of X with

$$(\underline{\dim}_\Delta(T))_i = \begin{cases} (\underline{\dim}_\Delta(X))_i & \text{if } 1 \leq i \leq i_t, \\ 0 & \text{otherwise.} \end{cases}$$

We have a short exact sequence $0 \longrightarrow T \longrightarrow X \longrightarrow U \longrightarrow 0$. Now $T \oplus Y$ is exceptional by induction, and $\mathrm{Hom}_{\mathbf{D}}(\Omega(Y), U) = 0 = \mathrm{Hom}_{\mathbf{D}}(\Omega(U), Y)$ which shows that $U \oplus Y$ is exceptional. Hence $X \oplus Y$ is exceptional. Let S be the submodule of X with

$$(\underline{\dim}_\Delta(S))_i = \begin{cases} (\underline{\dim}_\Delta(X))_i & \text{if } i_t \leq i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

There is a short exact sequence $0 \longrightarrow S \longrightarrow X \longrightarrow V \longrightarrow 0$. Now $S \oplus Z$ is exceptional by Proposition 5.3, and $V \oplus Z$ is exceptional since $\mathrm{Hom}_{\mathbf{D}}(\Omega(Z), V) = 0 = \mathrm{Hom}_{\mathbf{D}}(\Omega(V), Z)$. This proves that $X \oplus Z$ is exceptional. The case where i_t is a sink can be done similarly. This completes the proof that $M(\mathbf{d})$ is exceptional.

Let $\mathbf{b} = \sum_i d_i \underline{\dim}(\Delta(i))$. By Voigt's lemma [18] we know that the $\mathrm{GL}(\mathbf{b})$-orbit of $M(\mathbf{d})$ in $\mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{b})$ is open, and therefore dense, by Proposition 2.9. Hence $M(\mathbf{d})$ is unique up to isomorphism. □

Corollary 6.9. *Let N be a Δ-filtered module with Δ-dimension vector $\mathbf{d}$. Then N is contained in the Zariski closure of the $\mathrm{GL}(\mathbf{b})$-orbit of the exceptional Δ-filtered module $M(\mathbf{d})$, where $\mathbf{b} = \sum_i d_i \underline{\dim}(\Delta(i))$.*

Proof. This follows from the proof of the uniqueness of $M(\mathbf{d})$ in Theorem 6.1. □

Two elements $j, j' \in \{1, \ldots, n\}$ are called non-comparable if $j \not\preceq j'$ and $j' \not\preceq j$.

Corollary 6.10. *The map $J \mapsto M_J$ defines a bijection between isomorphism classes of indecomposable exceptional Δ-filtered modules and subsets $J \subseteq \{1, \ldots, n\}$ satisfying the following conditions:*

(1) *If $j \succeq i$ and $j' \succeq i$ for two non-comparable $j, j' \in J$, then $i \in J$.*
(2) *If $j \preceq i$ and $j' \preceq i$ for two non-comparable $j, j' \in J$, then $i \in J$.*

Proof. By Proposition 6.2, we see that M_J is indecomposable if and only if J satisfies the conditions (1) and (2). That M_J is exceptional follows from Theorem 6.1. □

Example 6.11. *Imitating the example in Section 8 in* [2]*, we can interpret the exceptional Δ-filtered module $M(\mathbf{d})$ in Example 6.3 as follows.*

(1) We take the Δ-composition factors of $M(\mathbf{d}^i)$ as vertieces and connect them by arrows: for any two composition factors $\Delta(i_j)$ and $\Delta(i_l)$ of an indecomposable direct summands of $M(\mathbf{d}^i)$, we draw a double arrow from $\Delta(i_j)$ to $\Delta(i_l)$, if there is a β-path from vertex i_j to vertex i_l and if this indecomposable direct summand has no composition factor $\Delta(i_s)$ such that $i_j \succ i_s \succ i_l$. In this way, we achieve the following diagrams.

$$\begin{array}{cccc}
 & & \Delta(4) & \Delta(4) \Longrightarrow \Delta(5) \\
\Delta(1) \Longrightarrow \Delta(2) & \Delta(2) \Longleftarrow\!\!=\!\!=\!\!= & \Delta(4) & \Delta(4) \Longrightarrow \Delta(5) \\
\Delta(2) & \Delta(2) \Longleftarrow \Delta(3) \Longleftarrow & \Delta(4) & \Delta(4)
\end{array}$$

There are three columns, representating the three Δ-filtered modules with a linear Δ-support, $M(\mathbf{d}^1)$, $M(\mathbf{d}^2)$ and $M(\mathbf{d}^3)$, respectively. From top to bottom: the first column is ordered from small to big according to $>_2$. The second column is ordered from big to small for those indecomposable direct summands of $M(\mathbf{d}^2)$ with the 2nd entry of its Δ-dimension vector non-zero, according to $>_2$ and from small to big according to $>_4$. Finally the third column is ordered from big to small according to $>_4$. We also make sure that the pieces to be glued are in the same row.

(2) Now identify the Δ-factors, at which the neighboring modules are to be glued, and we achieve the module $M(\mathbf{d})$ as follows.

$$\begin{array}{l}
\qquad\qquad\qquad\qquad\quad \Delta(4) \Longrightarrow \Delta(5) \\
\Delta(1) \Longrightarrow \Delta(2) \Longleftarrow\!\!=\!\!=\!\!= \Delta(4) \Longrightarrow \Delta(5) \\
\qquad\quad\;\; \Delta(2) \Longleftarrow \Delta(3) \Longleftarrow \Delta(4)
\end{array}$$

7. Seaweed Lie algebras and Richardson elements

In this section, we recall some basic defintions and results related to seaweed Lie algebras and Richardson elements. We also give examples on seaweed Lie algebras. We assume that $\mathfrak{g}$ is a reductive Lie algebra defined over an algebraically closed field k of characteristic zero.

