Universität zu Köln WS 2019/2020
Institut für Mathematik
Dozent: Dr. P. Gracar
Assistenten: A. Prévost, L. Schmitz

To hand in: never

0. Exercise sheet Probability II

(To work on during the second week)

Exercise 0.1

a) Compute the characteristic function of a random variable X with distribution $\mathrm{Bin}(n,p)$, for $n \in \mathbb{N}$ and $p \in (0,1)$.

b) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables, such that the law of X_n is $\mathrm{Uni}(1,\dots,n)$ for all $n \in \mathbb{N}$. What is the probability that $X_n = 1$ infinitely often?

c) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables, such that the law of X_n is $\mathrm{Geo}(\frac{\lambda}{n})$ for all $n \in \mathbb{N}$, $\lambda > 0$. Show that

$$\frac{X_n}{n} \quad \Longrightarrow \quad X \sim \mathrm{Exp}(\lambda).$$

d) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables, so that the density of X_n is $f_n(x) = 2xn^2 e^{-x^2n^2}\mathbb{1}_{(0,\infty)}(x)$ for all $n \in \mathbb{N}$, $\lambda > 0$. Show that

$$X_n \xrightarrow{\mathbb{P}} 0.$$

Exercise 0.2

Let $(X_n)_{n\in\mathbb{N}}$ and X be non-negative random variables in $\mathcal{L}^p(\Omega,\mathcal{F},\mathbb{P})$, such that $X_n \underset{n\to\infty}{\longrightarrow} X$ $\mathbb{P}$-a.s. In the first two questions, we assume that $p \in \mathbb{N}$.

a) Show that if $\|X_n - X\|_p \underset{n\to\infty}{\longrightarrow} 0$, then $\|X_n\|_p \underset{n\to\infty}{\longrightarrow} \|X\|_p$.

b) Show that

$$\int \left((X_n - X)^p\right)^- \,\mathrm{d}\mathbb{P} \underset{n\to\infty}{\longrightarrow} 0.$$

c) Show that, if $p = 1$,

$$\|X_n\|_1 \underset{n\to\infty}{\longrightarrow} \|X\|_1 \qquad \Longleftrightarrow \qquad \|X_n - X\|_1 \underset{n\to\infty}{\longrightarrow} 0.$$

d) If $p = \infty$, does the following equivalence always hold:

$$\|X_n\|_\infty \underset{n\to\infty}{\longrightarrow} \|X\|_\infty \qquad \Longleftrightarrow \qquad \|X_n - X\|_\infty \underset{n\to\infty}{\longrightarrow} 0 \ ?$$

Exercise 0.3

Let $X : (\Omega, \mathcal{F}) \to (E, \mathcal{E})$ and $Y : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be two random variables. The goal of this exercise is to show that:

Y is $\sigma(X)$-measurable $\Leftrightarrow \exists$ a $\mathcal{E} - \mathcal{B}(\mathbb{R})$-measurable function $f : E \to \mathbb{R}$ such that $Y = f \circ X$.

a) Show $\Leftarrow$.

b) Show $\Rightarrow$ when Y is a simple non-negative function.

c) Show $\Rightarrow$ when Y is non-negative.

d) Show $\Rightarrow$.