**Universität zu Köln** WS 2019/2020
Institut für Mathematik
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*To hand in: 06.11 during the exercise class*

# 4. Exercise sheet Probability II

(Doob's submartingale- and $\mathcal{L}^1$-inequality, martingale convergence, uniform integrability)

## Exercise 4.1 **(2 editing points)**

Let $(X_n)_{n\in\mathbb{N}_0}$ be a $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ martingale or a non-negative $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ submartingale and $|X|_n^* := \max_{0\leq k\leq n} |X_k|$.

a) Show the inequality

$$\mathbb{E}\big[|X|_n^*\big] \leq 1 + \mathbb{E}\big[|X_n| \ln\big(|X|_n^*\big) \mathbb{1}_{|X|_n^* \geq 1}\big],$$

where $0 \ln 0 := 0$.

b) Show the inequalities

$$x \ln y \leq x \ln x + \frac{y}{e} \quad \forall x, y > 0,$$

and

$$\mathbb{E}\big[|X|_n^*\big] \leq \frac{e}{e-1}\big(1 + \mathbb{E}\big[|X_n| \cdot \ln(|X_n|) \mathbb{1}_{|X_n|\geq 1}\big]\big).$$

## Exercise 4.2 **(4 editing points)**

Let $(X_n)_{n\in\mathbb{N}_0}$ be an $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ submartingale such that $\sup_{n\geq 0} \mathbb{E}[(X_n)^+] < \infty$, and let $M_n = \lim_{m\to\infty} \mathbb{E}[X_m^+ \,|\, \mathcal{F}_n]$.

a) Show that $(M_n)_{n\in\mathbb{N}_0}$ is well defined, i.e. $\lim_{m\to\infty} \mathbb{E}[X_m^+ \,|\, \mathcal{F}_n]$ exists, and is a non-negative $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ martingale.

b) Show that there exists random variables $M_\infty$ and $X_\infty$ in $\mathcal{L}^1$ such that

$$X_n \underset{n\to\infty}{\longrightarrow} X_\infty \text{ and } M_n \underset{n\to\infty}{\longrightarrow} M_\infty \ \mathbb{P}\text{-a.s.}$$

c) Assume that there exists a non-negative random variable $Y \in \mathcal{L}^1$ such that $X_n^+ \leq Y$ for all $n \in \mathbb{N}$. Show that

$$M_n = \mathbb{E}[X_\infty^+ \,|\, \mathcal{F}_n].$$

d) Let $(Y_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence of random variables with $\mathbb{P}(Y_1 = 1) = 1 - \mathbb{P}(Y_1 = -1) = \frac{1}{2}$, $S_n = \sum_{i=1}^n Y_i$ the associated random walk, and take $X_n = S_{n\wedge\tau_{-1}}$, where $\tau_{-1} = \inf\{n \in \mathbb{N} : S_n = -1\}$. Show that $\sup_{n\geq 0} \mathbb{E}[(X_n)^+] < \infty$ is verified, but

$$M_n \neq \mathbb{E}[X_\infty^+ \,|\, \mathcal{F}_n].$$

e) Assume that, either there exists a non-negative random variable $Y \in \mathcal{L}^1$ such that $X_n^+ \leq Y$ for all $n \in \mathbb{N}$, or $(X_n)_{n\in\mathbb{N}_0}$ is a non-negative martingale. Show that

$$M_\infty = X_\infty^+.$$

## Exercise 4.3 **(2 editing points)**

Let $\mathcal{S} \subset \mathcal{L}^1$ be an $L^1$-bounded family. Show that the property of $\mathcal{S}$ to be uniformly integrable is equivalent to each of the following conditions:

a) For every sequence $(A_n) \subset \mathcal{F}$ with $A_1 \supset A_2 \supset \ldots$ and $\mathbb{P}\big(\bigcap_{n=1}^\infty A_n\big) = 0$ we have

$$\lim_{n\to\infty} \left( \sup_{X\in\mathcal{S}} \mathbb{E}\left[|X|\mathbb{1}_{A_n}\right] \right) = 0;$$

b) for every sequence $(B_n) \subset \mathcal{F}$ of pairwise disjoint sets we have

$$\lim_{n\to\infty} \left( \sup_{X\in\mathcal{S}} \mathbb{E}\left[|X|\mathbb{1}_{B_n}\right] \right) = 0.$$

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**Remark:** Please write your name, Matrikel-number, group number and exercise number in the first row! If you need more than one paper, please staple all your sheets together!