Universität zu Köln WS 2019/2020
Institut für Mathematik
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To hand in: 20.11. during the exercise class

6. Exercise sheet Probability II

(Generalisation of law of large numbers and central limit theorem, backwards martingales)

Exercise 6.1 **(2 editing points)**

Prove Lemma 2.3.24: Let (X_n) and (Y_n) be two sequences of random variables satisfying the following conditions:

- There exists $c \in \mathbb{R}$ such that $X_n \xrightarrow{\mathbb{P}} c$ as $n \to \infty$;
- the sequences (Y_n) and $(X_n Y_n)$ are both uniformly integrable;
- $\mathbb{E}[Y_n] \longrightarrow 1$ as $n \to \infty$.

a) Show that for all $\varepsilon > 0$

$$\mathbb{E}[X_n Y_n \mathbb{1}_{|X_n - c| \geq \varepsilon}] \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{and} \qquad \mathbb{E}[Y_n \mathbb{1}_{|X_n - c| \geq \varepsilon}] \underset{n\to\infty}{\longrightarrow} 0.$$

b) Show that

$$\mathbb{E}[X_n Y_n] \underset{n\to\infty}{\longrightarrow} c.$$

Exercise 6.2 **(4 editing points)**

In a) and c) you are not allowed to use previously known laws of large numbers.

a) Let $(X_n) \subset \mathcal{L}^1$ be an i.i.d. sequence of random variables. Show the strong law of large numbers:

$$\frac{1}{n}\sum_{i=1}^{n} X_i \underset{n\to\infty}{\longrightarrow} \mathbb{E}[X] \ \mathbb{P}\text{-a.s. and in } \mathcal{L}^1.$$

Hint: Use backward martingales as well as Kolmogorov's 0-1-law.

b) Let $(X_n) \subset \mathcal{L}^2$ be a pairwise uncorrelated sequence of random variables such that for all $i \in \mathbb{N}$

$$\frac{1}{\alpha_n^2}\sum_{i=1}^{n} \mathrm{Var}(X_i) \underset{n\to\infty}{\longrightarrow} 0,$$

for some sequence $(\alpha_i)_{i\in\mathbb{N}}$ of positive numbers. Show that

$$\frac{1}{\alpha_n}\sum_{i=1}^{n} (X_i - \mathbb{E}[X_i]) \underset{n\to\infty}{\longrightarrow} 0 \ \text{ in probability and in } \mathcal{L}^2.$$

Hint: You can use the weak law of large numbers (Theorem 3.6.5 in the script of WTI).

c) Let $(X_n) \subset \mathcal{L}^2$ be an independent sequence of random variables such that for all $i \in \mathbb{N}$

$$\sum_{i=1}^{\infty} \frac{\mathrm{Var}(X_i)}{\alpha_i^2} < \infty,$$

for some positive, non-decreasing sequence $(\alpha_i)_{i\in\mathbb{N}}$ such that $\alpha_n \underset{n\to\infty}{\longrightarrow} \infty$. Show that

$$\frac{1}{\alpha_n} \sum_{i=1}^{n} (X_i - \mathbb{E}[X_i]) \underset{n\to\infty}{\longrightarrow} 0 \ \mathbb{P}\text{-a.s. and in } \mathcal{L}^2.$$

Hint: Study $M_n = \sum_{i=1}^{n} \frac{X_i - \mathbb{E}[X_i]}{\alpha_i}$ and use Kronecker's lemma: if $(u_k)_{k\geq 0}$ and $(b_k)_{k\geq 0}$ are two sequences such that $b_k > 0$ for all $k \in \mathbb{N}$, $\sum_{k\geq 1} u_k$ exists and is finite and b_k increases to ∞, then $\frac{1}{b_n}\sum_{k=1}^{n} u_k b_k \underset{n\to\infty}{\longrightarrow} 0$.

d) Let (Y_n) be an independent sequence of random variables such that $\mathbb{P}(Y_n = \frac{1}{\sqrt{n}}) = \mathbb{P}(Y_n = -\frac{1}{\sqrt{n}}) = \mathbb{P}(Y_n = 0) = \frac{1}{3}$. Show that

$$\frac{1}{\log(n)} \sum_{i=1}^{n} Y_i \underset{n\to\infty}{\longrightarrow} 0 \ \mathbb{P}\text{-a.s. and in } \mathcal{L}^2,$$

and that

$$\frac{1}{\sqrt{\log(n)}} \sum_{i=1}^{n} Y_i \underset{n\to\infty}{\longrightarrow} \mathcal{N}\Big(0, \frac{2}{3}\Big) \text{ in law.}$$

Exercise 6.3 **(2 editing points)**

Let (X_n), $n \in -\mathbb{N}_0$, be a sequence of random variables with $X_{-n} \longrightarrow X_{-\infty}$ $\mathbb{P}$-a.s. for $n \to +\infty$ and $|X_{-n}| \leq Z$ $\mathbb{P}$-a.s. for all n, where $Z \in \mathcal{L}^1$. Let $(\mathcal{F}_n)$, $n \in -\mathbb{N}_0$, be a decreasing sequence of sub-σ-algebras, i.e. $\mathcal{F}_{-n} \downarrow \mathcal{F}_{-\infty}$ for $n \to +\infty$, and let $W_N := \sup\{|X_{-n} - X_{-m}|,\ n, m \geq N\}$ for all $N \in \mathbb{N}_0$.

a) Show that for all $N \in \mathbb{N}_0$

$$\limsup_{n\to+\infty} \mathbb{E}\left[|X_{-n} - X_{-\infty}| \,|\, \mathcal{F}_{-n}\right] \leq \lim_{n\to+\infty} \mathbb{E}\left[W_N \,|\, \mathcal{F}_{-n}\right].$$

b) Show that

$$\mathbb{E}\left[X_{-n} \mid \mathcal{F}_{-n}\right] \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[X_{-\infty} \mid \mathcal{F}_{-\infty}\right] \ \mathbb{P}\text{-a.s.}$$

Remark: Please write your name, Matrikel-number, group number and exercise number in the first row! If you need more than one paper, please staple all your sheets together!