INDECOMPOSABLE REPRESENTATIONS FOR REAL ROOTS OF A WILD QUIVER

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Abstract. For a given quiver and dimension vector, Kac has shown that there is exactly one indecomposable representation up to isomorphism if and only if this dimension vector is a positive real root. However, it is not clear how to compute these indecomposable representations in an explicit and minimal way, and the properties of these representations are mostly unknown. In this note we study representations of a particular wild quiver. We define operations which act on representations of this quiver, and using these operations we construct indecomposable representations for positive real roots, compute their endomorphism rings and show that these representations are tree representations. The operations correspond to the fundamental reflections in the Weyl group of the quiver. Our results are independent of the characteristic of the field.

INTRODUCTION

A fundamental result in the representation theory of quivers is the discovery of the relation between the dimension vectors of the indecomposable representations and the positive roots of the corresponding Kac-Moody Lie-algebra. This discovery was first made for Dynkin quivers by Gabriel [4], where the dimension vectors correspond to positive roots of the corresponding semi-simple Lie-algebra, and then generalized to arbitrary quivers by Kac [5]. Moreover, Kac showed that for a given dimension vector, there is exactly one indecomposable representation up to isomorphism if and only if this dimension vector is a real root. At the conference ICRA XI in Mexico in 2004, Crawley-Boevey asked the following question: What is the dimension of the endomorphism ring of an indecomposable representation with dimension vector a real root? This question is the first motivation for the results in this paper.

An important concept in the representation theory of quivers is the generic representation [5][7][10]. A positive root $\mathbf{d}$ is called a Schur root if there exists a representation in the representation space $\mathrm{Rep}(Q, \mathbf{d})$ with trivial endomorphism ring. In this case, the generic representation is indecomposable. For Dynkin quivers all the positive roots are real Schur roots. For euclidean quivers (excluding Kronecker quiver), not all the positive real roots are Schur roots. However, in any case, the dimensions of the endomorphism rings of the indecomposable representation and the generic representation coincide. We will see that for wild quivers, the situation can be very different.

Crawley-Boevey [2] has defined reflection functors for deformed preprojective algebras and has used these functors to give an algorithm for computing indecomposable representations for positive real roots of quivers. But it is not clear how to compute indecomposable representations in an explicit and minimal way, independent of the characteristic of the field. Ringel [8] has shown that indecomposable representations for positive real Schur roots are tree representaions. That is, they can be given by $[0, 1]$-matrices using the smallest possible number of non-zero entries. Ringel's constructions of tree representations are not clearly independent of the characteristic of the field. Ringel [9] has asked the following question: Let $\mathbf{d}$ be a positive root. Is there an indecomposable tree representation with dimension vector $\mathbf{d}$? Our second motivation is related to the question of Ringel: Are indecomposable representations for positive real roots tree representations which can be given independent of the characteristic of the field?

In this paper we study the indecomposable representations for positive real roots of the wild quiver

$$\mathcal{Q} : 1 \xrightarrow{\alpha} 2 \underset{\gamma}{\overset{\beta}{\rightleftarrows}} 3$$

We give a complete and explicit construction of indecomposable tree representations of the quiver $\mathcal{Q}$ with dimension vector a real root. This construction, which is independent of the characteristic of the field, allows us to compute the dimension of the endomorphism rings of the representations. Moreover we construct generic representation for positive real roots of $\mathcal{Q}$, and compare the dimensions of the endomorphism rings of the generic and indecomposable representations.

The rest of the paper is organized as follows. In Section 1 we give notation and background. In Section 2 we recall some properties of the representations of the Euclidean subquiver of $\mathcal{Q}$. We compute the set of real roots of $\mathcal{Q}$ in Section 3. In Section 4 we give the three operations corresponding to the three fundamental reflections in the Weyl group. In Section 5 we show how these operations can be used to compute the indecomposable representations and the dimensions of their endomorphism rings. In Section 6 we show that our representations are tree representations which can be given independent of the characteristic of the field. We compute the canonical decomposition of a real root in Section 7 and compare the dimensions of the endomorphism rings of indecomposable and generic representations. In Section 8 we illustrate our construction by some examples.

1. Background and notation

1.1. **Representations of quivers.** A quiver $Q = (Q_0, Q_1, s, t)$ consists of a set of vertices Q_0, a set of arrows Q_1 and two maps $s, t : Q_1 \longrightarrow Q_0$, where for any arrow $\alpha \in Q_1$ $s(\alpha)$ is the starting vertex of α and $t(\alpha)$ is the terminating vertex of α.

A representation M of Q is a family of finite dimensional vector spaces $\{M_i\}_{i\in Q_0}$ together with a family of linear maps $\{M_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}\}_{\alpha\in Q_1}$. The vector $\underline{\dim} M = (\dim M_i)_{i\in Q_0}$ is called the dimension vector of M. Given two vectors $\mathbf{c} = (c_i)_{i\in Q_0}$ and $\mathbf{d} = (d_i)_{i\in Q_0}$, by $\mathbf{c} \geq \mathbf{d}$ we mean that $c_i \geq d_i$ for all $i \in Q_0$, and by $\mathbf{c} > \mathbf{d}$ we mean that $\mathbf{c} \geq \mathbf{d}$ and $\mathbf{c} \neq \mathbf{d}$. If Q has n vertices, then we identify dimension vectors with elements of $\mathbb{N}^n$.

A morphism $f : M \longrightarrow N$ between two representations M and N is a family of $\mathbf{k}$-linear maps $\{f_i : M_i \longrightarrow N_i\}_{i\in Q_0}$ such that $N_\alpha f_{s(\alpha)} = f_{t(\alpha)} M_\alpha$ for each $\alpha \in Q_1$. We say that f is an isomorphism if for each $i \in Q_0$ f_i is an isomorphism. The direct sum $M \oplus N$ is defined by $(M \oplus N)_i = M_i \oplus N_i$ for each $i \in Q_0$ and

$$(M \oplus N)_\rho = \begin{pmatrix} M_\rho & 0 \\ 0 & N_\rho \end{pmatrix}$$

for each $\rho \in Q_1$. We say that a representation M is indecomposable if $M \cong M^1 \oplus M^2$ implies $M^1 = 0$ or $M^2 = 0$. We denote by S_i the simple representation associated to vertex i.

1.2. **Representation spaces and canonical decomposition.** Given a dimension vector $\mathbf{d} = (d_i)_{i\in Q_0}$, we denote by $\mathrm{Rep}(Q, \mathbf{d})$ the space of representations of Q, given by

$$\mathrm{Rep}(Q, \mathbf{d}) = \prod_{\alpha\in Q_1} \mathrm{Hom}(\mathbf{k}^{d_{s(\alpha)}}, \mathbf{k}^{d_{t(\alpha)}}).$$

So $\mathrm{Rep}(Q, \mathbf{d})$ parameterizes the representations of Q with dimension vector $\mathbf{d}$. We do not distinguish a point in $\mathrm{Rep}(Q, \mathbf{d})$ from the associated representation of Q. There is a natural action of $\mathrm{Gl}(\mathbf{d}) = \prod_{i\in Q_0} \mathrm{Gl}(d_i)$ on $\mathrm{Rep}(Q, \mathbf{d})$ by conjugation,

$$(g \cdot M)_\alpha = g_{t(\alpha)} M_\alpha g^{-1}_{s(\alpha)},$$

such that there is a one-to-one correspondence between the $\mathrm{Gl}(\mathbf{d})$-orbits in $\mathrm{Rep}(Q,\mathbf{d})$ and the isomorphism classes of representations of Q with dimension vector $\mathbf{d}$.

Let $\mathrm{Ind}(Q,\mathbf{d})$ denote the subset of indecomposable representations in $\mathrm{Rep}(Q,\mathbf{d})$. There exists a unique decomposition of $\mathbf{d}=\mathbf{d}^1+\cdots+\mathbf{d}^s$ such that the set $\mathrm{Rep}(Q,\mathbf{d})_{\mathrm{gen}}=\{M\in \mathrm{Rep}(Q,\mathbf{d})|M\cong M^1\oplus\cdots\oplus M^s,\ \text{where } M^i\in\mathrm{Ind}(Q,\mathbf{d}^i) \text{ for } i=1,\ldots,s\}$ contains a dense open subset of $\mathrm{Rep}(Q,\mathbf{d})$ [5][7][10]. This decomposition is called the canonical decomposition of $\mathbf{d}$.

1.3. **Root systems and Kac' theorem.** In this subsection we forget the orientation of Q, we still denote the set of vertices by Q_0 and the set of edges by Q_1. We denote the Tits form of Q by $q(\mathbf{d})$, given by $q(\mathbf{d})=\sum_{i\in Q_0}d_i^2-\sum_{\alpha\in Q_1}d_{s(\alpha)}d_{t(\alpha)}$. By $<,>$ we denote the Euler form of Q defined by $<\mathbf{d},\mathbf{c}>=\sum_{i\in Q_0}d_ic_i-\sum_{\alpha\in Q_1}d_{s(\alpha)}c_{t(\alpha)}$ and by $(,)$ we denote the symmetric bilinear form of Q defined by $(\mathbf{d},\mathbf{c})=<\mathbf{d},\mathbf{c}>+<\mathbf{c},\mathbf{d}>$. Note that we have $q(\mathbf{d})=\frac{1}{2}(\mathbf{d},\mathbf{d})$.

Given a dimension vector $\mathbf{d}$, we denote by $\mathrm{supp}(\mathbf{d})$ the support of $\mathbf{d}$, which is the full subgraph of Q with the set of vertices $\mathrm{supp}(\mathbf{d})_0=\{i\in Q_0|d_i\neq 0\}$. We denote by $\mathbf{e}_i$ the simple root of Q associated to vertex i and we say that $\mathbf{e}_i$ is fundamental if there are no loops at i. Associated to each fundamental simple root $\mathbf{e}_i$, we define a fundamental reflection $\sigma_i(\mathbf{d})=\mathbf{d}-(\mathbf{d},\mathbf{e}_i)\mathbf{e}_i$. We denote by $\mathcal{W}$ the Weyl group of Q, generated by all the fundamental reflections. We say that two vectors $\mathbf{d}$ and $\mathbf{c}$ are $\mathcal{W}$-equivalent if there is an element $\sigma\in\mathcal{W}$ such that $\mathbf{d}=\sigma(\mathbf{c})$. Let $\mathcal{F}=\{\mathbf{d}\in\mathbb{N}^n|(\mathbf{d},\mathbf{e}_i)\leq 0 \text{ for any simple root } \mathbf{e}_i \text{ of } Q \text{ and } \mathrm{supp}(\mathbf{d}) \text{ is connected}\}$ be the fundamental set of Q.

We recall the definition of the root system of Q and two results on the root system [5], among them there is the well-known theorem on the relation between the indecomposable representations of quivers and the corresponding root systems. A vector $\mathbf{d}$ is called a real root of Q if $\mathbf{d}$ is $\mathcal{W}$-equivalent to a fundamental simple root. A vector $\mathbf{d}$ is called an imaginary root if $\mathbf{d}$ is $\mathcal{W}$-equivalent to a root in $\mathcal{F}$ or to the negative of a root in $\mathcal{F}$. A root $\mathbf{d}$ is always either positive, that is each $d_i\geq 0$ and $\mathbf{d}\neq 0$, or $\mathbf{d}$ is negative, that is $-\mathbf{d}$ is positive. Moreover, $\mathbf{d}$ is a positive root if and only if $-\mathbf{d}$ is a negative root. Note that the bilinear form $(,)$ of Q is $\mathcal{W}$-invariant. So we have $q(\mathbf{d})=1$ if $\mathbf{d}$ is a real root and $q(\mathbf{d})\leq 0$ if $\mathbf{d}$ is an imaginary root. A quiver Q is hyperbolic if it is wild and any proper sub-quiver of Q is a union of quivers of Dynkin and Euclidean type.

Proposition 1.1 (Kac [5])**.** *Let Q be a hyperbolic quiver. Then $\mathbf{d}\in\mathbb{N}^{Q_0}$ is a real root of Q if and only if $q(\mathbf{d})=1$.*

Theorem 1.2 (Kac [5])**.** *Let Q be a quiver and let $\mathbf{d}$ be a dimension vector.*

(1) *There is an indecomposable representation with dimension vector $\mathbf{d}$ if and only if $\mathbf{d}$ is a positive root.*
(2) *There is a unique indecomposable representation (up to isomorphism) with dimension vector $\mathbf{d}$ if and only if $\mathbf{d}$ is a positive real root.*
(3) *If $\mathbf{d}$ is a positive imaginary root then we need $1-q(\mathbf{d})$ parameters to parameterize the indecomposable representations with dimension vector $\mathbf{d}$.*

In the following, if we don't say otherwise, a root is always a positive root.

1.4. **Reflection functors and duality.** In this subsection we recall the definition of the BGP-reflection functors [1] defined on the category of representations $\mathrm{Rep}(Q)$ of a quiver Q. We denote by $\mathrm{Rep}(Q)\backslash\mathrm{add}S_i$ the subcategory containing the representations without S_i as a direct summand, where S_i is the simple representation associated to vertex i.