Definition 7.1. *A Lie subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ is called a seaweed Lie algebra in $\mathfrak{g}$ if there exists a pair $(\mathfrak{p}, \mathfrak{p}')$ of parabolic subalgebras of $\mathfrak{g}$ such that $\mathfrak{q} = \mathfrak{p} \cap \mathfrak{p}'$ and $\mathfrak{p} + \mathfrak{p}' = \mathfrak{g}$. We call such a pair of parabolic subalgebras weakly opposite.*

For example, take the pair consisting of a parabolic subalgebra and its opposite, or the pair consisting of $\mathfrak{g}$ and a parabolic subalgebra. Thus the set of seaweed Lie algebras contains all parabolic subalgebras and their Levi factors.

More generally, let us fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h}$ contained in $\mathfrak{b}$. Denote by R (resp. R^+, R^- and Π) the root system (resp. the set of positive roots, the set of negative roots and the set of simple roots) relative to $\mathfrak{h}$, $\mathfrak{b}$ and $\mathfrak{g}$. For $\alpha \in R$, denote by $\mathfrak{g}_\alpha$ the root subspace relative to α.

For $S \subset \Pi$, set

$$\mathbb{Z}S = \sum_{\alpha \in S} \mathbb{Z}\alpha \ , \ R_S = \mathbb{Z}S \cap R \ , \ R_S^{\pm} = R^{\pm} \cap R_S \text{ and } \mathfrak{p}_S^{\pm} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_S \cup R^{\pm}} \mathfrak{g}_\alpha$$

the standard parabolic subalgebra and its opposite relative to S.

Let $\mathbf{G}$ be a connected reductive algebraic group whose Lie algebra is $\mathfrak{g}$. The following result says that as in the case of parabolic subalgebras, any seaweed Lie algebra is conjugate to a "standard" one.

Proposition 7.2. [12, 15, 16]

1. *Let $S, T \subset \Pi$. Then $(\mathfrak{p}_S^-, \mathfrak{p}_T^+)$ is a pair of weakly opposite parabolic subalgebras. In particular, $\mathfrak{q}_{S,T} = \mathfrak{p}_S^- \cap \mathfrak{p}_T^+$ is a seaweed Lie algebra in $\mathfrak{g}$.*
2. *Any seaweed Lie algebra in $\mathfrak{g}$ is conjugate by $\mathbf{G}$ to a seaweed Lie algebra of the form $\mathfrak{q}_{S,T}$ with $S, T \subset \Pi$.*

Example 7.3. *For example, if $S = \Pi$ or $T = \Pi$, then $\mathfrak{q}_{S,T}$ is a parabolic subalgebra, while if $S = T$, then $\mathfrak{q}_{S,T}$ is a Levi factor of the parabolic subalgebra $\mathfrak{q}_{\Pi,T}$.*

Let us consider an example which is neither a parabolic subalgebra nor a reductive subalgebra. Let $\mathrm{gl}_n(k)$ be the set of n by n matrices, $(E_{ij})_{1 \le i,j \le n}$ the standard basis of $\mathrm{gl}_n(k)$ and $(E_{ij}^*)_{1 \le i,j \le n}$ its dual basis. For $1 \le i \le n$, denote $\varepsilon_i = E_{ii}^*$.

The set $\mathfrak{h}$ of diagonal matrices is a Cartan subalgebra of $\mathrm{gl}_n(k)$. The corresponding root system is $R = \{\varepsilon_i - \varepsilon_j; 1 \le i \neq j \le n\}$, and $\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = kE_{ij}$ for $1 \le i \neq j \le n$. Set $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \le i \le n-1$. Then $\Pi = \{\alpha_1, \ldots, \alpha_{n-1}\}$ is the set of simple roots for R with respect to $\mathfrak{b}$, the set of upper triangular matrices.

Let $n = 9$, $S = \Pi \setminus \{\alpha_1, \alpha_7\}$ and $T = \Pi \setminus \{\alpha_3, \alpha_4\}$. Then $\mathfrak{q}_{S,T}$ consists of matrices of the form :

$$\left(\begin{array}{c|cc|c|ccc|cc}
* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
* & * & * & * & * & * & * & 0 & 0 \\
* & * & * & * & * & * & * & 0 & 0 \\ \hline
0 & 0 & 0 & * & * & * & * & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & * & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & * & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & * & * & * & * & * \\
0 & 0 & 0 & 0 & * & * & * & * & *
\end{array} \right)$$

Set $R_{S,T}^+ = R_S^+ \setminus R_{S \cap T}$, $R_{S,T}^- = R_T^- \setminus R_{S \cap T}$ and $R_{S,T} = R_{S,T}^+ \cup R_{S,T}^-$. Thus $R_S^+ \cup R_T^- = R_{S,T} \cup R_{S \cap T}$, and we have

$$\mathfrak{q}_{S,T} = \mathfrak{n}_{S,T}^- \oplus \mathfrak{l}_{S,T} \oplus \mathfrak{n}_{S,T}^+$$

where $\mathfrak{n}^{\pm}_{S,T} = \bigoplus_{\alpha \in R^{\pm}_{S,T}} \mathfrak{g}_\alpha$ and $\mathfrak{l}_{S,T} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_{S\cap T}} \mathfrak{g}_\alpha = \mathfrak{q}_{S\cap T, S\cap T}$.

Then $\mathfrak{l}_{S,T}$ is reductive in $\mathfrak{g}$ and $\mathfrak{n}_{S,T} = \mathfrak{n}^+_{S,T} \oplus \mathfrak{n}^-_{S,T}$ is the nilpotent radical of $\mathfrak{q}_{S,T}$.

