Universität zu Köln WS 2019/2020
Institut für Mathematik
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To hand in: 16.10 during the exercise class

1. Exercise sheet Probability II

(Conditional expectation)

Exercise 1.1 **(2 editing points)**

Let $\Omega = \{-1, 0, 1\}$, $\mathcal{A} = \mathcal{P}(\Omega)$ and $\mathbb{P}(\{-1\}) = \mathbb{P}(\{0\}) = \mathbb{P}(\{1\}) = 1/3$. Consider

$$\mathcal{F} = \{\emptyset, \{1\}, \{-1, 0\}, \{-1, 0, 1\}\} \quad \text{and} \quad \mathcal{G} = \{\emptyset, \{-1\}, \{0, 1\}, \{-1, 0, 1\}\}.$$

and let $X : \Omega \to \mathbb{R}$, $X(\omega) = \omega$.

a) Calculate a version of

$$\mathbb{E}[X \mid \mathcal{F}] \quad \text{and} \quad \mathbb{E}[X \mid \mathcal{G}].$$

b) Calculate a version of

$$\mathbb{E}\big[\mathbb{E}[X \mid \mathcal{F}] \mid \mathcal{G}\big] \quad \text{and} \quad \mathbb{E}\big[\mathbb{E}[X \mid \mathcal{G}] \mid \mathcal{F}\big]$$

with the help of a).

Exercise 1.2 **(2 editing points)**

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra of $\mathcal{F}$.

a) Let $(X_n)_{n\in\mathbb{N}} \subset \mathcal{L}^1$ be a sequence of non-negative random variables. Use the conditional Monotone Convergence theorem for conditional expectations to prove the *conditional Lemma of Fatou*, i.e.

$$\mathbb{E}\big[\liminf_{n\to\infty} X_n \mid \mathcal{G}\big] \leq \liminf_{n\to\infty} \mathbb{E}\left[X_n \mid \mathcal{G}\right] \quad \mathbb{P}\text{-a.s.}$$

b) Let $X \in \mathcal{L}^r$, $r > 0$. Show the *conditional Markov inequality*, i.e. for all $\varepsilon > 0$ we have

$$\mathbb{P}\big[|X| \geq \varepsilon \mid \mathcal{G}\big] \leq \frac{\mathbb{E}\big[|X|^r \mid \mathcal{G}\big]}{\varepsilon^r} \quad \mathbb{P}\text{-a.s.}$$

For $X \in \mathcal{L}^2$ we define

$$\operatorname{Var}\big[X \,|\, \mathcal{G}\big] := \mathbb{E}\big[\,(X - \mathbb{E}\,[X \,|\, \mathcal{G}])^2 \,|\, \mathcal{G}\big].$$

Show the following statements:

c)
$$\operatorname{Var}[X \,|\, \mathcal{G}] = \mathbb{E}\left[X^2 \,|\, \mathcal{G}\right] - (\mathbb{E}[X \,|\, \mathcal{G}])^2 \quad \mathbb{P}\text{-a.s.},$$

d)
$$\operatorname{Var}[X] = \operatorname{Var}[\mathbb{E}[X \,|\, \mathcal{G}]] + \mathbb{E}[\operatorname{Var}[X \,|\, \mathcal{G}]] \quad \mathbb{P}\text{-a.s.}$$

Exercise 1.3 **(4 editing points)**

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. For a) - b) we assume $X, Y \in \mathcal{L}^1$ to be integrable random variables such that

$$\mathbb{E}[X \,|\, Y] = Y \ \mathbb{P}\text{-a.s.} \quad \text{and} \quad \mathbb{E}[Y \,|\, X] = X \ \mathbb{P}\text{-a.s.}$$

a) Show that for all $c \in \mathbb{R}$ we have
$$\mathbb{E}\big[(X - Y)\mathbb{1}_{\{Y \leq c < X\}}\big] = 0 = \mathbb{E}\big[(X - Y)\mathbb{1}_{\{X \leq c < Y\}}\big].$$

b) Show that $X = Y$ $\mathbb{P}$-a.s.

From now on we assume X, Y to be positive random variables, such that $\mathbb{P}\big(\frac{Y}{X} \in \{\frac{1}{2}, 2\}\big) = 1$ and $\mathbb{P}\big(X \in \{2^n : n \in \mathbb{N}_0\}\big) = 1$. We denote for all $n \in \mathbb{N}_0$

$$p_n := \mathbb{P}(X = 2^n, Y = 2^{n-1}), \quad q_n := \mathbb{P}(X = 2^n, Y = 2^{n+1}),$$

and we assume that $p_n + q_n > 0$ and $p_{n+1} + q_{n-1} > 0$ for all $n \in \mathbb{N}_0$.

c) Show that
$$\mathbb{E}\Big[\frac{Y}{X} \,|\, X = 2^n\Big] > 1 \ \text{ if and only if } \ q_n > \frac{1}{2} p_n$$
and
$$\mathbb{E}\Big[\frac{X}{Y} \,|\, Y = 2^n\Big] > 1 \ \text{ if and only if } \ p_{n+1} > \frac{1}{2} q_{n-1}.$$

d) Choose q_n, p_n, $n \in \mathbb{N}$, such that for these random variables X, Y we have
$$\mathbb{E}\big[X \,|\, Y\big] > Y \ \mathbb{P}\text{-a.s.} \quad \text{and} \quad \mathbb{E}\big[Y \,|\, X\big] > X \ \mathbb{P}\text{-a.s.}$$

e) Show that c) implies $\mathbb{E}[X] = \infty$.

Remark: Please write your name, Matrikel-number, group number and exercise number in the first row! If you need more than one paper, please staple all your sheets together!