A vertex $i\in Q_0$ is called a sink if there is no arrows starting at i and it is called a source if there is no arrows ending at j.

Let i be a source vertex and let $\{\alpha_{i_j}: i\longrightarrow i_j\}$ be all the arrows starting at i. We reverse all the arrows α_{i_j} and so i becomes a sink vertex. We denote the new quiver by Q' and the simple representation associated to vertex i by S'_i. We have a functor

$$F_i : \mathrm{Rep}(Q)\backslash \mathrm{add}S_i \longrightarrow \mathrm{Rep}(Q')\backslash \mathrm{add}S'_i$$

corresponding to the reflection σ_i by $F(\{M_j\}_j, \{M_\alpha\}_\alpha) = (\{M'_j\}_j, \{M'_\alpha\}_\alpha)$, where

$$M'_j = \begin{cases} M_j & \text{if } j \neq i, \\ \mathrm{cok}(M_i \xrightarrow{(M_{\alpha_{i_j}})_j} \oplus_j M_{i_j}) & \text{if } j = i. \end{cases}$$

and $M'_\alpha = M_\alpha$ if $\alpha \notin \{\alpha_{i_j}\}_j$ and $M'_{\alpha_{i_l}}$ is the composition of the natural embedding of M_{i_l} into $\oplus_j M_{i_j}$ with the projection from $\oplus_j M_{i_j}$ to M'_i. Note that $\underline{\dim} F_i M = \sigma_i \underline{\dim} M$ for $M \in \mathrm{Rep}(Q)\backslash \mathrm{add}S_i$.

Dually we have a reflection functor $F^i : \mathrm{Rep}(Q)\backslash \mathrm{add}S_i \longrightarrow \mathrm{Rep}(Q')\backslash \mathrm{add}S'_i$ if i is a sink vertex.

Proposition 1.3 (BGP [1]). *The functors F_i and F^i are equivalences.*

For a quiver Q, denote by Q^{op} the opposite quiver of Q. That is, Q^{op} is the quiver obtained by reversing the direction of every arrow in Q. Let $D : \mathrm{Rep}(Q, \mathbf{d}) \longrightarrow \mathrm{Rep}(Q^{\mathrm{op}}, \mathbf{d})$ be the duality.

2. Indecomposables for the subquiver of type $\widetilde{\mathbb{A}}_2$

We will now fix a set of representatives for the indecomposable representations for the positive real roots of the subquiver of type $\widetilde{\mathbb{A}}_1$ of

$$\begin{tikzcd} \mathcal{Q} : 1 \arrow[r, "\alpha"] & 2 \arrow[r, bend left, "\beta"] & 3 \arrow[l, bend left, "\gamma"] \end{tikzcd}$$

and discuss homomorphisms between them.

Recall that the set of real roots for $\widetilde{\mathbb{A}}_1$ is $\{(a, a+1), (a+1, a) | a \geq 0\}$. Denote by Y^a the indecomposable representation of $\mathcal{Q}$ with $\underline{\dim} Y^a = (0, a, a+1)$, $(Y^a)_\gamma = \begin{pmatrix} I_a & 0 \end{pmatrix}$ and $(Y^a)_\beta = \begin{pmatrix} 0 & I_a \end{pmatrix}^{\mathrm{tr}}$, where I_a is the $a \times a$ identity matrix. Denote by Y_a the indecomposable representation of $\mathcal{Q}$ with $\underline{\dim} Y_a = (0, a, a-1)$, $(Y_a)_\beta = \begin{pmatrix} I_{a-1} & 0 \end{pmatrix}$ and $(Y_a)_\gamma = \begin{pmatrix} 0 & I_{a-1} \end{pmatrix}^{\mathrm{tr}}$. Denote by $M(\lambda_1, \ldots, \lambda_a)$ the $a \times a$ matrix $(m_{ij})_{ij}$ with $m_{ij} = \lambda_l$ if $i - j + 1 = l$ and 0 else. The following lemma is easy.

Lemma 2.1. *Let $a > b$ be two natural numbers.*

(1) $\mathrm{End}(Y_a) = \{(0, M(\lambda_1, \ldots, \lambda_a), M(\lambda_1, \ldots, \lambda_{a-1})) | \lambda_i \in \mathbf{k}\}$.

(2) $\mathrm{Hom}(Y_a, Y_b) = \{(0, \begin{pmatrix} M(\lambda_1, \ldots, \lambda_b) & 0 \end{pmatrix}, \begin{pmatrix} M(\lambda_1, \ldots, \lambda_{b-1}) & 0 \end{pmatrix}) | \lambda_i \in \mathbf{k}\}$.

(3) $\mathrm{Hom}(Y_b, Y_a) = \{(0, \begin{pmatrix} 0 \\ M(\lambda_1, \ldots, \lambda_b) \end{pmatrix}, \begin{pmatrix} 0 \\ M(\lambda_1, \ldots, \lambda_{b-1}) \end{pmatrix}) | \lambda_i \in \mathbf{k}\}$.

Dually we have

Lemma 2.2. *Let $a > b$ be two natural numbers.*

(1) $\mathrm{End}(Y^a) = \{(0, M(\lambda_1, \ldots, \lambda_a), M(\lambda_1, \ldots, \lambda_{a+1})) | \lambda_i \in \mathbf{k}\}$.

(2) $\mathrm{Hom}(Y^a, Y^b) = \{(0, \begin{pmatrix} M(\lambda_1, \ldots, \lambda_b) & 0 \end{pmatrix}, \begin{pmatrix} M(\lambda_1, \ldots, \lambda_{b+1}) & 0 \end{pmatrix}) | \lambda_i \in \mathbf{k}\}$.

(3) $\mathrm{Hom}(Y^b, Y^a) = \{(0, \begin{pmatrix} 0 \\ M(\lambda_1, \ldots, \lambda_b) \end{pmatrix}, \begin{pmatrix} 0 \\ M(\lambda_1, \ldots, \lambda_{b+1}) \end{pmatrix}) | \lambda_i \in \mathbf{k}\}$.

3. Real roots of $\mathcal{Q}$

We compute the real roots for the quiver $\mathcal{Q}$.

Proposition 3.1. *The set $\{(n, \frac{m^2-1}{n} + n, \frac{m^2-1}{n} + n \pm m) | n \geq 1, m \geq 0, m^2 \equiv 1 (\mathrm{mod} n)\}$ is the set of all real roots $\mathbf{d} = (d_1, d_2, d_3)$ of $\mathcal{Q}$ with $d_1 \neq 0$. All other real roots of $\mathcal{Q}$ are real roots of its subquiver of type $\widetilde{\mathbb{A}}_2$.*

Proof. By Proposition 1.1 we know that a dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$ is a real root if and only if $q(\mathbf{d}) = 1$. So by calculating we see that the dimension vectors described in the statement are real roots. For the converse, assume we have $q(\mathbf{d}) = d_1^2 - d_1 d_2 + (d_2 - d_3)^2 = 1$. Thus $d_3 = d_2 \pm \sqrt{d_1 d_2 - d_1^2 + 1}$. Now let $n = d_1$ and $m = \sqrt{d_1 d_2 - d_1^2 + 1}$. Since d_3 is an integer, m is also an integer. If $d_1 = 0$ then $d_2 = d_3 \pm 1$ and so $\mathbf{d}$ is a real root of the subquiver of type $\widetilde{\mathbb{A}}_1$. Otherwise $d_2 = \frac{m^2-1}{n} + n$. So the real roots of $\mathcal{Q}$ are exactly those described in the statement. This finishes the proof. □

We let

$$\mathbf{d}_{n,m} = (n, \frac{m^2-1}{n} + n, \frac{m^2-1}{n} + n - m)$$

and

$$\mathbf{d}_n^m = (n, \frac{m^2-1}{n} + n, \frac{m^2-1}{n} + n + m)$$

denote real roots of $\mathcal{Q}$. We write $m = ns + r$, where $s, r \in \mathbb{Z}$ and $0 \le r < n$. To understand our formulas for the real roots of $\mathcal{Q}$, we discuss when $r^2 - 1$ is divisible by n.

Lemma 3.2. *Suppose $n = 2^a$ with $a \ge 1$.*

(1) *If $a = 1$, then $r = 1$.*
(2) *If $a = 2$, then $r = 1$ or 3.*
(3) *If $a \ge 3$, then $r = 1, 2^{a-1} - 1, 2^{a-1} + 1$ or $2^a - 1$.*

Proof. We prove (3). By the assumption we know that $r = 2b + 1$, where $b \ge 0$. Then $b(b+1)$ is divisible by 2^{a-2}. So we have $b = x2^{a-2}$, or $b = y2^{a-2} - 1$. Then the possible x is $0, 1$ and the possible y is $1, 2$. That is we have $r = 1, 2^{a-1} - 1, 2^{a-1} + 1$ or $2^a - 1$. □

Lemma 3.3. *Suppose that $n = p^a$ with $p \ne 2$ a prime number, then the possible r is 1 or $n - 1$.*

Proof. Since that the only possible prime common divisor of $r - 1$ and $r + 1$ is 2. So we have either $r - 1$ or $r + 1$ is divisble by n and so r is either 1 or $n - 1$. □

Using the prime factorization of n, the Chinese remainder theorem and the previous two lemmas we can compute all the remainders r with the property that $r^2 - 1$ divisible by n. It is difficult to write down a general formula, but we illustrate by computing the number of such r.

Proposition 3.4. *Let $n = p_1^{\mu_1} \dots p_\lambda^{\mu_\lambda} > 1$, where p_i are pairwise different prime numbers and each exponent μ_i is positive. By R we denote the number of r with $0 < r < n$ and $r^2 - 1$ divisible by n.*

(1) *One of p_i is 2. We may suppose that $p_1 = 2$.*
   i) *If $\mu_1 = 1$, then $R = 2^{\lambda-1}$.*
   ii) *If $\mu_1 = 2$, then $R = 2^\lambda$.*
   iii) *If $\mu_1 > 2$, then $R = 2^{\lambda+1}$.*
(2) *None of p_i is 2, then $R = 2^\lambda$.*

Proof. We prove (1)(i), the other cases can be done similarly. By the assumption we know that r must be odd and so we may assume that $r = 2b + 1$. Suppose that $\mu_1 = 1$. Then $2b(b+1)$ is divisible by $p_2^{\mu_2} \cdots p_\lambda^{\mu_\lambda}$. So we have that b is divisible by $p_{i_1}^{\mu_{i_1}} \cdots p_{i_\sigma}^{\mu_{i_\sigma}}$ and $b+1$ is divisible by $p_{j_1}^{\mu_{j_1}} \cdots p_{j_\varrho}^{\mu_{j_\varrho}}$, where $\{p_{i_l}\}_l \cup \{p_{j_l}\}_l = \{p_2, \dots, p_\lambda\}$ is a disjoint union and $\sigma, \varrho \ge 0$. By the Chinese Remainder Theorem we know that there exists a unique number N such that $0 < N < p_2^{\mu_2} \cdots p_\lambda^{\mu_\lambda}$ and $N \equiv 0 \pmod{p_{i_1}^{\mu_{i_1}} \cdots p_{i_\sigma}^{\mu_{i_\sigma}}}$ and $N \equiv -1 \pmod{p_{j_1}^{\mu_{j_1}} \cdots p_{j_\varrho}^{\mu_{j_\varrho}}}$. There are $\begin{pmatrix} \lambda - 1 \\ 0 \end{pmatrix} + \cdots + \begin{pmatrix} \lambda - 1 \\ \lambda - 1 \end{pmatrix}$ possible choices of such $\{p_{i_l}\}_l$ and $\{p_{j_l}\}_l$ with $\sigma, \varrho \ge 0$ and the N obtained in this way are all pairwise different. So in all there are $2^{\lambda-1}$ choices for r. □

4. Three operations

In this section we define three operations Σ_1, Σ_2 and Σ_3, corresponding to the three fundamental reflections in the Weyl group. Using these three operations we will be able to construct an indecomposable representation of $\mathcal{Q}$, for any real root, and compute the dimension of its endomorphism ring.

Let M be a representation of $\mathcal{Q}$. We denote by $\mathrm{Kr}(M)$ the restriction of M to the $\widetilde{\mathbb{A}}_1$ subquiver of $\mathcal{Q}$. That is, we have $\underline{\dim}(\mathrm{Kr}(M)) = (0, \dim M_2, \dim M_3), \mathrm{Kr}(M)_\beta = M_\beta$ and $\mathrm{Kr}(M)_\gamma = M_\gamma$. Let $n > 0$ and $m \geq 0$ be integers such that $m^2 - 1$ is divisible by n. Note that if $M \in \mathrm{Rep}(Q, \mathbf{d}_{n,m})$ and $\mathrm{Kr}(M) = \oplus_{i=1}^t Y_{a_i}$, then $t = m$ and $\Sigma_i a_i = (\mathbf{d}_n^m)_2$. Let

$$V_{n,m} = \{M \in \mathrm{Rep}(Q, \mathbf{d}_{n,m}) | \mathrm{Kr}(M) = \oplus_{j=1}^m Y_{a_j}, \Sigma_j a_j = (\mathbf{d}_{n,m})_2, a_j \geq 1, M_\alpha \text{ is injective}\}$$

and let

$$V_n^m = \{M \in \mathrm{Rep}(Q, \mathbf{d}_n^m) | \mathrm{Kr}(M) = \oplus_{j=1}^m Y^{a_j}, \Sigma_j a_j = (\mathbf{d}_n^m)_2, a_j \geq 1, M_\alpha \text{ is injective}\}$$

4.1. **Operation Σ_1.** Let $m > 0$. Our first operation

$$\Sigma_1 : V_{n,m} \longrightarrow V_{\frac{(m^2-1)}{n}, m}$$

is defined as follows. Let $M \in V_{n,m}$ with $\mathrm{Kr}(M) = \oplus_{j=1}^m Y_{a_j}$. Let I_a be the $a \times a$ identity matrix and let I^a be the $a \times a$ matrix with ones on the anti-diagonal and zero elsewhere. Let $g = (I_n, \oplus_{i=1}^m I^{a_i}, \oplus_{i=1}^m I^{a_i - 1})$.

