Homework 7

The homeworks are due on the Thursday of the week after the assignment was posted online[1]. Please hand in your homework at the beginning of the tutorial or bring it to the lecture on Thursday morning. You can work on and submit your homework in groups of two. Please staple your pages and write your names and matriculation numbers on the first page.

Problem 19 (10 pts.)

Show that $|z| = (z\overline{z})^{1/2} = \sqrt{\mathrm{N}_{\mathbb{C}/\mathbb{R}}(z)}$ is the only valuation of $\mathbb{C}$ which extends the absolute value $|\ |$ of $\mathbb{R}$.

Problem 20 (10 pts.)

Let k be a field and $K = k(t)$ the function field in one variable. Show that the valuations $\nu_{\mathfrak{p}}$ associated to the prime ideals $\mathfrak{p} = (p(t))$ of $k[t]$, together with the degree valuation ν_∞, are the only (up to equivalence) non-trivial valuations of K that are trivial on k. What are the corresponding valuation rings, maximal ideals and residue fields?
Hints: Start by using Exercise 22 to clarify the basics. Then make a case distinction according to whether a valuation ν satisfies $\nu(f) \geq 0$ for all $f \in k[t]$, or whether there exists $f \in k[t]$ with $\nu(f) < 0$.

Problem 21 (10 pts.)

Krull valuations. Let $\mathcal{O}$ be an arbitrary valuation ring with field of fractions K, and let $\Gamma = K^*/\mathcal{O}^*$. Prove that Γ is a totally ordered group if we define $x \pmod{\mathcal{O}^*} \geq y \pmod{\mathcal{O}^*}$ to mean $x/y \in \mathcal{O}$. Write Γ additively and show that the function (called *Krull valuation*)

$$\nu : K \to \Gamma \cup \{\infty\}$$

defined by $\nu(0) = \infty$, $\nu(x) = x \pmod{\mathcal{O}^*}$ for $x \in K^*$, satisfies the conditions

(1) if $\nu(x) = \infty$, then $x = 0$.

(2) $\nu(xy) = \nu(x) + \nu(y)$.

(3) $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$.

The following exercises will be discussed in the tutorial and you do not need to hand in solutions for them.

Exercise 21

Is the polynomial ring over a field a valuation ring (in its field of fractions)?

[1] This assignment is due Thursday, 28.11.19.

Exercise 22

In this exercise we prepare Problem 20. Let k be a field and $K = k(t)$ the function field in one variable over k. Recall that the polynomial ring $R = k[t] \subset K$ is a principal ideal domain with field of fractions K.

(i) Let $\mathfrak{p} = (p(t))$ be a prime ideal in R. Show that there is an associated (exponential) valuation $v_{\mathfrak{p}} : K \to \mathbb{Z} \cup \{\infty\}$.

(ii) Define an exponential valuation $\nu_\infty : K \to \mathbb{Z} \cup \{\infty\}$ using the degree map. **Hint:** Start on R and check that $\nu_\infty(f) = -\deg(f)$ satisties the required properties. Then extend this to K.

Exercise 23

Determine the valuation ring $\mathcal{O}$ in $\mathbb{Q}$, its maximal ideal $\mathfrak{P}$ and its residue field with respect to the p-adic exponential valuation ν_p. What is the relation between the ring $\mathcal{O}$ and $\mathbb{Z}_p$?

Exercise 24

Let k be a field and $k((T))$ the field of Laurent power series in the variable T. If $f = \sum_{n \geq n_0} a_n T^n$ with $a_{n_0} \neq 0$, define the order of f as $\nu(f) = n_0$. This defines an (exponential) valuation. Determine the valuation ring and show that it is a discrete valuation ring. What is its residue field?