For the seaweed Lie algebra in Example 7.3, $\mathfrak{l}_{S,T}$ consists of those elements with non-zero entries only on the diagonal blocks, while $\mathfrak{n}_{S,T}$ consists of those elements whose entries on the diagonal blocks are all zero, and $\mathfrak{n}^+_{S,T}$ (resp. $\mathfrak{n}^-_{S,T}$) consists of those elements in $\mathfrak{n}_{S,T}$ which are upper triangular (resp. lower triangular).

Let $\mathbf{P}^-_S$ and $\mathbf{P}^+_T$ be the parabolic subgroups of $\mathbf{G}$ whose Lie algebras are $\mathfrak{p}^-_S$ and $\mathfrak{p}^+_T$ respectively. Set $\mathbf{Q}_{S,T} = \mathbf{P}^-_S \cap \mathbf{P}^+_T$. Then $\mathbf{Q}_{S,T}$ is a closed subgroup of $\mathbf{G}$ whose Lie algebra is $\mathfrak{q}_{S,T}$, and $\mathfrak{n}_{S,T}$ is $\mathbf{Q}_{S,T}$-stable. For example, for the seaweed Lie algebra in Example 7.3, we may take $\mathbf{G} = \mathrm{GL}_9(k)$, and $\mathbf{Q}_{S,T}$ consists of elements of $\mathfrak{q}$ which are invertible.

Definition 7.4. *An element $x \in \mathfrak{n}_{S,T}$ is called a Richardson element of $\mathfrak{q}_{S,T}$ if the orbit closure $\overline{\mathbf{Q}_{S,T}.x} = \mathfrak{n}_{S,T}$ (or equivalently, $[x, \mathfrak{q}_{S,T}] = \mathfrak{n}_{S,T}$).*

We are concerned with the following question raised by Michel Duflo and Dmitri Panyushev independently :

Question 7.5. *Does $\mathfrak{q}_{S,T}$ has a Richardson element ?*

It is well-known that if $\mathfrak{q}_{S,T}$ is a parabolic subalgebra, then it has a Richardson element [14] (see also [16, Chapter 33] for a proof).

8. Seaweed Lie algebras in gl(V) and weakly opposite flags

Let $\mathfrak{g} = \mathrm{gl}(V)$, where V is a k-vector space of dimension $n > 0$, and $\mathbf{G} = \mathrm{GL}(V)$. In this section, we first explain the relations between elements in the nilpotent radical $\mathfrak{n}$ of a seaweed Lie algebra $\mathfrak{q}$ and Δ-filtered $k\tilde{Q}/\mathcal{I}$-modules, where Q is a quiver of type A determined by $\mathfrak{q}$. We then study the connection between $\mathbf{Q}$-orbits in $\mathfrak{n}$ and $\mathrm{GL}(\mathbf{d})$-orbits in the variety $\mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$, where $\mathbf{Q}$ is the closed subgroup of $\mathbf{G}$ associated to $\mathfrak{q}$ and $\mathbf{d}$ is a vector determined by $\mathfrak{q}$. At the end we explain how to find conveniently an explicit Richardson element from the exceptional Δ-filtered module $M(\mathbf{d})$ constructed in Section 6. Seaweed Lie algebras in $\mathfrak{g}$ has a nice description via stabilizers of certain pairs of partial flags.

Let Ξ be the set of sequences of integers $(a_i)_{0\le i\le r}$ (r not being fixed), verifying

$$0 = a_0 < a_1 < \cdots < a_r = n.$$

For $\mathbf{a} = (a_i)_{0\le i\le r} \in \Xi$, let $\mathcal{F}_{\mathbf{a}}$ denote the set of (partial) flags

$$0 = V_0 \subset V_1 \subset \cdots \subset V_r = V$$

such that $\dim V_i = a_i$.

Recall that given a partial flag $\mathbf{F}$, its stabilizer $\mathfrak{p}_{\mathbf{F}}$ (resp. $\mathbf{P}_{\mathbf{F}}$) in $\mathfrak{g}$ (resp. in $\mathbf{G}$) is a parabolic subalgebra (resp. parabolic subgroup), and this induces a bijection between the set of all partial flags and the set of parabolic subalgebras of $\mathfrak{g}$ (resp. the set of parabolic subgroups of $\mathbf{G}$).

Let $B = (e_1, \ldots, e_n)$ be a basis for V. A partial flag $\mathbf{F} = (V_i)_{0\le i\le r}$ will be called B-upper if $V_i = \mathrm{Vect}(e_1, \ldots, e_{\dim V_i})$ for $1 \le i \le r$. Similarly, $\mathbf{F}$ is B-lower if $\mathbf{F}$ is $(e_n, \ldots, e_1)$-upper.

A pair of partial flags $(\mathbf{F}, \mathbf{F}')$ is called weakly opposite if there exists a basis B such that $\mathbf{F}$ is B-upper and $\mathbf{F}'$ is B-lower.

It follows from Proposition 7.2 that the map

$$(\mathbf{F}, \mathbf{F}') \mapsto \mathfrak{p}_{\mathbf{F}} \cap \mathfrak{p}_{\mathbf{F}'}$$

induces a bijection between the set of pairs of weakly opposite flags and the set of seaweed Lie algebras in $\mathfrak{g}$.

We shall associate to a pair of weakly oppposite flags a quiver of type A and a dimension vector. Let us start with an example.