Now let Σ_1 be given by

$$\Sigma_1 M = \Gamma D F_1 g M,$$

where Γ is induced by the graph isomorphism $\beta \mapsto \gamma$ and $\gamma \mapsto \beta$, D is the duality and F_1 is the reflection functor at vertex 1.

That is, $\Sigma_1 M$ is given by $(\Sigma_1 M)_i = (F_1 g M)_i$ and $(\Sigma_1 M)_\alpha = ((F_1 g M)_\alpha)^{tr}$, $(\Sigma_1 M)_\beta = ((F_1 g M)_\gamma)^{tr} = M_\beta$ and $(\Sigma_1 M)_\gamma = ((F_1 g M)_\beta)^{tr} = M_\gamma$. In particular, $\mathrm{Kr}(\Sigma_1 M) = \mathrm{Kr}(M)$.

There is no canonical choice of a basis for $F_1 g M$, but we shall see in Section 6 that we may choose it in a minimal way, and independently of the ground field.

Proposition 4.1. *Let $M \in V_{n,m}$ be indecomposable. Then $\Sigma_1(M)$ is indecomposable and* $\dim \mathrm{End}(M) = \dim \mathrm{End}(\Sigma_1 M)$.

Proof. The proof follows Proposition 1.3. □

4.2. **Operation Σ_3.** Let $M \in V_{n,m}$. Assume that $Kr(M) = \oplus_{i=1}^m Y_{a_i}$. The operation

$$\Sigma_3 : V_{n,m} \longrightarrow V_n^m$$

is given by $\mathrm{Kr}(\Sigma_3 M) = \oplus_{i=1}^m Y^{a_i}$.

Proposition 4.2. *Let $M \in V_{n,m}$ be indecomposable. Then $\Sigma_3 M$ is indecomposable and* $\dim \mathrm{End}(\Sigma_3(M)) = \dim \mathrm{End}(M) + m^2$.

Proof. We denote by Ψ_{a_i, a_j} the surjective linear map

$$\Psi_{a_i, a_j} : \mathrm{Hom}(Y^{a_i}, Y^{a_j}) \longrightarrow \mathrm{Hom}(Y_{a_i}, Y_{a_j})$$

induced by the isomorphisms $Y_{a_i} \cong \mathrm{rad} Y^{a_i} / \mathrm{soc} Y^{a_i}$.

Let $\Psi : \mathrm{End}(\oplus_{i=1}^m Y^{a_i}) \longrightarrow \mathrm{End}(\oplus_{i=1}^m Y_{a_i})$ be the surjective algebra homomorphism $\Psi = (\Psi_{a_i, a_j})_{i,j}$. Note that if $f = (f_1, f_2, f_3) \in \mathrm{End}(M)$, then $(0, f_2, f_3) \in \mathrm{End}(\mathrm{Kr}(M))$. So we get an induced map

$$\Phi : \mathrm{End}(\Sigma_3(M)) \longrightarrow \mathrm{End}(M)$$

given by $\Phi(f_1, f_2, f_3) = (f_1, f_2, f_3')$ where $(0, f_2, f_3') = \Psi(0, f_2, f_3)$. We can easily see that Φ is also surjective.

Now $(f_1, f_2, f_3) \in \ker \Phi$ if and only if $f_1 = 0$ and $(0, f_2, f_3) \in \ker \Psi$. For if $f' = (0, f_2, f_3) \in \ker \Psi$ then f' maps the top of $\oplus_i Y^{a_i}$ to the socle of $\oplus_i Y^{a_i}$ showing that $f_2 (\Sigma_3 M)_\alpha = 0$. Hence

$(0, f_2, f_3) \in \ker\Phi$. The other implication is trivial. Also $\ker\Phi^2 = 0$, hence $\mathrm{End}(\Sigma_3(M))$ is local if $\mathrm{End}(M)$ is local. This proves the first part.

We have $\dim\mathrm{End}(\Sigma_3(M)) = \dim\mathrm{End}(M) + \dim\ker\Phi$. Now the second part follows since $\dim\ker\Psi_{a_i,a_j} = 1$ and therefore $\dim\ker\Phi = \sum_{i,j}\dim\ker\Psi_{a_i,a_j} = m^2$. □

4.3. **Operation Σ_2.** We define the operation

$$\Sigma_2 : V_n^m \longrightarrow V_{n,n+m}$$

Let $M \in V_n^m$. We fix a basis $\mathbf{x}_1, \ldots, \mathbf{x}_n$ for the vector space $\mathbf{k}^n$ at vertex 1. We let $\mathbf{y}_{i,1}, \ldots, \mathbf{y}_{i,a_i}$ be a basis for $\mathbf{k}^{a_i} = (Y^{a_i})_2$ and let $\mathbf{y}_{i,0}, \mathbf{y}_{i,1}, \ldots, \mathbf{y}_{i,a_i}, \mathbf{y}_{i,a_i+1}$ be a basis for $\mathbf{k}^{a_i+2} = (Y_{a_i+2})_2$. The construction of $\Sigma_2 M$ is based on this fixed basis. By $\mathbf{e}_{a,i}$ we mean a $a \times 1$ matrix with 1 at the ith-row and zero elsewhere. Write

$$M_\alpha : \mathbf{k}^n \xrightarrow{(\phi_{ij})_{ij}} \oplus_{i=1}^m \mathbf{k}^{a_i} = \mathbf{k}^{(\mathbf{d}_n^m)_2},$$

where

$$\phi_{ij} = \begin{pmatrix} z_{ij,1} & \ldots & z_{ij,a_i} \end{pmatrix}^{\mathrm{tr}} : \mathbf{k}\mathbf{x}_j \longrightarrow \mathbf{k}^{a_i}$$

That is, $\phi_{ij} = \sum_{l=1}^{a_i} z_{ij,l}\mathbf{e}_{a_i,l}$. Let

$$\Phi_{ij} = \begin{pmatrix} 0 & z_{ij,1} & \ldots & z_{ij,a_i} & 0 \end{pmatrix}^{\mathrm{tr}} : \mathbf{k}\mathbf{x}_j \longrightarrow \mathbf{k}^{a_i+2},$$

that is $\Phi_{ij} = \sum_{l=1}^{a_i} z_{ij,l}\mathbf{e}_{a_i+2,l+1}$. The representation $\Sigma_2 M$ in $V_{n,m+n}$ is defined as

(1) $\mathrm{Kr}(\Sigma_2 M) = \oplus_{i=1}^m Y_{a_i+2} \oplus S_2^{\oplus n}$.

(2) $(\Sigma_2 M)_\alpha = \begin{pmatrix} (\Phi_{ij})_{ij} \\ I_n \end{pmatrix} : \mathbf{k}^n \longrightarrow \oplus_{i=1}^m \mathbf{k}^{a_i+2} \oplus \mathbf{k}^{\oplus n} = \mathbf{k}^{(\mathbf{d}_{n,n+m})_2}$.

Proposition 4.3. *Let $M \in V_n^m$ be indecomposable. Then $\Sigma_2(M)$ is indecomposable and $\dim\mathrm{End}(\Sigma_2(M)) = \dim\mathrm{End}(M) + m^2 + mn$.*

Proof. Let $M \in V_n^m$ be indecomposable. As in the proof of Proposition 4.2, the isomorphism

$$\mathrm{Kr}(M) \cong \mathrm{radKr}(\Sigma_2 M)/\mathrm{socKr}(\Sigma_2 M)$$

induce a surjective linear map $\Psi : \mathrm{End}(\mathrm{Kr}(\Sigma_2 M)) \longrightarrow \mathrm{End}(\mathrm{Kr}M)$. Suppose that $f = (f_1, f_2, f_3)$ is an endomorphism in $\mathrm{End}(\Sigma_2(M))$. Then we have (1) $(0, f_2, f_3) \in \mathrm{End}(\mathrm{Kr}(\Sigma_2 M))$ and (2) $f_2(\Sigma_2 M)_\alpha = (\Sigma_2 M)_\alpha f_1$, and so we get an induced linear map

$$\Phi : \mathrm{End}(\Sigma_2 M) \longrightarrow \mathrm{End}(M)$$

given by $\Phi(f_1, f_2, f_3) = (f_1, f_2', f_3)$ where $(0, f_2', f_3) = \Psi(0, f_2, f_3)$.

We need to study Φ in more detail. We may without loss of generality assume that $\mathrm{Kr}(M) = \oplus_{i=1}^m Y^{a_i}$ with $a_1 \le a_2 \le \cdots \le a_m$.

By (1), following our choice of basis and Lemma 2.1 we can write

$$f_2 = \begin{pmatrix} (M_{ij})_{ij} & (G_{ij})_{ij} \\ & \\ (H_{ij})_{ij} & (c_{ij})_{ij} \end{pmatrix},$$

where $(M_{ij})_{ij}$ is an $m \times m$ matrix and each M_{ij} is of the form

$$M_{ij} = \begin{cases} M(\lambda_{ii,1}, \ldots, \lambda_{ii,a_i}) & \text{if } i = j, \\ \begin{pmatrix} M(\lambda_{ij,1}, \ldots, \lambda_{ij,a_i}) & 0 \end{pmatrix} & \text{if } i < j, \\ \begin{pmatrix} 0 \\ M(\lambda_{ij,1}, \ldots, \lambda_{ij,a_j}) \end{pmatrix} & \text{if } i > j, \end{cases}$$

$(G_{ij})_{ij}$ is $m \times n$ matrix over $\mathbf{k}$ with

$$G_{ij} = \begin{pmatrix} 0 & \ldots & 0 & \mu_{ij} \end{pmatrix}^{tr}$$

H_{ij} is an $n \times m$ matrix over $\mathbf{k}$ with

$$H_{ij} = \begin{pmatrix} \eta_{ij} & 0 & \dots & 0 \end{pmatrix},$$

and $(c_{ij})_{ij}$ is an $n \times n$ matrix over $\mathbf{k}$. Finally, we write $f_1 = (x_{ij})_{ij}$ as a $n \times n$ matrix over $\mathbf{k}$.

By (2) we have:

(I) $\sum_{l=1}^m M_{il}\Phi_{lj} + G_{ij} = \sum_{l=1}^n \Phi_{il}x_{lj}$.

(II) $\sum_{l=1}^m H_{il}\Phi_{lj} + c_{ij} = x_{ij}$.

Note that $H_{il}\Phi_{lj} = 0$ for any i, j and l, hence by (II) we have that $(c_{ij})_{ij} = f_1$. Dividing the matrices into blocks and calculating give us

$$\sum_{l=1}^m M_{il}\Phi_{lj} = \sum_{l=1}^m \begin{pmatrix} y_{il,1} & 0 & 0 \\ P_{il} & M'_{il} & 0 \\ y_{il,a_l+2} & Q_{il} & y_{il,1} \end{pmatrix} \begin{pmatrix} 0 \\ \phi_{lj} \\ 0 \end{pmatrix} = \sum_{l=1}^m \begin{pmatrix} 0 \\ M'_{il}\phi_{lj} \\ Q_{il}\phi_{lj} \end{pmatrix}$$

So by (I) we have

$$\begin{pmatrix} 0 \\ \\ \sum_{l=1}^m M'_{il}\phi_{lj} \\ \\ \sum_{l=1}^m Q_{il}\phi_{lj} \end{pmatrix} + G_{ij} = \begin{pmatrix} 0 \\ \\ \sum_{l=1}^n \phi_{il}x_{lj} \\ \\ 0 \end{pmatrix}$$

Therefore

$$\begin{cases} \sum_{l=1}^m M'_{il}\phi_{lj} = \sum_{l=1}^n \phi_{il}x_{lj}. \text{ That is, } (M'_{ij})_{ij}M_\alpha = M_\alpha f_1 \\ \\ \sum_{l=1}^m Q_{il}\phi_{lj} + \mu_{ij} = 0. \end{cases}$$

Hence $f = (f_1, f_2, f_3)$ is an endomorphism of $\Sigma_2 M$ if and only if $\Phi(f) = (f_1, (M'_{ij})_{ij}, f_3)$ is an endomorphism of M, (3) $\mu_{ij} = -\sum_{l=1}^m Q_{il}\phi_{lj}$ and $(c_{ij}) = f_1$. Now if $g = (g_1, g_2, g_3) \in \mathrm{End}(M)$ then

$$(g_1, \begin{pmatrix} g'' & (G_{ij})_{ij} \\ 0 & f_1 \end{pmatrix}, g_3) \in \Phi^{-1}(g),$$

where $(0, g'', g_3) \in \Psi^{-1}(0, g_2, g_3)$ and G_{ij} is determined by the equation (3). This shows that Φ is surjective.

