Universität zu Köln WS 2019/2020
Institut für Mathematik
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To hand in: 23.10 during the exercise class

2. Exercise sheet Probability II

(Martingale, Doob decomposition, Quadratic variation)

Exercise 2.1 **(2 editing points)**

Let $X_1, X_2, \dots$ be a sequence of i.i.d random variables with

$$p := \mathbb{P}(X_1 = 1) = 1 - \mathbb{P}(X_1 = -1), \qquad p \in [0,1],$$

and denote by $S_0 = 0$, $S_n := \sum_{i=1}^n X_i$ the corresponding random walk and $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$, $n \in \mathbb{N}$.

a) Give an expression for a (a.s. uniquely determined) version of $\mathbb{E}[S_{n+1} \mid \mathcal{F}_n]$ and $\mathbb{E}\left[S_{n+1}^2 \mid \mathcal{F}_n\right]$, respectively.

b) For which values of p is $(S_n)_{n\in\mathbb{N}_0}$ and $(S_n^2)_{n\in\mathbb{N}_0}$, respectively, a martingale, submartingale, or supermartingale with respect to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$?

c) Find the quadratic variation process $(< S >_n)_{n\in\mathbb{N}_0}$ of $(S_n)_{n\in\mathbb{N}_0}$.

Exercise 2.2 **(2 editing points)**

Let $Y_1, Y_2, \dots$ be a sequence of i.i.d random variables with $\mathbb{P}(Y_1 = 1) = \mathbb{P}(Y_1 = -1) = \frac{1}{2}$, and let $X_0 = 0$ and for all $n \geq 1$

$$X_n = \begin{cases} Y_1 & \text{if } X_{n-1} = 0 \\ X_{n-1} + Y_n & \text{otherwise,} \end{cases}$$

that is $(X_n)_{n\in\mathbb{N}}$ behave like a random walk as long as it doesn't hit 0, and always jump as Y_1 just after hitting 0. We also define $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$.

a) Show that $(X_n)_{n\in\mathbb{N}_0}$ is an $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ adapted process with $\mathbb{E}[X_{n+1} \mid X_n] = X_n$ for all $n \in \mathbb{N}_0$.

b) Show that $(X_n)_{n\in\mathbb{N}_0}$ is not an $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ martingale.

c) Find a Doob decomposition for $(X_n)_{n\in\mathbb{N}_0}$, i.e. give an $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ martingale $(M_n)_{n\in\mathbb{N}_0}$ and an $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ adapted process $(A_n)_{n\in\mathbb{N}_0}$ such that $X_n = M_n + A_n$ for all $n \in \mathbb{N}_0$.

Exercise 2.3 **(4 editing points)**

Let X be a random variable in $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{G} \subset \mathcal{F}$ be a σ-Algebra, and $\varphi : \mathbb{R} \to \mathbb{R}$ be a strictly convex function.

a) Show that for any random variable X, $\mathbb{P}$-a.s,

$$\mathbb{P}(X = \mathbb{E}[X \,|\, \mathcal{G}] \,|\, \mathcal{G}) = 1 \quad \Longleftrightarrow \quad \mathbb{E}[\varphi(X) \,|\, \mathcal{G}] = \varphi(\mathbb{E}[X \,|\, \mathcal{G}])$$

Hint: You can use that for all $y \in \mathbb{R}$, there exists $m \in \mathbb{R}$ such $\varphi(x) > m(x-y) + \varphi(y)$ for all $x \neq y$.

Let $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ be a filtration, $(M_n)_{n\in\mathbb{N}_0}$ be an $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ martingale and $(Y_n)_{n\in\mathbb{N}_0}$ be a non-negative, bounded and $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$-adapted process.

b) Show that a quadratic variation of $(Y \bullet M)_{n\in\mathbb{N}_0}$ is given by

$$< (Y \bullet M) >_n = (Y^2 \bullet < M >)_n \text{ for all } n \in \mathbb{N}_0.$$

c) Let $(\tilde{Y}_n)_{n\in\mathbb{N}_0}$ be a non-negative, bounded and $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$-adapted process. Show that

$$(< (\tilde{Y} \bullet M) >_n)_{n\in\mathbb{N}_0} = (< (Y \bullet M) >_n)_{n\in\mathbb{N}_0} \ \mathbb{P}\text{-a.s.} \Leftrightarrow ((\tilde{Y} \bullet M)_n)_{n\in\mathbb{N}_0} = ((Y \bullet M)_n)_{n\in\mathbb{N}_0} \ \mathbb{P}\text{-a.s.}$$

d) Give an example of an $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ martingale $(M_n)_{n\in\mathbb{N}_0}$ and of two non-negative, bounded and $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$-adapted processes $(Y_n)_{n\in\mathbb{N}_0}$ and $(\tilde{Y}_n)_{n\in\mathbb{N}_0}$ such that

$$\mathbb{P}(M_n = 0 \text{ for all } n \in \mathbb{N}_0) < 1 \text{ and } \mathbb{P}(Y_n = \tilde{Y}_n \text{ for all } n \in \mathbb{N}_0) < 1,$$
$$\text{but } (Y \bullet M)_n = (\tilde{Y} \bullet M)_n \text{ for all } n \in \mathbb{N}_0.$$

Remark: Please write your name, Matrikel-number, group number and exercise number in the first row! If you need more than one paper, please staple all your sheets together!