Universität zu Köln WS 2019/2020
Institut für Mathematik
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To hand in: 13.11. during the exercise class

5. Exercise sheet Probability II

(Martingale convergence theorems)

Exercise 5.1 **(4 editing points)**

Let $(M_n)_{n\in\mathbb{N}_0} \subset \mathcal{L}^2$ be a martingale w.r.t. $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ with $M_0 = 0$ $\mathbb{P}$-a.s, $(< M >_n)_{n\in\mathbb{N}_0}$ its square variation process and $\tau_x := \inf\{n \in \mathbb{N}_0 : < M >_{n+1} > x^2\}$.

a) For each $m \in \mathbb{N}$, prove the inequality

$$\mathbb{P}\Big(\sup_{0\le n\le m} |M_n| > x\Big) \le \mathbb{P}\left(< M >_m > x^2\right) + \mathbb{P}\Big(\sup_{0\le n\le m} |M_{n\wedge\tau_x}| > x\Big).$$

b) Show that for all $m \in \mathbb{N}$

$$\mathbb{P}\big(\sup_{0\le n\le m} |M_{n\wedge\tau_x}| \ge x\big) \le x^{-2}\mathbb{E}\Big[< M >_m \wedge\, x^2\Big] \quad \forall m \in \mathbb{N}_0 \text{ and } x > 0$$

c) Show that for all $m \in \mathbb{N}$

$$\int_{(0,\infty)} x^{-2}\mathbb{E}\Big[< M >_m \wedge\, x^2\Big]\,\mathrm{d}x = 2\mathbb{E}\Big[< M >_m^{1/2}\Big].$$

d) Show that for all $m \in \mathbb{N}$

$$\mathbb{E}\Big[\sup_{0\le n\le m} |M_n|\Big] \le 3\mathbb{E}\Big[< M >_m^{1/2}\Big].$$

e) Let $(X_n)_{n\in\mathbb{N}} \subset \mathcal{L}^2$ be i.i.d. random variables with $\mathbb{E}[X_1] = 0$ and $S_n := \sum_{i=1}^n X_i$. Let $\tau : \Omega \to \mathbb{N}$ be a stopping time with $\tau \in \mathcal{L}^{1/2}$. Show that $S_\tau \in \mathcal{L}^1$ and $\mathbb{E}[S_\tau] = 0$.

Exercise 5.2 **(4 editing points)**

Prove or refute the following statements.

a) If (M_n) is a martingale, then there exists a random variable M_∞ such that

$$M_n \underset{n\to\infty}{\longrightarrow} M_\infty \ \mathbb{P}\text{-a.s.}$$

b) If (M_n) is a square integrable martingale such that $< M >_\infty \in \mathcal{L}^1$, then there exists a random variable $M_\infty \in \mathcal{L}^2$ such that

$$M_n^2 - < M >_n \underset{n\to\infty}{\longrightarrow} M_\infty^2 - < M >_\infty \ \mathbb{P}\text{-a.s. and in } \mathcal{L}^1.$$

c) If (M_n) is an $\mathcal{L}^1$-bounded martingale, then there exists a random variable $M_\infty \in \mathcal{L}^1$ such that

$$M_n \underset{n\to\infty}{\longrightarrow} M_\infty \text{ in } \mathcal{L}^1.$$

d) If (M_n) is a non-negative $\mathcal{L}^p$-bounded submartingale, $p > 1$, then there exists a random variable $M_\infty \in \mathcal{L}^p$ such that

$$M_n \underset{n\to\infty}{\longrightarrow} M_\infty \ \mathbb{P}\text{-a.s. and in } \mathcal{L}^p.$$

e) If (M_n) is an $\mathcal{L}^p$-bounded submartingale, $p > 1$, then there exists a random variable $M_\infty \in \mathcal{L}^p$ such that

$$M_n \underset{n\to\infty}{\longrightarrow} M_\infty \ \mathbb{P}\text{-a.s. and in } \mathcal{L}^p.$$

Hint: Take $M_n = -e^{S_n - an}$, where (S_n) is the simple random walk on $\mathbb{Z}$, and choose $a > 0$ in a smart way.

f) If (M_n) is an $\mathcal{L}^p$-bounded submartingale, $p > 1$, then there exists a random variable $M_\infty \in \mathcal{L}^p$ such that

$$M_n \underset{n\to\infty}{\longrightarrow} M_\infty \ \mathbb{P}\text{-a.s. and in } \mathcal{L}^q \text{ for all } q < p.$$

Remark: Please write your name, Matrikel-number, group number and exercise number in the first row! If you need more than one paper, please staple all your sheets together!