8.1. **A guiding example.** Take the seaweed Lie algebra in Example 7.3. It corresponds to a pair of weakly opposite flags $(\mathbf{F},\mathbf{F}') \in \mathcal{F}_{\mathbf{a}} \times \mathcal{F}_{\mathbf{b}}$ where $\mathbf{a} = (0,3,4,9)$ and $\mathbf{b} = (0,2,8,9)$. Let $B = (e_i)_{1\leq i\leq 9}$ be a basis of V such that $\mathbf{F}$ is B-upper and $\mathbf{F}'$ is B-lower. In this basis B, the elements of $\mathfrak{q} = \mathfrak{p}_{\mathbf{F}} \cap \mathfrak{p}_{\mathbf{F}'}$ has the following matrix form

$$\begin{array}{c} E_1 \\ \hline E_2 \\ \\ \hline E_3 \\ \hline \\ E_4 \\ \\ \hline E_5 \\ \\ \end{array} \left(\begin{array}{c|cc|c|ccc|cc} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & 0 & 0 \\ \hline 0 & 0 & 0 & * & * & * & * & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & * & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * & * \end{array}\right),$$

where we set

$$E_1 = \mathrm{Vect}(e_1)\ ,\ E_2 = \mathrm{Vect}(e_2,e_3)\ ,\ E_3 = \mathrm{Vect}(e_4)\ ,\ E_4 = \mathrm{Vect}(e_5,e_6,e_7)\ ,\ E_5 = \mathrm{Vect}(e_8,e_9).$$

The Levi factor $\mathfrak{l}$ of $\mathfrak{q}$ is then $\mathrm{gl}(E_1) \times \cdots \times \mathrm{gl}(E_5)$.

For $1 \leq i \leq 5$, we set Y_i to be the sum of the subspaces E_k corresponding to non-zero blocks in the i-th block-column. So

$$Y_1 = E_1 \oplus E_2\ ,\ Y_2 = E_2\ ,\ Y_3 = E_2 \oplus E_3\ ,\ Y_4 = E_2 \oplus E_3 \oplus E_4 \oplus E_5\ ,\ Y_5 = E_5.$$

We define a quiver Q with the set of vertices $Q_0 = \{1,2,3,4,5\}$, and we have an arrow from i to $i \pm 1$ if $Y_i \subset Y_{i\pm 1}$. Thus, Q is a quiver of type A_5 in the following orientation :

$$1 \xleftarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} 4 \xleftarrow{\alpha_4} 5$$

Now let $\mathfrak{n}$ denote the nilpotent radical of $\mathfrak{q}$. Given $x \in \mathfrak{n}$, in the basis B, the matrix of x has zero entries on the diagonal blocks. We have by inspection that

$$x(Y_1) \subset Y_2\ ,\ x(Y_2) = 0\ ,\ x(Y_3) \subset Y_2\ ,\ x(Y_4) \subset Y_3 \oplus Y_5\ ,\ x(Y_5) = 0.$$

We denote by β_i the reverse arrow of α_i in $\tilde{Q}$. We define the following representation M^x of the double $\tilde{Q}$ of this quiver Q : the vector space M^x_i is Y_i, $M^x_{\alpha_i}$ is just the canonical injection, while $M^x_{\beta_i}$ is the projection onto $Y_{s(\alpha_i)}$ with respect to the direct sum decomposition above of the restriction of x to $Y_{t(\alpha_i)}$. For example $M^x_{\beta_3} = \pi_1 \circ x|_{Y_4}$, where $\pi_1 : Y_3 \oplus Y_5 \to Y_3$ is the canonical projection with respect to this direct sum decomposition.

It is now a straightforward verification that M^x is a representation of $(\tilde{Q},\mathcal{I})$, and since all the α_i are injective, it follows from Proposition 2.4 that M^x is Δ-filtered.

We have therefore a map from $\mathfrak{n}$ to $\mathrm{Rep}_\Delta(\tilde{Q},\mathcal{I},\mathbf{d})$ where $\mathbf{d} = (\dim Y_i)_{1\leq i\leq 5}$.

8.2. **The formal construction.** We shall now describe this construction formally.

Let us fix $\mathbf{a} = (a_i)_{0\leq i\leq r}$, $\mathbf{b} = (b_j)_{0\leq j\leq r'} \in \Xi$, and $(\mathbf{F},\mathbf{F}') \in \mathcal{F}_{\mathbf{a}} \times \mathcal{F}_{\mathbf{b}}$ a pair of weakly opposite flags. Let $B = (e_1,\ldots,e_n)$ be a basis of V such that $\mathbf{F}$ is B-upper and $\mathbf{F}'$ is B-lower. Then $\mathbf{F} = (V_i)_{0\leq i\leq r}$ and $\mathbf{F}' = (V'_j)_{0\leq j\leq r'}$ where $V_0 = V'_0 = \{0\}$ and $V_i = \mathrm{Vect}(e_1,\ldots,e_{a_i})$ for $i > 0$, $V'_j = \mathrm{Vect}(e_{n-b_j+1},\ldots,e_n)$ for $j > 0$.

Let $0 = m_0 < m_1 < \cdots < m_l = n$ be integers such that $\{m_0,\ldots,m_l\}$ is the union of the a_i and $n - b_j$, $0 \leq i \leq r$, $0 \leq j \leq r'$. So for each $0 \leq s \leq l$, either $\mathrm{Vect}(e_1,\ldots,e_{m_s})$ is some V_i or $\mathrm{Vect}(e_{m_s+1},\ldots,e_n)$ is some V'_j.

For $1 \leq s \leq l$, set

$$E_s = \mathrm{Vect}(e_{m_{s-1}+1},\ldots,e_{m_s}),\ \text{and}\ Y_s = V_{u(s)} \cap V'_{\ell(s)},$$

where $u(s) = \min\{i; E_s \subseteq V_i\}$ and $\ell(s) = \min\{j; E_s \subseteq V'_j\}$.