Now if f is in the kernel of Φ then $f_1 = 0$ and therefore $(c_{ij}) = 0$, also $f_3 = 0$ and so $Q_{il} = 0$ and therefore $G_{ij} = 0$, and finally, $f'_2 = 0$. Thus the kernel of Φ are the maps $(0, f_2, 0)$ where

$$f_2 = \begin{pmatrix} (M_{ij})_{ij} & 0 \\ \\ (H_{ij})_{ij} & 0 \end{pmatrix}$$

and the image of M_{ij} is in the socle of Y^{a_j}. Clearly $\ker\Phi$ is nilpotent and so $\Sigma_2 M$ is indecomposable. Moreover the M_{ij} contribute m^2 basis elements to $\ker\Phi$ and the H_{ij} contribute mn basis elements. Therefore $\dim\mathrm{End}(\Sigma_3(M)) = \dim\mathrm{End}(M) + m^2 + mn$. This finishes the proof of Proposition 4.3. □

Remark 4.4. *It is possible to define inverse operations to Σ_2 and Σ_3. The operation Σ_1 is its own inverse. To make the definitions of our operations complete, we also define $\Sigma_3 Y_a = Y^a, \Sigma_2 Y^a = Y_{a+2}$ and we let $\Sigma_1 Y_a$ be given by $Kr\Sigma_1 Y_a = Y_a$ and $(\Sigma_1 Y_a)_\alpha = I_a$.*

5. Indecomposables and the dimensions of their endomorphism rings

In this section we give a concrete constuction of the indecomposable representations with dimension vector a real root of $\mathcal{Q}$, using the operations Σ_1, Σ_2 and Σ_3 defined in the previous section. We also give a formula for the dimension of the endomorphism ring of the indecomposables we construct. This gives an answer for $\mathcal{Q}$ to the question proposed by Crawley-Boevey, which we mentioned in the introduction. Moreover this formula is independent of the ground field. Before giving the general construction, let us first see an easy example. More examples on the construction will be given using pictures in the last section.

Example 5.1. *(1) We have* $\mathbf{d}_{1,m} = (1, m^2, m^2-m) = (\sigma_2\sigma_3)^{m-1}\sigma_1\mathbf{d}_{0,1}$, *where by* $\mathbf{d}_{0,1}$ *we mean the simple root* $(0,1,0)$. *Let* $M(\mathbf{d}_{1,m})$ *be a representation in* $V_{1,m}$ *with* $\mathrm{Kr}(M(\mathbf{d}_{1,m})) = \oplus_{i=1}^m Y_{2i-1}$ *and* $M(\mathbf{d}_{1,m})_\alpha = (\mathbf{e}_{2m-1,m}, \mathbf{e}_{2(m-1),m-1}, \ldots, \mathbf{e}_{3,2}, \mathbf{e}_{1,1})^{\mathrm{tr}}$. *In particular we have*

$$M(\mathbf{d}_{1,1}) = k \xrightarrow{1} k \underset{\longleftarrow}{\overset{\longrightarrow}{}} 0$$

Obviously $M(\mathbf{d}_{1,1})$ *is indecomposable with trivial endormorphism ring. By the definition of the operations* Σ_2 *and* Σ_3, *we have* $M(\mathbf{d}_{1,m}) = (\Sigma_2\Sigma_3)^{m-1}M(\mathbf{d}(1,1))$. *Therefore* $M(\mathbf{d}_{1,m})$ *is indecomposable. By Proposition 4.3 and Proposition 4.2 we have* $\dim \mathrm{End} M(\mathbf{d}_{1,m}) = \dim \mathrm{End} M(\mathbf{d}_{1,m-1}) + 2(m-1)^2 + (m-1)$ *and so inductively we have* $\dim \mathrm{End} M(\mathbf{d}_{1,m}) = \dim \mathrm{End} M(\mathbf{d}_{1,1}) + \sum_{i=1}^{m-1}(2i^2+i) = \frac{2m^3}{3} - \frac{m^2}{2} - \frac{m}{6} + 1$. *Let* $M(\mathbf{d}_1^m) = \Sigma_3(M(\mathbf{d}_{1,m}))$. *Then by Proposition 4.2* $M(\mathbf{d}_1^m)$ *is indecomposable and* $\dim \mathrm{End} M(\mathbf{d}_1^m) = \frac{2m^3}{3} + \frac{m^2}{2} - \frac{m}{6} + 1$.

The following two results are obvious.

Lemma 5.2. *Let* n *and* m *be two natural numbers and let* $m = ns + r$ *with* $s, r \in \mathbb{N}$ *and* $0 < r < n$.

1. $\sigma_1 \mathbf{d}_{n,r} = \mathbf{d}_{\frac{r^2-1}{n}, r}$.
2. $(\sigma_3\sigma_2)^s \mathbf{d}_{n,m} = \mathbf{d}_{n,r}$ *for* $n > 1$.
3. $(\sigma_3\sigma_2)^{s-1}\mathbf{d}_{1,m} = \mathbf{d}_{1,1}$.
4. $\sigma_3(\mathbf{d}_n^m) = \mathbf{d}_{n,m}$.

Proposition 5.3. *Let* n, m *be two natural numbers.*

1. *If* $\frac{m^2-1}{n} + n - m$ *is even, then the real roots* $\mathbf{d}_{n,m}$ *and* $\mathbf{d}_n^m$ *are* $\mathcal{W}$*-equivalent to* $\mathbf{e}_2$.
2. *If* $\frac{m^2-1}{n} + n - m$ *is odd, then the real roots* $\mathbf{d}_{n,m}$ *and* $\mathbf{d}_n^m$ *are* $\mathcal{W}$*-equivalent to* $\mathbf{e}_3$.
3. *The two simple roots* $\mathbf{e}_1$ *and* $\mathbf{e}_2$ *are* $\mathcal{W}$*-equivalent.*

We now start our general construction of indecomposable representations for real roots and calculate the dimension of their endomorphism rings.

Theorem 5.4. *Let* $\mathbf{d}_{n,m}$ *be a real root of* $\mathcal{Q}$. *Suppose that* $m = ns+r$ *with* $r, s \in \mathbb{N}$, $0 < r < n$ *and* $n > 1$. *Let* N *be an indecomposable representation in* $V_{\frac{r^2-1}{n}, r} \subseteq \mathrm{Rep}(\mathcal{Q}, \mathbf{d}_{\frac{r^2-1}{n}, r})$ *and let* $M = (\Sigma_2\Sigma_3)^s\Sigma_1(N)$. *Then*

1. M *is an indecomposable representation in* $V_{n,m} \subseteq \mathrm{Rep}(\mathcal{Q}, \mathbf{d}_{n,m})$.
2. $\dim \mathrm{End} M = \dim \mathrm{End} N + \frac{2n^2}{3}s^3 + (2rn - \frac{n^2}{2})s^2 + (2r^2 - rn - \frac{n^2}{6})s$.
3. $\Sigma_3 M$ *is an indecomposable representation in* $V_n^m \subseteq \mathrm{Rep}(\mathcal{Q}, \mathbf{d}_n^m)$.
4. $\dim \mathrm{End}\Sigma_3 M = \dim \mathrm{End} N + \frac{2n^2}{3}s^3 + (2rn + \frac{n^2}{2})s^2 + (2r^2 + rn - \frac{n^2}{6})s + r^2$

Proof. We prove (1) by induction on s. If $s = 0$, then $(\Sigma_2\Sigma_3)^s\Sigma_1 = \Sigma_1$. So it follows from Proposition 4.1. Now assume that $s > 0$. Let M be an indecomposable representation in $V_{n,m'} \subseteq \mathrm{Rep}(\mathcal{Q}, \mathbf{d}_{n,m'})$, where $m' = n(s-1) + r$. By Propositions 4.2 and 4.3 we have $\Sigma_3(M) \in V_n^{m'}$ and so we can apply Σ_2 on $\Sigma_3(M)$. Moreover $\Sigma_2\Sigma_3(M)$ is indecomposable. Now the result follows from induction hypothesis.

We now prove (2). By Propositions 4.1, 4.3 and 4.2 we have $\dim \mathrm{End}(\Sigma_2\Sigma_3)\Sigma_1(N) = \dim \mathrm{End}N + r^2 + (r^2+nr)$ and $(\Sigma_2\Sigma_3)\Sigma_1(N) \in V_{n,r+n} \subseteq \mathrm{Rep}(\mathcal{Q}, \mathbf{d}_{n,r+m})$. So we have

$$\dim \mathrm{End}M = \dim \mathrm{End}N + \sum_{i=0}^{s-1}(2(r+ni)^2 + n(r+ni))$$
$$= \dim \mathrm{End}N + \frac{2n^2}{3}s^3 + (2rn - \frac{n^2}{2})s^2 + (2r^2 - rn - \frac{n^2}{6})s.$$

The proof of (3) and (4) follows from Proposition 4.2.

□

Corollary 5.5. *Let M be an indecomposable representation in $V_{n,m} \subseteq \mathrm{Rep}(\mathcal{Q}, \mathbf{d}_{n,m})$. Let $N = (\Sigma_2\Sigma_3)^s M$. Then $\dim \mathrm{End}N = \frac{2n^2}{3}s^3 + o(s^3)$ for $s \gg 0$.*

Let $\mathbf{d}_{n,m}$ be a real root with $n > 0$. Denote $n_0 = n, m_0 = m = n_0 s_0 + r_0$, where $n_0, s_0 \in \mathbb{N}$ and $0 \le r_0 < n_0$. Define $\tau_1 = \begin{cases} \sigma_1(\sigma_3\sigma_2)^{s_0-1} & \text{if } n_0 = 1 \\ \sigma_1(\sigma_3\sigma_2)^{s_0} & \text{if } n_0 > 1 \end{cases}$. So we have $\tau_1(\mathbf{d}_{n,m}) = \begin{cases} \mathbf{d}_{0,1} = (0,1,0) & \text{if } n_0 = 1 \\ \mathbf{d}_{\frac{r^2-1}{n}, r} & \text{if } n_0 > 1 \end{cases}$. We define recursively a sequence of real roots $\{\mathbf{d}_{n_i,m_i}\}_i$ and a sequence of reflections $\{\tau_i\}_i$. Write $m_{i-1} = n_{i-1}s_{i-1} + r_{i-1}$ with $s_{i-1}, r_{i-1} \in \mathbb{N}$ and $0 \le r_{i-1} < n_{i-1}$. Let $\tau_i = \begin{cases} \sigma_1(\sigma_3\sigma_2)^{s_{i-1}-1} & \text{if } n_{i-1} = 1 \\ \sigma_1(\sigma_3\sigma_2)^{s_{i-1}} & \text{if } n_{i-1} > 1 \end{cases}$. Then $\mathbf{d}_{n_i,m_i} = \tau_i(\mathbf{d}_{n_{i-1},m_{i-1}})$. That is we have $n_i = \frac{r_{i-1}^2 - 1}{n_{i-1}}$ and $m_i = r_{i-1}$ if $n_{i-1} \neq 1$; and $n_i = 0$ and $m_i = 1$ if $n_{i-1} = 1$. This sequence $\{\mathbf{d}_{n_i,m_i}\}_i$ stops when the first $n_i = 0$ appears. Note that $n_i = 0$ if and only if $\sigma_1\tau_i\mathbf{d}_{n_{i-1},m_{i-1}} = (n_{i-1}, n_{i-1}, n_{i-1}-1)$, that is $r_{i-1} = 1$, so in this case $\mathbf{d}_{n_i,m_i} = \mathbf{d}_{0,m_i} = (0, n_{i-1}, n_{i-1}-1)$. Inductively we have the following proposition.

Proposition 5.6. *Let $\mathbf{d}_{n,m}$ be a real root and let $\{\tau_i\}_i$ and $\{\mathbf{d}_{n_i,m_i}\}_i$ be defined as above. Then*

(1) *$0 \le n_i < n_{i-1}$.*
(2) *There exists t such that $n_t = 0$ and $m_t = n_{t-1}$. That is $d_{n_t,m_t} = (0, n_{t-1}, n_{t-1}-1)$.*

Given a reflection $\tau = \sigma_1(\sigma_3\sigma_2)^s$, by $O(\tau)$ we mean the operation $(\Sigma_2\Sigma_3)^s\Sigma_1$.

