Universität zu Köln WS 2019/2020
Institut für Mathematik
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To hand in: 30.10 during the exercise class

3. Exercise sheet Probability II

(Stopping time, optional stopping theorem)

Exercise 3.1 **(2 editing points)**

In this exercise we show Theorem 2.1.16. You can use that if τ is a stopping time, then $\tau \wedge k$ is a bounded stopping time for all $k \in \mathbb{N}$.

a) Let $\tau : \Omega \to \mathbb{N}_0$ be a (finite) stopping time w.r.t. $(\mathcal{F}_n)$ such that $\tau \in \mathcal{L}^1$. Let $(M_n)_{n\in\mathbb{N}_0}$ be a martingale w.r.t. $(\mathcal{F}_n)$ fulfilling $|M_n - M_{n-1}| \leq K$, $n \in \mathbb{N}$, for some constant $K > 0$. Show that $M_\tau \in \mathcal{L}^1$ and $\mathbb{E}[M_\tau] = \mathbb{E}[M_0]$.

b) A superstitious player tosses a fair coin, wins one euro on tail and loses one euro on head. To avoid bad luck, he stops exactly after the first time he flipped 66 times tail in a row. On average, does superstition help the player to win money at the end of the game?

c) A smarter (and infinitely rich) player first bets 1 euro, loses his bet on head and wins his bet on tail, then doubles the previous bet every time he tosses the coin, and stops the first time his total gain is larger than his total loss. Show that the smarter player always wins at the end of the game. Which hypothesis(es) of a) is (are) not fulfilled?

Exercise 3.2 **(2 editing points)**

Let $X_1, X_2, \ldots \in \mathcal{L}^1$ be a sequence of i.i.d. random variables with $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}$. Denote by $S_0 = 0$ and $S_n := \sum_{i=1}^n X_i$, $n \in \mathbb{N}$ the corresponding random walk, and by $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. We define the following random variables:

- $T_1 = \min\{n \geq 0 : S_n = 2019\}$,
- $T_2 = \min\{n \geq T_1 + 1 : S_n = 2019\} \wedge 10000$,
- $T_3 = \max\{0 \leq n \leq 2019 : S_n = 1\}$,
- $T_4 = \min\{n \geq 0 : S_n = S_{n-2020}\}$,
- $T_5 = \min\{n \geq 0 : S_n = S_{n+2020}\}$,
- $T_6 = \min\{0 \leq n \leq 2019 : S_n \leq S_m \text{ for all } m \in \{0, \ldots, 2019\}\}$,

with the convention $\min\{\varnothing\} = \infty$ and $\max\{\varnothing\} = 0$.

a) For each $i \in \{1, \dots, 6\}$, prove or refute that T_i is a stopping time w.r.t. $(\mathcal{F}_n)$.

b) For each $i \in \{1, \dots, 6\}$, prove or refute that $\mathbb{E}[S_{T_i}] = 0$.
Hint: you can use without proof that $\mathbb{P}(T_1 < \infty) = 1$, see Example 2.1.15.b).

Exercise 3.3 **(4 editing points)**

Let $X_1, X_2, \ldots \in \mathcal{L}^1$ be a sequence of i.i.d. random variables and denote $S_n := \sum_{i=1}^n X_i$ the corresponding random walk. Further let $\tau : \Omega \to \mathbb{N}$ be a (finite) stopping time w.r.t. $(\sigma(S_n))$.

a) Show that, if τ is bounded, then

$$\mathbb{E}\left[S_\tau\right] = \mathbb{E}\left[\tau\right] \mathbb{E}\left[X_1\right].$$

b) Show that, if $X_1 \geq 0$ $\mathbb{P}$-a.s or if $\tau \in \mathcal{L}^1$, then

$$\mathbb{E}\left[S_\tau\right] = \mathbb{E}\left[\tau\right] \mathbb{E}\left[X_1\right].$$

We now assume that $\mathbb{P}(X_1 = 1) = 1 - \mathbb{P}(X_1 = -1) = p > \frac{1}{2}$ and we take $\tau_m = \inf\{n \in \mathbb{N}_0 : S_n \geq m\}$, $m \in \mathbb{N}$. Show that

c) $\mathbb{P}(\tau_m < \infty) = 1$,
Hint: You can use the the weak law of large number.

d) $\mathbb{E}[\tau_m \mathbb{1}_{\tau_m \leq k}] \leq \frac{m}{2p-1}$ for all $k \in \mathbb{N}$,

e) $\mathbb{E}[\tau_m] = \frac{m}{2p-1}$.

Remark: Please write your name, Matrikel-number, group number and exercise number in the first row! If you need more than one paper, please staple all your sheets together!