By definition, for $1 \leq s \leq l$, we have either $E_1 \oplus \cdots \oplus E_s = V_i$ for some $1 \leq i \leq r$, or $E_{s+1} \oplus \cdots \oplus E_l = V'_j$ for some $1 \leq j \leq r'$. Clearly, all the V_i (resp. V'_j) are obtained in this way. So we can find $p \leq s \leq q$ such that

$$V_{u(s)} = E_1 \oplus \cdots \oplus E_q \ , \ V'_{\ell(s)} = E_p \oplus \cdots \oplus E_l \text{ and } Y_s = E_p \oplus \cdots \oplus E_q.$$

In particular, $Y_s \supseteq E_s$.

Lemma 8.1. *Let $1 \leq s \leq l-1$. Then we have one of the following 3 disjoint configurations:*

(1) $Y_s \cap Y_{s+1} = \{0\}$.
(2) $Y_s \subsetneq Y_{s+1}$.
(3) $Y_{s+1} \subsetneq Y_s$.

Moreover, for $1 \leq s \leq l$ we have,

$$\begin{cases} Y_s = Y_{s-1} \oplus E_s & \text{if } Y_{s-1} \subseteq Y_s \text{ and } Y_{s+1} \not\subseteq Y_s, \\ Y_s = Y_{s-1} \oplus E_s \oplus Y_{s+1} & \text{if } Y_{s-1} \subseteq Y_s \text{ and } Y_{s+1} \subseteq Y_s, \\ Y_s = Y_{s+1} \oplus E_s & \text{if } Y_{s-1} \not\subseteq Y_s \text{ and } Y_{s+1} \subseteq Y_s, \\ Y_s = E_s & \text{if } Y_{s-1} \not\subseteq Y_s \text{ and } Y_{s+1} \not\subseteq Y_s, \end{cases}$$

where by convention, we set $Y_0 = Y_{l+1} = \{0\}$.

Proof. Observe first that by the definition of the E_s, the sequence $(u(s))_{s=1,\dots,l}$ (resp. $(l(s))_{s=1,\dots,l}$) is weakly increasing (resp. weakly decreasing), and moreover, if $u(s+1) > u(s)$ (resp. $l(s) > l(s+1)$), then $u(s+1) = u(s) + 1$ (resp. $l(s) = l(s+1) + 1$).

Recall that $E_s = \text{Vect}(e_{m_{s-1}+1}, \dots, e_{m_s})$, and that either $\text{Vect}(e_1, \dots, e_{m_s}) = V_i$ for some i, or $\text{Vect}(e_{m_s+1}, \dots, e_n) = V'_j$ for some j. In the first case, we have $i = u(s)$ and $u(s+1) > u(s)$, while in the second case, we have $j = l(s) - 1 = l(s+1)$. Thus $(u(s), l(s)) \neq (u(s+1), l(s+1))$.

Now let $1 \leq s \leq l-1$. Let $p \leq s \leq q$ be such that $V_{u(s)} = E_1 \oplus \cdots \oplus E_q$ and $V'_{l(s)} = E_p \oplus \cdots \oplus E_l$. So $Y_s = E_p \oplus \cdots \oplus E_q$.

We have 3 (disjoint) possibilities :

(1) $u(s) = u(s+1)$. Then $l(s) = l(s+1) + 1$, and hence $Y_{s+1} = V_{u(s+1)} \cap V'_{l(s+1)} \subseteq V_{u(s)} \cap V'_{l(s)} = Y_s$. Furthermore, since $l(s+1) < l(s)$, $E_s \not\subseteq V'_{l(s+1)}$. It follows that $V'_{l(s+1)} = E_{s+1} \oplus \cdots \oplus E_l$. Thus
$$Y_{s+1} = E_{s+1} \oplus \cdots \oplus E_q \subsetneq E_p \oplus \cdots \oplus E_q = Y_s.$$
(2) $l(s) = l(s+1)$. Then $u(s+1) = u(s) + 1$, and a similar argument shows that $V_{u(s)} = E_1 \oplus \cdots \oplus E_s$, and
$$Y_s = E_p \oplus \cdots \oplus E_s \subsetneq E_p \oplus \cdots \oplus E_{m_{u(s+1)}} = Y_{s+1}.$$
(3) $u(s) < u(s+1)$ and $l(s) > l(s+1)$. Then from the above arguments, we have $V_{u(s)} = E_1 \oplus \cdots \oplus E_s$, and $V'_{l(s+1)} = E_{s+1} \oplus \cdots \oplus E_l$. Hence $Y_s \cap Y_{s+1} = \{0\}$.

We have therefore proved the first part of the lemma. The second part follows easily from the 3 cases above and the fact that $E_s \subseteq Y_s$. □

As in the example, we define a quiver Q whose vertices are the integers $\{1, \dots, l\}$, and for $i = 1, \dots, l-1$, we have an arrow

$$i \xrightarrow{\alpha_i} i+1 \text{ if } Y_i \subset Y_{i+1} \text{ or } i \xleftarrow{\alpha_i} i+1 \text{ if } Y_i \supset Y_{i+1}.$$

Note that Q is not necessarily connected, however each of its connected components is of type A.

Now let $\mathfrak{q} = \mathfrak{p}_{\mathbf{F}} \cap \mathfrak{p}_{\mathbf{F}'}$ be the seaweed Lie algebra associated to $(\mathbf{F}, \mathbf{F}')$. Then its Levi factor $\mathfrak{l}$ is $\text{gl}(E_1) \times \cdots \times \text{gl}(E_l)$.

