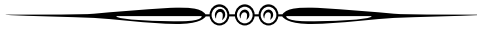


*To hand in: never*

## 0. Exercise sheet Probability II

(To work on during the second week)



### Exercise 0.1

- Compute the characteristic function of a random variable  $X$  with distribution  $\text{Bin}(n, p)$ , for  $n \in \mathbb{N}$  and  $p \in (0, 1)$ .
- Let  $(X_n)_{n \in \mathbb{N}}$  be a sequence of independent random variables, such that the law of  $X_n$  is  $\text{Uni}(1, \dots, n)$  for all  $n \in \mathbb{N}$ . What is the probability that  $X_n = 1$  infinitely often?
- Let  $(X_n)_{n \in \mathbb{N}}$  be a sequence of random variables, such that the law of  $X_n$  is  $\text{Geo}(\frac{\lambda}{n})$  for all  $n \in \mathbb{N}$ ,  $\lambda > 0$ . Show that

$$\frac{X_n}{n} \implies X \sim \text{Exp}(\lambda).$$

- Let  $(X_n)_{n \in \mathbb{N}}$  be a sequence of random variables, so that the density of  $X_n$  is  $f_n(x) = 2xn^2e^{-x^2n^2}\mathbf{1}_{(0, \infty)}(x)$  for all  $n \in \mathbb{N}$ ,  $\lambda > 0$ . Show that

$$X_n \xrightarrow{\mathbb{P}} 0.$$

### Exercise 0.2

Let  $(X_n)_{n \in \mathbb{N}}$  and  $X$  be non-negative random variables in  $\mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$ , such that  $X_n \xrightarrow[n \rightarrow \infty]{} X$   $\mathbb{P}$ -a.s. In the first two questions, we assume that  $p \in \mathbb{N}$ .

- Show that if  $\|X_n - X\|_p \xrightarrow[n \rightarrow \infty]{} 0$ , then  $\|X_n\|_p \xrightarrow[n \rightarrow \infty]{} \|X\|_p$ .
- Show that

$$\int ((X_n - X)^p)^- d\mathbb{P} \xrightarrow[n \rightarrow \infty]{} 0.$$

- Show that, if  $p = 1$ ,

$$\|X_n\|_1 \xrightarrow[n \rightarrow \infty]{} \|X\|_1 \iff \|X_n - X\|_1 \xrightarrow[n \rightarrow \infty]{} 0.$$

d) If  $p = \infty$ , does the following equivalence always hold:

$$\|X_n\|_\infty \xrightarrow[n \rightarrow \infty]{} \|X\|_\infty \quad \iff \quad \|X_n - X\|_\infty \xrightarrow[n \rightarrow \infty]{} 0 ?$$

### Exercise 0.3

Let  $X : (\Omega, \mathcal{F}) \rightarrow (E, \mathcal{E})$  and  $Y : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$  be two random variables. The goal of this exercise is to show that:

$Y$  is  $\sigma(X)$ -measurable  $\iff \exists$  a  $\mathcal{E} - \mathcal{B}(\mathbb{R})$ -measurable function  $f : E \rightarrow \mathbb{R}$  such that  $Y = f \circ X$ .

- a) Show  $\Leftarrow$ .
- b) Show  $\Rightarrow$  when  $Y$  is a simple non-negative function.
- c) Show  $\Rightarrow$  when  $Y$  is non-negative.
- d) Show  $\Rightarrow$ .