

## Homework 12

The homeworks are due on the Thursday of the week after the assignment was posted online<sup>1</sup>. Please hand in your homework at the beginning of the tutorial or bring it to the lecture on Thursday morning. You can work on and submit your homework in groups of two. Please staple your pages and write your names and matriculation numbers on the first page.

### Problem 34 (10 pts.)

Let  $L \subset \mathbb{R}^n$  be a lattice of full rank.

- Show that there is a matrix  $A \in \text{GL}_n(\mathbb{R})$  such that  $L = AZ^n$ .
- Show that if  $L = AZ^n$ , then the dual lattice  $L'$  of  $L$  is given by  $L' = A^*Z^n$ , where  $A^* = (A^T)^{-1}$ .

### Problem 35 (10 pts.)

*In this problem we prove some properties of Dirichlet characters that were already mentioned or used in class.*

- Show that the set  $G$  of Dirichlet characters  $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \rightarrow \mathbb{C}$  modulo  $m$  is a group with respect to multiplication.
- Show that  $|G| = \varphi(m)$ .
- Show that

$$\sum_{n \pmod{m}} \chi(n) = \begin{cases} \varphi(m) & \text{if } \chi = \chi^0, \\ 0 & \text{otherwise,} \end{cases}$$

where  $\chi^0$  denotes the trivial character modulo  $m$ .

**Hints:** For  $\chi \neq \chi^0$ , there exists  $k \in \mathbb{Z}$  with  $\text{gcd}(k, m) = 1$ , such that  $\chi(k) \neq 1$ . Then multiply the sum above by  $\chi(k)$ .

### Problem 36 (10 pts.)

*Poisson summation formula for finite groups. Some definitions:* Let  $G$  be an arbitrary group. A complex-valued function  $f$  defined on  $G$  is called a *character* of  $G$  if  $f$  has the multiplicative property  $f(ab) = f(a)f(b)$  for any  $a, b \in G$  and if there exists some  $c \in G$  with  $f(c) \neq 0$ . It is easy to see that if  $f$  is a character on a *finite* group  $G$  with identity  $e$ , then  $f(e) = 1$  and each  $f(a)$  is a root of unity. A *finite abelian* group  $G$  of order  $n$  has exactly  $n$  distinct characters. If multiplication of characters is defined by  $(f_i f_j)(a) = f_i(a)f_j(a)$  for any  $a \in G$ , then the set of characters of  $G$  forms an abelian group of order  $n$ , which is usually

<sup>1</sup>This assignment is due Thursday, 16.01.20.

denoted by  $\widehat{G}$ . Note that, alternatively but equivalently,  $\widehat{G}$  is then the dual group of  $G$ , i.e., the set of homomorphisms  $G \rightarrow S^1$  with the group law given by pointwise multiplication of functions.

If  $G$  is a finite abelian group, the *Fourier transform* of a function  $f : G \rightarrow \mathbb{C}$  is the function  $\widehat{f} : \widehat{G} \rightarrow \mathbb{C}$  given by

$$\widehat{f}(\chi) = \sum_{g \in G} f(g) \overline{\chi}(g).$$

One then has the *Fourier inversion formula*

$$f(x) = \frac{1}{|G|} \sum_{\chi \in \widehat{G}} \widehat{f}(\chi) \chi(x),$$

which tells us how to recover  $f$  from its Fourier transform.

**Your tasks:** You **do not** need to prove anything up to here! Rather, just familiarize yourself with these notions. What you **do** need to prove is the following version of Poisson summation, and illustrate it by an example.

- a) Let  $G$  be a finite abelian group and  $H \subset G$  a subgroup. Show that for any function  $f : G \rightarrow \mathbb{C}$  we have

$$\frac{1}{|H|} \sum_{h \in H} f(h) = \frac{1}{|G|} \sum_{\chi \in H^\perp} \widehat{f}(\chi),$$

where  $H^\perp = \{\chi \in \widehat{G} : \chi = 1 \text{ on } H\}$ .

**Hints:** You might want to use (and prove) the identities

$$\sum_{g \in G} \chi(g) = \begin{cases} |G| & \text{if } \chi = \mathbf{1}_G, \\ 0 & \text{if } \chi \neq \mathbf{1}_G, \end{cases} \quad \sum_{\chi \in \widehat{G}} \chi(g) = \begin{cases} |G| & \text{if } g = 1, \\ 0 & \text{if } g \neq 1, \end{cases}$$

and the *orthogonality relations*

$$\sum_{g \in G} \chi_1(g) \overline{\chi_2}(g) = \begin{cases} |G| & \text{if } \chi_1 = \chi_2, \\ 0 & \text{if } \chi_1 \neq \chi_2, \end{cases} \quad \sum_{\chi \in \widehat{G}} \chi_1(g) \overline{\chi_2}(g) = \begin{cases} |G| & \text{if } g_1 = g_2, \\ 0 & \text{if } g_1 \neq g_2. \end{cases}$$

Try to prove that if  $\delta_g : G \rightarrow \{0, 1\}$  is given by  $\delta_g(x) = 1$  if  $x = g$ , and  $\delta_g(x) = 0$  otherwise, then

$$\delta_g(x) = \frac{1}{|G|} \sum_{\chi \in \widehat{G}} \overline{\chi}(g) \chi(x).$$

To prove the Poisson summation formula for  $f$ , it is then enough to verify it for  $\delta_g$ .

- b) Let  $G = \mathbb{Z}/8\mathbb{Z}$  and  $f : G \rightarrow \mathbb{C}$  be given by  $f(0) = f(4) = 5$ ,  $f(1) = f(5) = 3$  and  $f(2) = f(3) = f(6) = f(7) = 1$ . Compute  $H^\perp$  and the values  $\widehat{f}(n)$ .