Let $\mathfrak{n}$ be the nilpotent radical of $\mathfrak{q}$. For $x \in \mathfrak{n}$, we have that $x(E_s) \subseteq \bigoplus_{i \neq s} E_i$ for all $s = 1, \dots, l$. Since elements of $\mathfrak{q}$ stabilizes the two flags, it follows from Lemma 8.1 that

$$\begin{cases} x(Y_s) \subseteq Y_{s-1} & \text{if } Y_{s-1} \subseteq Y_s \text{ and } Y_{s+1} \not\subseteq Y_s, \\ x(Y_s) \subseteq Y_{s-1} \oplus Y_{s+1} & \text{if } Y_{s-1} \subseteq Y_s \text{ and } Y_{s+1} \subseteq Y_s, \\ x(Y_s) \subseteq Y_{s+1} & \text{if } Y_{s-1} \not\subseteq Y_s \text{ and } Y_{s+1} \subseteq Y_s, \\ x(Y_s) = \{0\} & \text{if } Y_{s-1} \not\subseteq Y_s \text{ and } Y_{s+1} \not\subseteq Y_s. \end{cases}$$

Clearly, any x verifying these conditions is in $\mathfrak{n}$.

Again we denote by β_i the reverse arrow of α_i in $\tilde{Q}$. We define a representation M^x of the double $\tilde{Q}$ of the quiver Q as follows : $M_i^x = Y_i$, $M_{\alpha_i}^x$ is just the canonical injection, and $M_{\beta_i}^x$ is the projection to $Y_{s(\alpha_i)}$ with respect to the direct sum decompositions in Lemma 8.1 of the restriction of x to $Y_{t(\alpha_i)}$.

We verify easily that $M^x \in \mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$ where $\mathbf{d} = (\dim Y_i)_{1 \leq i \leq l}$.

Let us denote by $\mathbf{Q} = \mathbf{P_F} \cap \mathbf{P_{F'}}$. Then $\sigma \in \mathbf{Q}$ if and only if σ stabilizes both flags, or equivalently, σ stabilizes Y_s for $s = 1, \dots, l$.

Theorem 8.2. *There is a bijection between the* $\mathbf{Q}$*-orbits in* $\mathfrak{n}$ *and the* $\mathrm{GL}(\mathbf{d})$*-orbits in* $\mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$.

In particular, $\mathfrak{n}$ *contains an open* $\mathbf{Q}$*-orbit if and only if* $\mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$ *contains an open* $\mathrm{GL}(\mathbf{d})$*-orbit.*

Proof. Without loss of generalities, we may fix the underlying vector space to be $(Y_i)_{1 \leq i \leq l}$. Recall from Proposition 2.9 that R^α, the set of elements M in $\mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$ such that M_{α_i} is the canonical injection for each i, meets every $\mathrm{GL}(\mathbf{d})$-orbit in $\mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$.

By the construction above, we have an injective map

$$\Phi : \mathfrak{n} \longrightarrow R^\alpha \ , \ x \mapsto M^x.$$

(1) Let $M \in R^\alpha$. Recall that $V = E_1 \oplus \cdots \oplus E_l$, so any $v \in V$ is written uniquely as a sum $v = v_1 + \cdots + v_l$ where $v_i \in E_i$ for $1 \leq i \leq l$.

Since M_{α_j} is the canonical injection for all j and by the definition of $\mathcal{I}$, we may define

$$x(v_i) = \begin{cases} M_{\beta_j}(v_i) & \text{if } E_i \subset Y_{s(\beta_j)}, \\ 0 & \text{otherwise.} \end{cases}$$

Extending by linearity to v, we obtain an endomorphism x of V which clearly sends $Y_{s(\beta_j)}$ into $Y_{t(\beta_j)}$. So the relations in $\mathcal{I}$ imply that $x \in \mathfrak{n}$. It is now straightforward to check that $\Phi(x) = M$, and hence Φ is surjective.

Consequently, the map Φ is a bijection.

(2) Let $\sigma \in \mathrm{GL}(\mathbf{d})$ be such that $\sigma(R^\alpha) \subset R^\alpha$. Then this is equivalent to the condition that for any i, we have

$$\begin{array}{ccc} Y_{s(\alpha_i)} & \xrightarrow{M_{\alpha_i}} & Y_{t(\alpha_i)} \\ {\scriptstyle \sigma_{s(\alpha_i)}}\downarrow & & \downarrow{\scriptstyle \sigma_{t(\alpha_i)}} \\ Y_{s(\alpha_i)} & \xrightarrow{M_{\alpha_i}} & Y_{t(\alpha_i)} \end{array}$$

It follows that $\sigma_{s(\alpha_i)}$ is the restriction of $\sigma_{t(\alpha_i)}$ to $Y_{s(\alpha_i)}$. This allows us to define a (unique) element $\tilde{\sigma} \in \mathbf{G}$ such that $\sigma|_{Y_j} = \sigma_j$ for all $j \in Q_0$. Clearly, $\tilde{\sigma} \in \mathbf{Q}$.

Conversely, given $\tau \in \mathbf{Q}$, then its restrictions to Y_i gives an element of $\mathrm{GL}(\mathbf{d})$ which clearly stabilizes R^α. We may therefore identify $\mathbf{Q}$ with the stabilizer of R^α in $\mathrm{GL}(\mathbf{d})$. It follows that if $M \in R^\alpha$, then the intersection of the $\mathrm{GL}(\mathbf{d})$-orbit of M in $\mathrm{Rep}(\tilde{Q}, \mathcal{I}, \mathbf{d})$ with R^α is

$$(\mathrm{GL}(\mathbf{d}).M) \cap R^\alpha = \mathbf{Q}.M.$$

Consequently, there is a bijective correspondence between $\mathbf{Q}$-orbits in R^α and $\mathrm{GL}(\mathbf{d})$-orbits in $\mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$.

(3) By Point (1) and a direct check, the map Φ is a $\mathbf{Q}$-equivariant bijection, and so there is a one-to-one correspondence between $\mathbf{Q}$-orbits in $\mathfrak{n}$ and $\mathbf{Q}$-orbits in R^α which in turn, is in bijection with $\mathrm{GL}(\mathbf{d})$-orbits in $\mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$ by Point (2).