Theorem 5.7. *Let $\mathbf{d}_{n,m}$ be a real root of $\mathcal{Q}$. Let $\tau_1, \ldots, \tau_t$ be reflections as above such that $\tau_t \ldots \tau_1(\mathbf{d}_{n,m}) = (0, n_{t-1}, n_{t-1}-1)$. Then*

(1) *$M = O(\tau_1)\ldots O(\tau_t)(Y_{n_{t-1}})$ is an indecomposable representation in $V_{n,m} \subseteq \mathrm{Rep}(Q, \mathbf{d}_{n,m})$.*
(2) *$\dim \mathrm{End}M = n_{t-1} + \sum_{i=0}^{t-1}(\frac{2n_i^2}{3}s_i^3 + (2m_in_i - \frac{n_i^2}{2})s_i^2 + (2m_i^2 - m_in_i - \frac{n_i^2}{6})s_i)$.*
(3) *$\Sigma_3(M)$ is an indecomposable representation in $V_n^m \subseteq \mathrm{Rep}(Q, \mathbf{d}_n^m)$.*
(4) *$\dim \mathrm{End}\Sigma_3 M = m^2 + n_{t-1} + \sum_{i=0}^{t-1}(\frac{2n_i^2}{3}s_i^3 + (2m_in_i - \frac{n_i^2}{2})s_i^2 + (2m_i^2 - m_in_i - \frac{n_i^2}{6})s_i)$.*

Proof. The proof follows from Theorem 5.4. □

6. Tree representations and [0, 1]-matrices

Using results of Schofield in [11], Ringel has proven that exceptional representations are tree representations in [8]. In this section we prove directly that the indecomposable representations of $\mathcal{Q}$ with dimension vector a real root are tree representations. From Section 5 we see that the exceptional indecomposable representations for $\mathcal{Q}$ have respectively dimension vector $(1,0,0), (0,1,0), (0,0,1)$ and $(1,1,0)$. All other indecomposable representations are not exceptional.

It is known that any tree representation is isomorphic to a tree representation with only $[0,1]$-matrices (see [8]). So in particular we can use $[0,1]$-matrices for all indecomposable representations for $\mathcal{Q}$ with dimension vector a real root. Since they are tree representations, these are also matrices with a minimal number of non-zero entries. Our construction is independent of the characteristic of the ground field.

We recall some definitions from [8]. Let M be a representation in $\mathrm{Rep}(Q, \mathbf{d})$. We denote by $\mathcal{B}_i$ a fixed basis of the vector space M_i at vertex i and $\mathcal{B} = \cup_i \mathcal{B}_i$. The coefficient quiver $\Gamma_{M,\mathcal{B}}$ of M with respect to $\mathcal{B}$ has the basis elements of $\mathcal{B}$ as its set of vertices, and there is an arrow from $b \in \mathcal{B}_i$ to $b' \in \mathcal{B}_j$ if $M_\alpha(b, c) \neq 0$, where $M_\alpha(b, c)$ is the element at the intersection of the bth row and the cth column in M_α and $\alpha : i \to j \in Q_1$. A representation of Q over $\mathbf{k}$ is called a tree representation if there exists a basis $\mathcal{B}$ of M such that the coefficient quiver $\Gamma(M, \mathcal{B})$ is a tree.

We say that a vertex i is of degree x, denoted by $\deg(i) = x$, if there are x arrows incident to i. We first consider tree representations M of

$$\mathbb{A}_2 : 1 \xrightarrow{\alpha} 2$$

with $\Gamma(M, \mathcal{B})$ satisfying the following:

(1) If $\mathbf{x}_i \in \mathcal{B}_1$ then $\deg(\mathbf{x}_i) > 1$.
(2) If $\mathbf{y}_i \in \mathcal{B}_2$ then $\deg(\mathbf{y}_i) \leq 2$.
(3) If $\mathbf{x}_i \in \mathcal{B}_1$ then $\mathcal{N}_{\mathbf{x}_i} = \{\mathbf{y}_j | M_\alpha(\mathbf{x}_i, \mathbf{y}_j) \neq 0\}$ contains at most two elements of degree 2.

Denote by $\mathcal{A}$ the set of all the coefficient quivers of representations of $\mathbb{A}_2$, and which satisfy (1), (2) and (3) above.

Lemma 6.1. *Let M be a representation with $\Gamma(M, \mathcal{B})$ in $\mathcal{A}$. Then there exists an $\mathbf{x}_i \in \mathcal{B}_1$ such that $\mathcal{N}_{\mathbf{x}_i}$ contains at most one element with degree two. If $|\mathcal{B}_1| > 1$, then there exists an $\mathbf{x}_i$ in $\Gamma(M, \mathcal{B})$ such that $\mathcal{N}_{\mathbf{x}_i}$ contains exactly one element with degree two.*

Proof. Let

$$\rho = (\rho_0 \cdots \to \mathbf{y}_{j_{i-1}} \leftarrow \mathbf{x}_{j_i} \to \mathbf{y}_{j_{i+1}} \leftarrow \dots \rho_1)$$

be a (non-oriented) path in $\Gamma(M, \mathcal{B})$ of maximal length with no repeated vertices. The length of ρ is at least two, where length means the number of arrows in ρ.

Now $\deg(\rho_0) = 1$, otherwise ρ is not of maximal length. So $\rho_0 \in \mathcal{B}_2$ and we write $\rho = \rho_0 \leftarrow \mathbf{x}_{i_t} \to \mathbf{y}_{i_{t+1}} \dots$. If $\mathcal{N}_{\mathbf{x}_{i_t}}$ contains at most one vertex of degree two, then we are done. If not, then there is a vertex of degree two $\mathbf{y} \in \mathcal{N}_{\mathbf{x}_{i_t}}$ which is different from $\mathbf{y}_{i_{t+1}}$. We can make a new path

$$\rho' = \mathbf{y} \leftarrow \mathbf{x}_{i_t} \to \mathbf{y}_{i_{t+1}} \cdots$$

which has the same length as ρ, but which is not maximal. This is a contradiction, and so we are done. $\square$

Let $\mathcal{S}$ be a subset of $\mathcal{B}_2$ containing all the vertices of degree two and some vertices of degree one such that each $\mathbf{x}_i$ in $\mathcal{B}_1$ is connected exactly to two elements in $\mathcal{S}$. Since we only consider those trees in $\mathcal{A}$, such an $\mathcal{S}$ exists. We call the vertices contained in $\mathcal{S}$ simple vertices. Note that since $\Gamma(M, \mathcal{B})$ is a tree and $\deg(\mathbf{y}_i) \leq 2$ for any $\mathbf{y}_i \in \mathcal{B}_2$, there are exactly $a-1$ elements in $\mathcal{B}_2$ with degree two, where $a = |\mathcal{B}_1|$. Therefore the set $\mathcal{S}$ contains the $a-1$ elements of degree two and two elements of degree one.

Proposition 6.2. *Let M be a representation of $\mathbb{A}_2$ with $\Gamma(M, \mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2)$ in $\mathcal{A}$. Let $\mathcal{S}$ be a set of simple vertices of $\Gamma(M, \mathcal{B})$. There exists a basis ${\mathcal{B}'}_1$ of $\mathrm{Cok}(M_\alpha)$, such that*

(1) *The degree of $\mathbf{y} \in \mathcal{B}_2$ is one in $\Gamma(DF_1M, \mathcal{B}' = \mathcal{B}'_1 \cup \mathcal{B}_2)$ if and only if $\mathbf{y} \in \mathcal{S}$.*
(2) *Suppose $\mathbf{y} \in \mathcal{S}$ is of degree one in $\Gamma(M, \mathcal{B})$ and in $\Gamma(DF_1M, \mathcal{B}')$, the vertex $\mathbf{y}$ is contained in $\mathcal{N}_{\mathbf{x}}$ for some vertex $\mathbf{x}$ in $\mathcal{B}'_1$. Then $\mathcal{N}_{\mathbf{x}}$ contains at most one element with degree two in $\Gamma(DF_1M, \mathcal{B}')$.*
(3) *For any $\mathbf{x}' \in \mathcal{B}'_1$, the set $\mathcal{N}_{\mathbf{x}'} \subseteq \Gamma(DF_1M, \mathcal{B}')$ contains at most one simple vertex in $\mathcal{S}$ with degree one, except when $|\mathcal{B}'_1| = 1$.*
(4) *$\Gamma(DF_1M, \mathcal{B}'_1 \cup \mathcal{B}_2)$ is a tree in $\mathcal{A}$.*

Proof. By changing basis we may assume that M is given by $[0,1]$-matrices, with the same coefficient quiver. Set $\mathcal{B}_1 = \{\mathbf{x}_1, \ldots, \mathbf{x}_a\}$ and $\mathcal{B}_2 = \{\mathbf{y}_1, \ldots, \mathbf{y}_b\}$. We prove this proposition by induction on a. We first consider the case $a = 1$. We may assume that $M_\alpha = \begin{pmatrix} 1 & \ldots & 1 \end{pmatrix}^{\mathrm{tr}}$. Thus we have a tree representation in $\mathrm{Rep}(\mathbb{A}_2, (1, b))$ with $\deg \mathbf{y}_i = 1$ for any $\mathbf{y}_i$ in $|\mathcal{B}_2|$. We may assume that $\mathcal{S} = \{\mathbf{y}_1, \mathbf{y}_b\}$. Choose $\mathcal{B}'_1 = \{\mathbf{y}_1 - \mathbf{y}_2, \mathbf{y}_2 - \mathbf{y}_3, \ldots, \mathbf{y}_{b-1} - \mathbf{y}_b\}$ as a basis of $\mathrm{Cok} M_\alpha$. It is clear that DF_1M is a tree representation in $\mathcal{A}$ and it satisfies (1)-(3) in the statement.

Now suppose that $a > 1$. By Lemma 6.1, there is an $\mathbf{x}_{i_0}$ in $\mathcal{B}_1$ with $\mathcal{N}_{\mathbf{x}_{i_0}}$ containing exactly one element of degree two. Assume that $\mathcal{N}_{\mathbf{x}_{i_0}} = \{\mathbf{y}_{j_0}, \mathbf{y}_{j_1}, \ldots, \mathbf{y}_{j_s}\}$ with $\deg(\mathbf{y}_{j_0}) = 2$ and $\deg(\mathbf{y}_{j_l}) = 1$ for $l > 0$. Since $\deg \mathbf{x}_{i_0}$ is bigger than one, we have $s > 0$. By the definition of $\mathcal{S}$, we may suppose that $\mathbf{y}_{j_s}$ is one of the two elements of degree one in $\mathcal{S}$. Consider the full subtree T of $\Gamma(M, \mathcal{B})$ with the set of vertices $T_0 = \mathcal{B} \backslash \{\mathbf{x}_{i_0}, \mathbf{y}_{j_1}, \ldots, \mathbf{y}_{j_s}\}$. Note that it is still a tree in $\mathcal{A}$. We denote the corresponding representation by N. Let $\mathcal{S}' = \mathcal{S} \backslash \{\mathbf{y}_{j_s}\}$. Then $\mathcal{S}'$ is a set of simple vertices for T and $\mathbf{y}_{j_0}$ has degree one in T.

By the induction, we can choose a basis $\mathcal{B}'_1 = \{\mathbf{x}'_i\}_{i=1}^{b-s-(a-1)}$ for $\mathrm{Cok}(N_\alpha)$ satisfying (1)-(3) in the statement. In particular $\mathbf{y}_{j_0}$ is of degree 1 in $\Gamma(DF_1N, \mathcal{B}'_1 \cup \mathcal{B}'_2)$, where $\mathcal{B}'_2 = \mathcal{B}_2 \backslash \{\mathbf{y}_{j_l}\}_{l>0}$, and $\mathbf{y}_{j_0} \in \mathcal{N}_{\mathbf{x}'_{l_0}}$ with $\mathcal{N}_{\mathbf{x}'_{l_0}}$ containing at most one element of degree two in $\Gamma(DF_1N, \mathcal{B}'_1 \cup \mathcal{B}'_2)$. We write $\mathbf{x}'_i = \sum_{l=1}^{b} a_{il}\mathbf{y}_l$. We construct a basis $\mathcal{B}''_1 = \{\mathbf{x}''_i\}_{i=1}^{b-a}$ for $\mathrm{Cok}(M_\alpha)$ as

$$\mathbf{x}''_l = \begin{cases} \mathbf{y}_{i_l} - \mathbf{y}_{i_{l+1}} & \text{if } 1 \le l \le s-1 \\ \mathbf{x}'_{l-s+1} - a_{l-s+1, i_0}\mathbf{y}_{i_1} & \text{if } s \le l \le b-a. \end{cases}$$

By the linear independence of $\{\mathbf{x}'_i\}_i$ and $\{\mathbf{y}_{i_l} - \mathbf{y}_{i_{l+1}}\}_{l=1}^{s-1}$, we have that $\mathbf{x}''_1, \ldots, \mathbf{x}''_{b-a}$ are linear independent. By reordering the basis $\{\mathbf{y}_i\}_i$ we can write

$$M_\alpha = \begin{pmatrix} X & 0 \\ Y & N_\alpha \end{pmatrix},$$

where $X = \begin{pmatrix} 1 & \ldots & 1 \end{pmatrix}^{\mathrm{tr}}$ and $Y = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix}^{\mathrm{tr}}$. Let $Z = \begin{pmatrix} Z_1 & \ldots & Z_{b-a} \end{pmatrix}^{\mathrm{tr}}$, be a $(b-a) \times b$ matrix with each row $Z_i = (c_{i1}, \ldots, c_{ib})$, given by $\mathbf{x}''_i = \sum_{l=1}^{b} c_{il}\mathbf{y}_l$. That is $Z : \mathbf{k}^b \longrightarrow \mathbf{k}^{b-a}$ is the natural projection from M_2 to $\mathrm{Cok}(M_\alpha)$. Now by the construction we have $ZM_\alpha = 0$. Therefore $Z : \mathbf{k}^b \longrightarrow \mathbf{k}^{b-a}$ is F_1M.

