Homework 1

The homeworks are due on the Thursday of the week after the assignment was posted online[1]. You can either hand in your homework at the beginning of the tutorial or put it in the mailbox labeled "Algebraic Number Theory II" in Room 301 of the Mathematical Institute by 16:00. You can work on and submit your homework in groups of two. Please staple your pages and write your names and matriculation numbers on the first page.

Problem 1 (10 pts.)

Let $\| \ \|_1$ and $\| \ \|_2$ be two norms on a field F. Prove that $\| \ \|_1 \sim \| \ \|_2$ if and only if there exists $\alpha > 0$ such that $\|x\|_1 = \|x\|_2^\alpha$ for all $x \in F$.

Problem 2 (10 pts.)

If $0 < \rho < 1$, prove that the function $\| \ \|$ defined on $\mathbb{Q}$ by

$$\|x\| = \begin{cases} \rho^{\nu_p(x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is a non-Archimedean norm. Prove that this norm is equivalent to $| \ |_p$. What happens if $\rho = 1$? What about $\rho > 1$?

Problem 3 (10 pts.)

Product formula. Prove that for $x \in \mathbb{Q}$, $x \neq 0$,

$$\prod_p |x|_p = 1,$$

where the product is taken over all primes p, including $p = \infty$.

The following exercises will be discussed in the tutorial and you do not need to hand in solutions for them.

Exercise 1

Prove that $| \ |_p$ is not equivalent to $| \ |_q$ if p and q are distinct primes.

Exercise 2

For $x \in \mathbb{Q}$ define $\|x\| = |x|^\alpha$ for a fixed positive real number α, where $| \ |$ denotes the usual absolute value. Show that $\| \ \|$ is a norm if and only if $\alpha \leq 1$ (in which case it is equivalent to $| \ |$.)

[1] This assignment is due Thursday, 17.10.19.

Exercise 3

Prove that two equivalent norms on a field F are either both Archimedean or both non-Archimedean.

Exercise 4

Compute:

(i) $\nu_2(128)$

(ii) $\nu_2(128/7)$

(iii) $\nu_3(54)$

(iv) $\nu_3(10^9)$

(v) $\nu_3(7/9)$

(vi) $\nu_3(13.23)$

(vii) $\nu_7(-13.23)$