Finally, if $\mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$ has an open $\mathrm{GL}(\mathbf{d})$-orbit, then R^α has an open $\mathbf{Q}$-orbit, and so has $\mathfrak{n}$ by the above correspondence.

Conversely, if $\mathfrak{n}$ has an open $\mathbf{Q}$-orbit, then R^α has an open $\mathbf{Q}$-orbit, say Ω. Then

$$\overline{\mathrm{GL}(\mathbf{d}).\Omega} \supset \mathrm{GL}(\mathbf{d}).\overline{\Omega} = \mathrm{GL}(\mathbf{d}).R^\alpha = \mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d}).$$

Hence $\mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$ has an open $\mathrm{GL}(\mathbf{d})$-orbit. □

It follows from Theorem 8.2 and Theorem 6.1 that Question 7.5 has a positive answer for seaweed Lie algebras in $\mathrm{gl}(V)$.

Corollary 8.3. *Let $\mathfrak{q}$ be a seaweed Lie algebra in $\mathrm{gl}(V)$. Then $\mathfrak{q}$ has a Richardson element.*

Remark 8.4. *Given any integer p, we set $\mathfrak{n}_p = \{x \in \mathfrak{n}; \dim \mathbf{Q}.x = p\}$, and similarly, $R_p = \{M \in \mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d}); \dim \mathrm{GL}(\mathbf{d}).M = p\}$. Then via the bijection in Theorem 8.2, we obtain readily that*

$$\dim \mathfrak{n}_p - p = \dim R_{\dim \mathrm{GL}(\mathbf{d}) - \dim \mathbf{Q} + p} - (\dim \mathrm{GL}(\mathbf{d}) - \dim \mathbf{Q} + p)$$

whenever $\mathfrak{n}_p$ is non-empty. It follows that the modality of the $\mathbf{Q}$-action on $\mathfrak{n}$ is the same as the modality of the $\mathrm{GL}(\mathbf{d})$-action on $\mathrm{Rep}_\Delta(\tilde{Q}, \mathcal{I}, \mathbf{d})$ (See [17] for a definition of the modality of an action).

Remark 8.5. *Using the procedure described in Example 6.11 to construct an explicit exceptional Δ-filtered module with a given Δ-dimension vector, we obtain a procedure for constructing an explicit Richardson element, imitating the construction given in [2] for parabolic subalgebras (see also [1]).*

We first determine the quiver Q and the vector $\mathbf{e} = (\dim E_i)_{i=1,\dots,l}$ associated to a given seaweed Lie algebra. Then we construct the exceptional Δ-filtered $k\tilde{Q}/\mathcal{I}$-module $M(\mathbf{e})$ and draw an arrow diagram associated to $M(\mathbf{e})$ as in Example 6.11. Then we replace each double arrow $\Delta(i) \Longrightarrow \Delta(j)$ by $\Delta(i) \longleftarrow \Delta(j)$. Finally we replace column-wise the $\Delta(i)$'s by numbers 1 to n, starting on the left and from the bottom to the top. Set

$$x = \sum_{i \to j} E_{ij},$$

where the sum runs over all arrows in the diagram. Then x is a Richardson element of $\mathfrak{q}$.

For example, the seaweed Lie algebra $\mathfrak{q}$ of Example 7.3 gives us the quiver Q and the Δ-dimension vector $\mathbf{e}$ as in Example 6.11. From the diagram associated to $M(\mathbf{e})$ in Example 6.11, we obtain the following arrow diagram.

$$7 \longleftarrow 9$$

$$1 \longleftarrow 3 \longrightarrow 6 \longleftarrow 8$$

$$2 \longrightarrow 4 \longrightarrow 5$$

Thus

$$x = E_{31} + E_{24} + E_{45} + E_{36} + E_{86} + E_{97} = \left(\begin{array}{c|cc|c|ccc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

is a Richardson element of $\mathfrak{q}$.

Remark 8.6. *In this matrix setting, it is possible to extend with the same proofs Theorem 6.1 and Theorem 8.2 (in the case of stabilizers of weakly opposite flags) when the field is algebraically closed of non-zero characteristic.*

9. Remarks on seaweed Lie algebras of other types

In this section, we shall discuss the existence of Richardson elements in a simple Lie algebra $\mathfrak{g}$ of type other than A_n.

Let us use the notations in Section 7, and fix $S, T \subset \Pi$. Recall that we have the decomposition

$$\mathfrak{q}_{S,T} = \mathfrak{n}_{S,T}^- \oplus \mathfrak{l}_{S,T} \oplus \mathfrak{n}_{S,T}^+ \ , \ \mathfrak{n}_{S,T} = \mathfrak{n}_{S,T}^+ \oplus \mathfrak{n}_{S,T}^-.$$

We have already observed in Section 7 that $\mathfrak{n}_{S,T}$ is the nilpotent radical of $\mathfrak{q}_{S,T}$, and that $\mathfrak{l}_{S,T} = \mathfrak{q}_{S\cap T, S\cap T}$ is a reductive subalgebra of $\mathfrak{g}$.

We have a nice "decomposition" of $\mathfrak{q}_{S,T}$ as a sum of two parabolic subalgebras of reductive subalgebras of $\mathfrak{g}$. More precisely, we have :

Proposition 9.1. *Set* $\mathfrak{p}_{S,T}^{\pm} = \mathfrak{l}_{S,T} \oplus \mathfrak{n}_{S,T}^{\pm}$. *Then*

(1) $\mathfrak{p}_{S,T}^+ = \mathfrak{q}_{S,S\cap T}$ *is a parabolic subalgebra of the reductive Lie algebra* $\mathfrak{q}_{S,S}$.
(2) $\mathfrak{p}_{S,T}^- = \mathfrak{q}_{S\cap T,T}$ *is a parabolic subalgebra of the reductive Lie algebra* $\mathfrak{q}_{T,T}$.
(3) $\mathfrak{q}_{S,T} = \mathfrak{p}_{S,T}^+ + \mathfrak{p}_{S,T}^-$ *and* $\mathfrak{p}_{S,T}^+ \cap \mathfrak{p}_{S,T}^- = \mathfrak{l}_{S,T}$.
(4) $[\mathfrak{n}_{S,T}^+, \mathfrak{n}_{S,T}^-] = \{0\}$.