By the construction we know that $\deg(\mathbf{y}_{j_s})$ and $\deg(\mathbf{y}_{j_0})$ are 1 in $\Gamma(DF_1M, \mathcal{B}''_1 \cup \mathcal{B}_2)$. By induction the degree of other vertices in $\mathcal{S}$ is one. Note that those vertices of degree two in $\Gamma(DF_1N, \mathcal{B}'_1 \cup \mathcal{B}'_2)$ are still of degree two in $\Gamma(DF_1M, \mathcal{B}''_1 \cup \mathcal{B}_2)$ and by induction there are $b - s - (a-1) - 1$ of them. By the choosing of the $\mathbf{x}''_i$ we have exactly $s-1$ other vertices $\mathbf{y}_{i_1}, \ldots, \mathbf{y}_{i_{s-1}}$ of degree two. Therefore in all there are $b - a - 1$ vertices with degree two in $\Gamma(DF_1M, \mathcal{B}''_1 \cup \mathcal{B}_2)$ and so there are $a+1$ vertices of degree one in $\Gamma(DF_1M, \mathcal{B}''_1 \cup \mathcal{B}_2)$. These are exactly the vertices in $\mathcal{S}$ and so (1) follows.

We know that $\mathbf{y}_{j_0}$ is of degree one in $\Gamma(DF_1N, \mathcal{B}'_1 \cup \mathcal{B}'_2)$ and $\mathbf{y}_{j_0} \in \mathcal{N}_{\mathbf{x}'_{l_0}}$. That is, a_{l_0, i_0} is the unique non-zero element in $\{a_{i, i_0}\}_i$. We first consider $s = 1$. In this case both $\mathbf{y}_{j_0}$ and $\mathbf{y}_{j_1}$ are connected to $\mathbf{x}''_{l_0}$ in $\Gamma(DF_1M, \mathcal{B}''_1 \cup \mathcal{B}_2)$ and both $\mathbf{y}_{j_0}$ and $\mathbf{y}_{j_1}$ are of degree one in $\Gamma(DF_1M, \mathcal{B}''_1 \cup \mathcal{B}_2)$. So $\mathcal{N}_{\mathbf{x}''_{s+l_0-1}}$ still contains at most one element of degree two. Moreover if $|\mathcal{B}'_1| > 1$, $\mathcal{N}_{\mathbf{x}''_{s+l_0-1}}$ contains a unique simple vertex $\mathbf{y}_{j_1}$ with degree one, since $\mathbf{y}_{j_0}$ is simple of degree one in $\Gamma(DF_1N, \mathcal{B}'_1 \cup \mathcal{B}'_2)$. So in this case by induction both (2), (3) and (4) are fulfilled. Now suppose that $s > 1$. Then $\mathbf{y}_{j_1}$ is of degree two in $\Gamma(DF_1M, \mathcal{B}''_1 \cup \mathcal{B}_2)$. It is clear that the edge from $\mathbf{x}_{s+l_0-1}$ to $\mathbf{y}_{j_1}$ does not increase the number of degree-one-elements in $\mathcal{N}_{\mathbf{x}''}$ for any $\mathbf{x}'' \in \mathcal{B}''_1$. Therefore (3) follows by induction. Now if $|\mathcal{B}'_1| = 1$, that is, $\mathcal{B}'_1 = \{\mathbf{x}'_{l_0}\}$, then $\mathbf{y}_{j_1}$ is the unique degree-two-element in $\mathcal{N}_{\mathbf{x}''_{s+l_0-1}}$. If $|\mathcal{B}'_1| > 1$, by (3) we know that there is no other simple degree-one-element in $\mathcal{N}_{\mathbf{x}'_{l_0}}$, besides $\mathbf{y}_{j_0}$ and so there is there is no simple degree-one-element in $\mathcal{N}_{\mathbf{x}''_{s+l_0-1}}$. This finishes the proof of (2).

Now the remainder is to prove that $\Gamma(DF_1M, \mathcal{B}_1'' \cup \mathcal{B}_2)$ is a tree in $\mathcal{A}$. It is clear that the full subtree T' with the set of vertices $\{\mathbf{x}_i''\}_{i=1}^{s-1} \cup \{\mathbf{y}_{i_l}\}_{l=1}^{s}$ is a tree in $\mathcal{A}$. By induction we know that $\Gamma(DF_1N, \mathcal{B}_1' \cup \mathcal{B}_2)$ is a tree in $\mathcal{A}$. By the definition of $\{\mathbf{x}_i''\}_i$, there is exactly one edge, produced by the choosing of $\mathbf{x}_{s+l_0-1}''$, connecting the two trees T' and $\Gamma(DF_1N, \mathcal{B}_1' \cup \mathcal{B}_2)$ together. This connection gives one more degree-two-element $\mathbf{y}_{i_1}$ in $\mathcal{N}_{\mathbf{x}_{s+l_0-1}''}$, when compared with degree-two-elements in $\mathcal{N}_{\mathbf{x}_t'}$, which contains at most one degree-two-element. Therefore $\Gamma(DF_1M, \mathcal{B}_1'' \cup \mathcal{B}_2)$ is a tree and it is a tree in $\mathcal{A}$. This finishes the proof of this proposition. □

Theorem 6.3. *The indecomposable representations of $\mathcal{Q}$ with dimension vector a real root are tree representations.*

Proof. Let N be an indecomposable representation for a real root. We want to find a basis $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ such that the coefficient quiver $\Gamma(N, \mathcal{B})$ is a tree. It is clear that the coefficient quiver of the indecomposable representations Y^a and Y_a are trees. Now let the dimension vector of N be $d_{n,m}$ or d_n^m for integers $n > 0$ and $m \geq 0$. We will show by induction that $\Gamma(N, \mathcal{B})$ is a tree where

(*) each connected component of the restriction $\Gamma_{\mathbb{A}_2}(N, \mathcal{B})$ of $\Gamma(N, \mathcal{B})$ to the full subquiver of type $\mathbb{A}_2$ is
  1. (1) a tree in $\mathcal{A}$ or an isolated vertex from $\mathcal{B}_2$, if $m > n$, or
  2. (2) a tree in $\mathcal{A}$ or a tree of type $\mathbb{A}_2 : \mathbf{x}_i \to \mathbf{y}_j$ if $m = 1 = n$ or $m \leq n$.

By Theorem 5.7 we may assume that $N \in V_n^m$ or $V_{m,n}$ and that M can be constructed from Y_a, for some a, using the operations Σ_1, Σ_2 and Σ_3. It is easy to see that if $N = \Sigma_1 Y_a \in V_{a,1}$, then we can find a basis such that $\Gamma(N, \mathcal{B})$ is a tree satisfying (*).

Let N be a representation in $V_{n,m}$ or V_n^m with a basis $\mathcal{B}$ such that $\Gamma(N, \mathcal{B})$ is a tree satisfying (*). Let $T_1, \dots, T_l$ be the connected components of $\Gamma_{\mathbb{A}_2}(N, \mathcal{B})$.

We first consider the operations Σ_2 and Σ_3. By counting the dimension increased and the number of new edges we can see that Σ_2 and Σ_3 send a tree to a tree. So we only need to show that they preserve the property (*).

By the definition of Σ_3, we can see that

$$\Gamma_{\mathbb{A}_2}(N, \mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3) = \Gamma_{\mathbb{A}_2}(\Sigma_3 N, \mathcal{B}' = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3')$$

and so Σ_3 preserves the property (*).

By the definition of Σ_2, the connected components of $\Gamma_{\mathbb{A}_2}(\Sigma_2 N, \mathcal{B}' = \mathcal{B}_1 \cup \mathcal{B}_2' \cup \mathcal{B}_3)$ are $T_1', \dots, T_l', T_{l+1}', \dots, T_{l+2m}'$, where $T_{l+1}', \dots, T_{l+2m}'$ are isolated vertices $\{\mathbf{y}_i', \mathbf{y}_i''\}_{1 \leq i \leq m}$ and each T_i' for $1 \leq i \leq l$ is obtained by attaching an arrow $\mathbf{x}_j \to \mathbf{y}(\mathbf{x}_j)$ for each vertex $\mathbf{x}_j \in T_i \cap \mathcal{B}_1$. Here $\mathcal{B}_2' = \mathcal{B}_2 \cup \{\mathbf{y}_i', \mathbf{y}_i''\}_{1 \leq i \leq m} \cup \{\mathbf{y}(\mathbf{x}_i)\}_{1 \leq i \leq n}$. In particular, if $T_i \in \mathcal{A}$ then $T_i' \in \mathcal{A}$ and if T_i is of type $\mathbb{A}_2$ then $T_i' \in \mathcal{A}$ too. Note that $\sigma_2(\mathbf{d}_n^m) = \mathbf{d}_{n,m+n}$ and $m + n$ is strictly bigger than n. So $\Gamma(\Sigma_2 N, \mathcal{B}')$ is a tree satisfying (*).

We now consider the operation Σ_1. By Theorem 5.7 we may assume that $m > n$. We may also suppose that $N \in V_{n,m}$. Let N^i be the representation corresponding to T_i. We may assume that $T_1, \dots, T_t \in \mathcal{A}$ and $T_{t+1}, \dots, T_l$ are isolated vertices from $\mathcal{B}_2$. That is, N^i is the simple representation S_2 for $i > t$. By Proposition 6.2 there is a basis ${\mathcal{B}_1^i}' \subseteq \mathcal{B}_2^i$ for $\mathrm{Cok} N_\alpha^i$, where $\mathcal{B}_1^i \cup \mathcal{B}_2^i \subseteq \mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for N^i, such that $\Gamma(DF_1 N^i, {\mathcal{B}_1^i}' \cup \mathcal{B}_2^i)$ is a tree in $\mathcal{A}$ for $1 \leq i \leq t$. Trivially, $\Gamma(DF_1 N^i, {\mathcal{B}_1^i}' \cup \mathcal{B}_2^i)$ is a tree of type $\mathbb{A}_2$ for $i > t$.

Since γ and g in the definition of Σ_1 induce isomorphisms on the coefficient quivers, the connected components of $\Gamma_{\mathbb{A}_2}(\Sigma_1 N, \mathcal{B}' = {\mathcal{B}_1}' \cup \mathcal{B}_2 \cup \mathcal{B}_3)$ are up to isomorphism the trees $\Gamma_{\mathbb{A}_2}(DF_1 N^i, {\mathcal{B}_1^i}' \cup \mathcal{B}_2^i \cup \mathcal{B}_3^i)$. Moreover we have $\sigma_1(\mathbf{d}_{n,m}) = \mathbf{d}_{\frac{m^2-1}{n},m}$ and $\frac{m^2-1}{n} > m$, since $m > n$. Therefore $\Gamma_{\mathbb{A}_2}(\Sigma_1 N, \mathcal{B}')$ is a tree satisfying the property (*).

By counting the number of arrows and edges, we now show that $\Gamma(\Sigma_1 N, \mathcal{B}')$ is in fact a tree. By removing the vertices in $\mathcal{B}_1$, we get a subquiver of $\Gamma(N, \mathcal{B})$ which we denote by K.

Denote by $\underline{\dim} N^i = (d_1^i, d_2^i, 0)$. We have the number $|\Gamma(N, \mathcal{B})_1|$ of edges of $|\Gamma(N, \mathcal{B})|$ is:

$$|\Gamma(N,\mathcal{B})_1| = \sum_{i=1}^{s} |T_1^i| + |K_1| = \sum_{i=1}^{s} (d_1^i + d_2^i - 1) + |K_1| = \dim N - 1,$$

since $\Gamma(N,\mathcal{B})$ is a tree. Now

$$\begin{aligned}
|\Gamma(\Sigma_1 N, \mathcal{B}_1' \cup \mathcal{B}_2 \cup \mathcal{B}_3)_1| &= \sum_{i=1}^{s} |\Gamma(DF_1 N^i, (\mathcal{B}_1^i)' \cup \mathcal{B}_2^i))| + |K_1| \\
&= \sum_{i=1}^{s} (d_2^i - d_1^i + d_2^i - 1) + |K_1| \\
&= \sum_{i=1}^{s} (d_1^i + d_2^i - 1 + (d_2^i - 2d_1^i)) + |K_1| \\
&= \sum_{i=1}^{s} ((d_1^i + d_2^i - 1) + |K_1|) + \sum_{i=1}^{s} (d_2^i - 2d_1^i) \\
&= (\dim N - 1) + d_2 - 2d_1 = \dim \Sigma_1 N - 1.
\end{aligned}$$

The coefficient quiver $\Gamma(\Sigma_1 N, \mathcal{B}')$ is connected, since N is indecomposable and so is $\Sigma_1 N$. Hence, $\Gamma(\Sigma_1 N, \mathcal{B}')$ is a tree. By induction and Theorem 5.7 all indecomposable representation of $\mathcal{Q}$ for real roots are tree representations. $\square$

7. Generic representations and the dimensions of their endomorphism rings

In this section we study the canonical decomposition of real roots of $\mathcal{Q}$ and the dimensions of the endomorphism rings of generic representations. We first consider the case $\mathbf{d}_{n,m}$. Note that operation Σ_1 can be also defined for any representations of $\mathcal{Q}$ with no direct summand S_1. By the definition of operation Σ_1, we know that Σ_1 preserves the dimension of the endomorphism ring of generic representations. So we may assume that $m \geq n$.