Proof. This is straightforward. □

By Richardson's Theorem on the existence of Richardson elements in parabolic subalgebras, $\mathfrak{p}_{S,T}^{\pm}$ has a Richardson element in $\mathfrak{n}_{S,T}^{\pm}$. It follows easily from part (4) of Proposition 9.1 and by considering projections with respect to the direct sum decomposition $\mathfrak{n}_{S,T} = \mathfrak{n}_{S,T}^+ \oplus \mathfrak{n}_{S,T}^-$ that if $\mathfrak{q}_{S,T}$ has a Richardson element x, then there exist Richardson elements $x_{\pm}$ of $\mathfrak{p}_{S,T}^{\pm}$ such that $x = x_+ + x_-$. Note that if we take arbitrary Richardson elements of $\mathfrak{p}_{S,T}^{\pm}$, their sum need not be a Richardson element of $\mathfrak{q}_{S,T}$. This explains in part why there is a constraint in how to glue exceptional Δ-filtered modules with linear Δ-support in Section 6.

Let us suppose that $\mathfrak{q}_{S,T}$ has a Richardson element x, and let $x_{\pm}$ be Richardson elements of $\mathfrak{p}_{S,T}^{\pm}$ such that $x = x_+ + x_-$. Using the action of the group $\mathbf{Q}_{S,T}$, we may fix one of the elements, say x_+.

By definition, we have $[x, \mathfrak{q}_{S,T}] = \mathfrak{n}_{S,T}$. By part (4) of Proposition 9.1, we deduce that $[x_{\pm}, \mathfrak{q}_{S,T}] \subseteq \mathfrak{n}_{S,T}^{\pm}$. It follows that to obtain $\mathfrak{n}_{S,T}^- \subseteq [x, \mathfrak{q}_{S,T}]$, it is necessary that

$$[x_-, \mathfrak{q}_{S,T}^{x_+}] = \mathfrak{n}_{S,T}^-$$

where $\mathfrak{q}_{S,T}^{x_+} = \{z \in \mathfrak{q}_{S,T}; [z, x_+] = 0\}$.

Using the above necessary condition, we found a seaweed Lie algebra which does not admit any Richardson element. Let $\mathfrak{g}$ be a simple Lie algebra of type E_8. In the numbering of simple roots of [16, Chapter 18], let

$$S = \Pi \setminus \{\alpha_8\}\ ,\ T = \Pi \setminus \{\alpha_4, \alpha_5\}.$$

A straightforward computation gives

$$\dim \mathfrak{q}_{S,T} = 81\ ,\ \dim \mathfrak{l}_{S,T} = 22\ ,\ \dim \mathfrak{n}^+_{S,T} = 56\ ,\ \dim \mathfrak{n}^-_{S,T} = 3.$$

Let $(x_i)_{1\leq i\leq 56}$ be the basis of $\mathfrak{n}^+_{S,T}$ consisting of root vectors ordered in a certain way given by our routine "Seaweed" in GAP4. We set

$$x_+ = \sum_{i=1}^{56} p_i x_i$$

where p_i is the i-th prime number. The function "LieCentralizer" then gives $\dim \mathfrak{q}^{x_+}_{S,T} = 25$. Thus $[\mathfrak{q}_{S,T}, x_+] = [\mathfrak{p}^+_{S,T}, x_+] = \mathfrak{n}^+_{S,T}$. So x_+ is a Richardson element of $\mathfrak{p}^+_{S,T}$.

Next, "BasisVectors (Basis ($\mathfrak{q}^{x_+}_{S,T}$))" provides a basis $(b_i)_{1\leq i\leq 25}$ of $\mathfrak{q}^{x_+}_{S,T}$. We then check that $[b_i, \mathfrak{n}^-_{S,T}] = 0$ if $1 \leq i \leq 22$, $[b_{23}, \mathfrak{n}^-_{S,T}] = [b_{24}, \mathfrak{n}^-_{S,T}] = kz$ for some $z \in \mathfrak{n}^-_{S,T}$, and $(\mathrm{ad} b_{25})|_{\mathfrak{n}^-_{S,T}} = -1/2$.

Since $\mathfrak{n}^-_{S,T}$ is abelian, we deduce from the above that for all $x_- \in \mathfrak{n}^-_{S,T}$, we have

$$[x_-, \mathfrak{q}^{x_+}_{S,T}] \subseteq \mathrm{Vect}(x_-, z),$$

so $\dim[x_-, \mathfrak{q}^{x_+}_{S,T}] \leq 2$. In particular, $\mathfrak{q}_{S,T}$ does not admit a Richardson element.

Corollary 9.2. *There exists a seaweed Lie algebra in the simple Lie algebra of type E_8 which does not admit a Richardson element.*

Remarks 9.3. *(1) By taking generic elements, we have checked by using* GAP4 *that if $\mathfrak{g}$ is a simple Lie algebra of rank ≤ 7, then any seaweed Lie algebra contained in $\mathfrak{g}$ has a Richardson element.*

(2) Using a type by type consideration and the fact that abelian ideals of parabolic subalgebras are spherical [13]*, we can show that if the nilpotent radical of a seaweed Lie algebra is abelian, then it has a Richardson element.*

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