Proposition 7.1. *For any real root $\mathbf{d}_{n,m}$ with $m \geq n$, its subroot $(n, d_2 - m + n, d_2 - m)$ is a Schur root.*

Proof. Denote the dimension vector $(n, d_2 - m + n, d_2 - m)$ by $\mathbf{d}$. If $m = n = 1$, then $\mathbf{d}_{n,m} = (1, 1, 0)$ and it is obviously a Schur root; otherwise we have $d_2 - m \geq n + 1$. In the following we only consider the second case. We onstruct a representation M in $\mathrm{Rep}(\mathcal{Q}, \mathbf{d})$ and then prove that its endomorphism ring is trivial. Let M be a representation in $\mathrm{Rep}(\mathcal{Q}, \mathbf{d})$ satisfying:

(1) $\mathrm{Kr}(M) = \oplus_{i=1}^{d_2-m} M(\lambda_i) \oplus S_2^{\oplus n}$, where $M(\lambda_i)$ is an indecomposable representation in $\mathrm{Rep}(\mathcal{Q}, (0, 1, 1))$ with $M(\lambda_i)_\beta = \lambda_i$, $M(\lambda_i)_\gamma = 1$ and $\lambda_i \neq \lambda_j \in \mathbf{k}^*$ for $i \neq j$.

(2) $M_\alpha = \begin{pmatrix} I_n & A^{\mathrm{tr}} & B^{\mathrm{tr}} & I_n \end{pmatrix}^{\mathrm{tr}}$, where I_n is the identity matrix, $A = \begin{pmatrix} 1 & \ldots & 1 \end{pmatrix}$ and B is a $(d_2 - m - n - 1) \times n$ matrix with 1 at each entry of the last column and 0 elsewhere.

We now prove that M is indecomposable with trivial endomorphism ring. Let $f = (f_1, f_2, f_3) \in \mathrm{End}(M)$. Then $(0, f_2, f_3) \in \mathrm{End}(\mathrm{Kr}(M))$. It is clear that $\mathrm{Hom}(M(\lambda_i), M(\lambda_j)) = \delta_{ij}\mathbf{k}$, $\mathrm{Hom}(M(\lambda_i), S_2) = 0$ and $\mathrm{Hom}(S_2, M(\lambda_i)) = 0$. So we can write

$$f_2 = \begin{pmatrix} X_1 & 0 & 0 & 0 \\ 0 & X_2 & 0 & 0 \\ 0 & 0 & X_3 & 0 \\ 0 & 0 & 0 & Y \end{pmatrix} \text{ and } f_3 = \begin{pmatrix} X_1 & 0 & 0 \\ 0 & X_2 & 0 \\ 0 & 0 & X_3 \end{pmatrix}$$

where $X_1 = \mathrm{diag}(\mu_1, \ldots, \mu_n) \in \mathrm{End}(\oplus_{i=1}^n M(\lambda_i))$, $X_2 = \mu_{n+1} \in \mathrm{End}(M(\lambda_{n+1}))$, and $X_3 = \mathrm{diag}(\mu_{n+2}, \ldots, \mu_{d_2-m}) \in \mathrm{End}(\oplus_{i=n+2}^{d_2-m} M(\lambda_i))$ are diagonal matrices, and Y an $n \times n$ matrix in $\mathrm{End}(S_2^{\oplus n})$. Now f_1 and f_2 should satisfy:

$$\begin{pmatrix} I_n \\ A \\ B \\ I_n \end{pmatrix} f_1 = \begin{pmatrix} X_1 & 0 & 0 & 0 \\ 0 & X_2 & 0 & 0 \\ 0 & 0 & X_3 & 0 \\ 0 & 0 & 0 & Y \end{pmatrix} \begin{pmatrix} I_n \\ A \\ B \\ I_n \end{pmatrix}$$

So we have (i) $f_1 = Y = X_1$, (ii) $AX_1 = X_2A$ and (iii) $BX_1 = X_2B$. By (ii) we get $\mu_i = \mu_{n+1}$ for $1 \leq i \leq n$ and by (iii) we get $\mu_j = \mu_{n+1}$ for $j \geq n+2$. Therefore we have $f_1 = \mu_{n+1}I_n$ and $f_2 = \mu_{n+1}I_{d_2-m+n}$ and so $\mathrm{End}(M) = \mathbf{k}$. This finishes the proof. □

Theorem 7.2. *The canonical decomposition of* $\mathbf{d}_{n,m}$ *is* $(n, d_2-m+n, d_2-m)+\overbrace{\mathbf{e}_2 + \cdots + \mathbf{e}_2}^{m-n}$.

We need some preparation to prove this theorem. We first recall a well known result on the Euler form of Q.

Proposition 7.3 (Ringel)**.** *Let N be a representation in* $\mathrm{Rep}(Q,\mathbf{d})$ *and let L be a representation in* $\mathrm{Rep}(Q,\mathbf{c})$. *Then* $<\mathbf{d}, \mathbf{c}>= \dim \mathrm{Hom}(M,N) - \dim \mathrm{Ext}^1(M,N)$.

Let $\mathrm{ext}(\mathbf{d},\mathbf{c}) = \min\{\dim \mathrm{Ext}^1(N,L) | N \in \mathrm{Rep}(Q,\mathbf{d}) \text{ and } L \in \mathrm{Rep}(Q,\mathbf{c})\}$.

Theorem 7.4 (Kac[6], see also [10][3])**.** *The decomposition* $\mathbf{d} = \mathbf{d}^1+\cdots+\mathbf{d}^s$ *is the canonical decomposition if and only if each* $\mathbf{d}^i$ *is a Schur root and* $ext(\mathbf{d}^i, \mathbf{d}^j) = 0$ *for* $i \neq j$.

Let $\mathbf{d} \in \mathbb{N}^{Q_0}$ and let $\mathbf{d} = \mathbf{d}^1 + \cdots + \mathbf{d}^s$ be the canonical decomposition of $\mathbf{d}$. Denote by $\mathrm{Rep}(Q,\mathbf{d})_{\mathrm{gen}} = \{M \in \mathrm{Rep}(Q,\mathbf{d}) | M \cong \oplus M^i, \text{ where } M^i \in \mathrm{Ind}(Q,\mathbf{d}^i)\}$. We call a representation $M \in \mathrm{Rep}_{\mathrm{gen}}(Q,\mathbf{d})$ a generic representation if $\dim \mathrm{End}(M)$ is minimal.

Proof of Theorem 7.2. By Proposition 7.1 we know that all terms appearing in the decomposition are Schur root. Denote by $\mathbf{d} = (n, d_2-m+n, d_2-m)$. By Theorem 7.4 we need only to show that $\mathrm{ext}(\mathbf{d},\mathbf{e}_2) = \mathrm{ext}(\mathbf{e}_2,\mathbf{d}) = 0$. Let M be a generic representation in $\mathrm{Rep}(\mathcal{Q},\mathbf{d})$, as constructed in the proof Proposition 7.1.

By the construction in the proof Proposition 7.1 we have $\mathrm{Hom}(M,S_2) = 0$. Now following Proposition 7.3 and $<\mathbf{d},\mathbf{e}_2>= d_2-m+n-n-(d_2-m) = 0$ we have $\mathrm{Ext}(M,S_2) = 0$ and so $\mathrm{ext}(\mathbf{d},\mathbf{e}_2) = 0$. Again the construction in the proof Proposition 7.1 we have that $\mathrm{Hom}(S_2,M)$ is n-dimensional. Using Proposition 7.3 and $<\mathbf{e}_2,\mathbf{d}>= n$ we have $\mathrm{Ext}(S_2,M) = 0$ and so $\mathrm{ext}(\mathbf{e}_2,\mathbf{d}) = 0$. This completes the proof. □

Corollary 7.5. *Let M be a generic representation in* $\mathrm{Rep}(\mathcal{Q},\mathbf{d}_{n,m})$. *Then the dimension of its endomorphism ring* $\mathrm{End}(M)$ *is* $1 + m(m-n)$.

The corollary follows from the proof of Theorem 7.2. Here we give a different proof, following the relation between the Euler form of Q and the number of parameters of $\mathrm{Ind}(Q,\mathbf{d})$ given by Theorem 1 in [5].

Proof. Use the same notation as in the proof of Theorem 7.2. Let N be a generic representation in $\mathrm{Rep}(\mathcal{Q},\mathbf{d}_{n,m})$. By Theorem 7.2, we know that N belongs to a $1-q(\mathbf{d})-$parameter-family. We have $q(\mathbf{d}) = n^2 - nd_2 + nm$. Therefore we have $\dim \mathcal{O}_N + 1 - (n^2 - nd_2 + nm) = \dim \mathrm{Rep}(\mathcal{Q},\mathbf{d}_{n,m})$, where $\mathcal{O}_N$ is the orbit of N. On the other hand we have $\dim \mathcal{O}_N = \dim \mathrm{Rep}(\mathcal{Q},\mathbf{d}_{n,m}) + q(\mathbf{d}_{n,m}) - \dim \mathrm{End}(N)$. Therefore $\dim \mathrm{End}(N) = 1 + m(m-n)$.

□

With similar arguments as in the proof of Theorem 7.2 and in the proof of Corollary 7.5 we have:

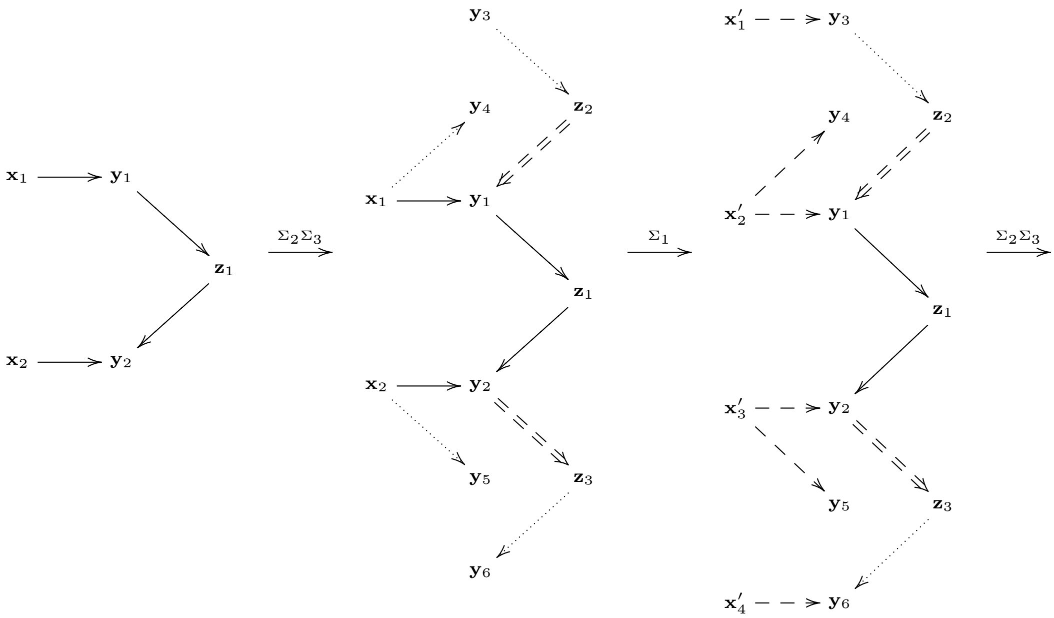
Theorem 7.6. *The canonical decomposition of the real root* $\mathbf{d}_n^m$ *is* $(n, d_2, d_2)+\overbrace{\mathbf{e}_3 \cdots + \mathbf{e}_3}^{m}$. *The dimension of the endomorphism ring of a generic representation is* $1 + m^2$.

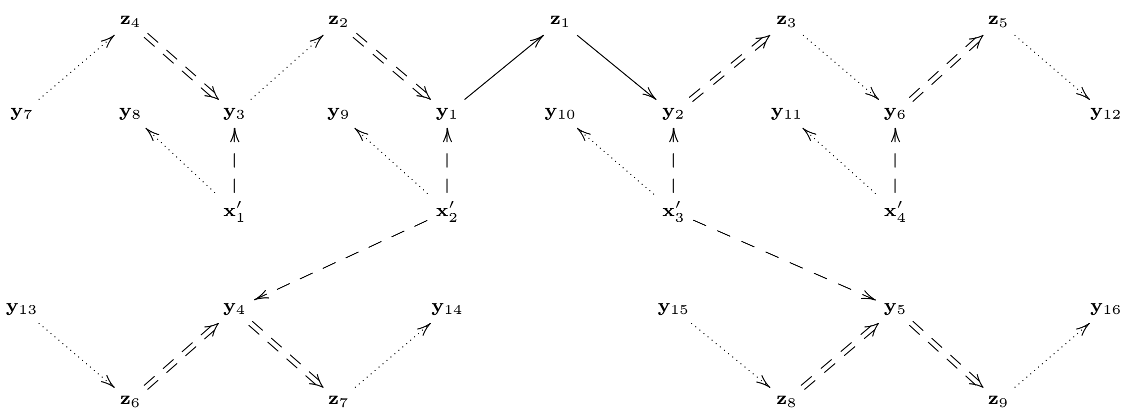
Remark 7.7. *By results in Section 5 and Section 7 we see that for $m = ns + r$ with $s \gg 0$, the dimension of the endomorphism ring of an indecomposable representation in* $\mathrm{Rep}(\mathcal{Q}, \mathbf{d}_{n,m})$ *or* $\mathrm{Rep}(\mathcal{Q}, \mathbf{d}_n^m)$ *is much larger than the dimension for the corresponding generic representations.*

8. Examples

In this section we show two examples on how the coefficient quiver grows according to the three operations. In the pictures the arrows $==\Rrightarrow$, $\cdots\cdots>$ and $--\succ$ correspond to operations Σ_3, Σ_2 and Σ_1, respectively.

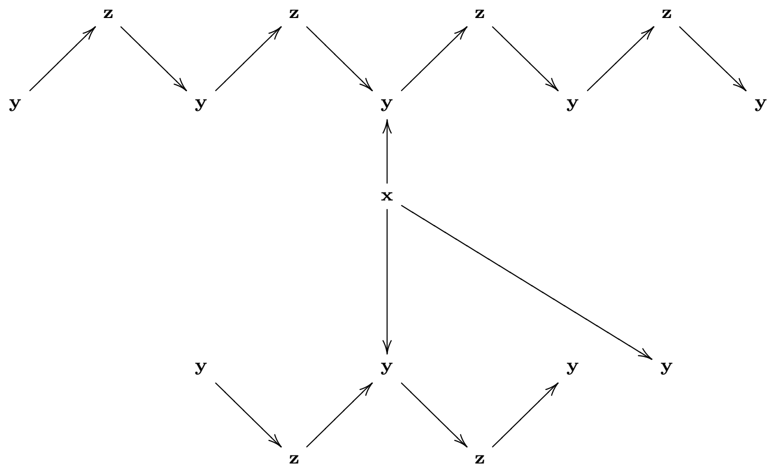
We first consider the operations on the tree $M : (I_2, (1\,0)^{\mathrm{tr}}, (0\,1))$ in $\mathrm{Rep}(\mathcal{Q}, \mathbf{d}_{2,1})$. The quivers in the following are in turn the coefficient quiver of M, $\Sigma_2\Sigma_3(M)$, $\Sigma_1\Sigma_2\Sigma_3(M)$ and $\Sigma_2\Sigma_3\Sigma_1\Sigma_2\Sigma_3(M)$. The vertex marked by $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are the basis elements of the vector spaces at vertex 1, 2 and 3, respectively. At the end we get a tree $\Sigma_2\Sigma_3\Sigma_1\Sigma_2\Sigma_3(M)$ in $\mathrm{Rep}(\mathcal{Q}, \mathbf{d}_{4,7})$. Here $\mathbf{d}_{4,7} = (4, 16, 9)$.



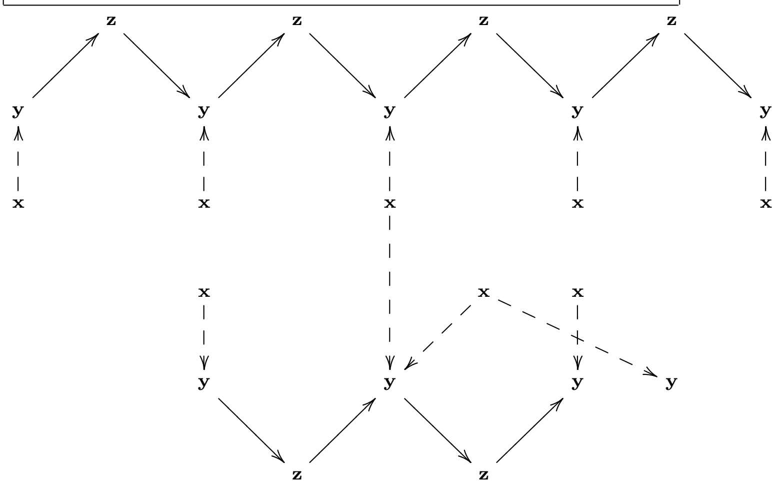


Our second example starts with M an indecomposable representation in $\mathrm{Rep}(\mathcal{Q}, \mathbf{d}_{1,3})$ as constructed in Example 5.1. We show the growing of the coefficient quiver of $\Sigma_1 M$, $\Sigma_2\Sigma_3\Sigma_1 M$ and then $\Sigma_1\Sigma_2\Sigma_3\Sigma_1 M$, respectively. We use the same notation as in the previous example. In particular, at the last step of operation, we show how to choose a good basis such that the operation Σ_1 send a tree in $\mathcal{A}$ to a tree in $\mathcal{A}$. In $\Gamma(\Sigma_2\Sigma_3\Sigma_1(M), \mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z})$ the subquiver containing the vertices underlined is a tree in $\mathcal{A}$ and among them the double underlined vertices are the simple vertices. From the coefficient quiver $\Gamma(\Sigma_1\Sigma_2\Sigma_3\Sigma_1(M), \mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z})$ we see that these simple vertices are of degree 1.

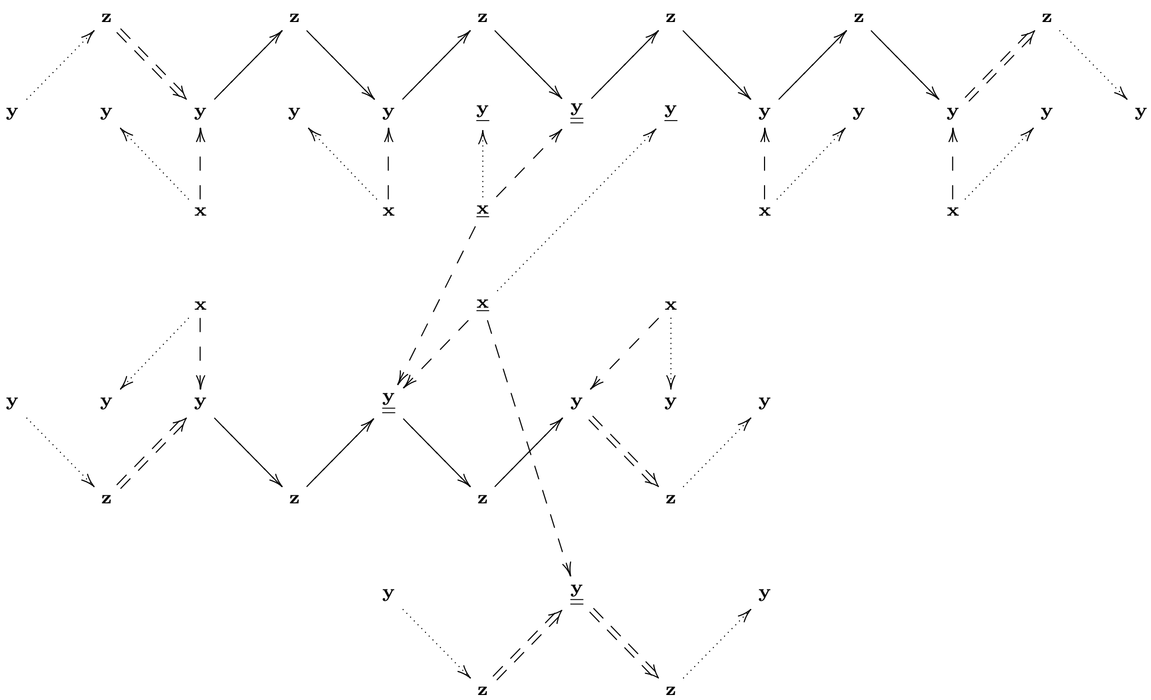
$\boxed{\Gamma(M, \mathcal{X} \cup \mathcal{Y} \cap \mathcal{Z}),\ \text{where } M \in \mathrm{Rep}(\mathcal{Q}, \mathbf{d}_{1,3}) \text{ and } \mathbf{d}_{1,3} = (1, 9, 6)}$:



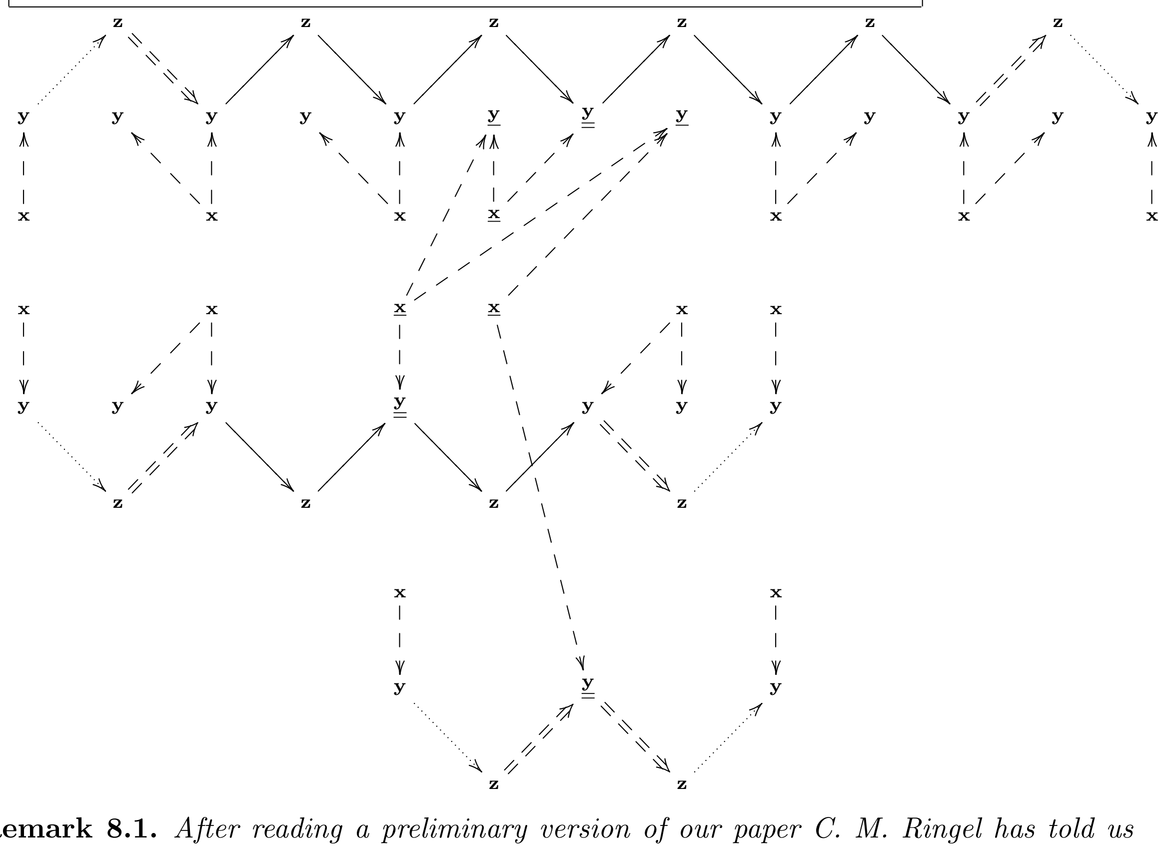
$\boxed{\Gamma(\Sigma_1(M), \mathcal{X} \cup \mathcal{Y} \cap \mathcal{Z}),\ \text{where } \Sigma_1 M \in \mathrm{Rep}(\mathcal{Q}, \mathbf{d}_{8,3}) \text{ and } \mathbf{d}_{8,3} = (8, 9, 6)}$:



$\boxed{\Gamma(\Sigma_2\Sigma_3\Sigma_1(M), \mathcal{X} \cup \mathcal{Y} \cap \mathcal{Z}),\ \text{where } \Sigma_2\Sigma_3\Sigma_1(M) \in \mathrm{Rep}(\mathcal{Q}, \mathbf{d}_{8,11}) \text{ and } \mathbf{d}_{8,11} = (8, 23, 12)}$:



$\boxed{\Gamma(\Sigma_1\Sigma_2\Sigma_3\Sigma_1(M), \mathcal{X}\cup\mathcal{Y}\cap\mathcal{Z}),\ \text{where } \Sigma_1\Sigma_2\Sigma_3\Sigma_1(M)\in \mathrm{Rep}(\mathcal{Q},\mathbf{d}_{15,11}) \text{ and } \mathbf{d}_{15,11}=(15,23,12)}$:



Remark 8.1. *After reading a preliminary version of our paper C. M. Ringel has told us that our construction of the three operations Σ_1, Σ_2 and Σ_3 can be generalized to quivers where our A_1 subquiver is replaced by b arrows from vertex 2 to vertex 3 and c arrows from 3 to 2, for arbitrary positive integers b and c. The indecomposable representations still have the tree property. However, it is not yet clear how to make these generalized constructions independent of the characteristic of the field.*

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