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To hand in: 27.11. during the exercise class

7. Exercise sheet Probability II

(Discrete Black-Sholes formula and Itô formula, regular conditional probability)

Exercise 7.1 **(2 editing points)**

Consider the situation in section 2.5 for $N = 1$. We let $C_a := \big((1+a)S_0 - K\big)^+$ and $C_b := \big((1+b)S_0 - K\big)^+$, $a < r < b$.
Find the hedging strategy for the option $(S_1 - K)^+$, i.e. find (A_1, V_1) such that $X_1 = (S_1 - K)^+$.
Show that for this strategy we have

$$X_0 = \frac{(r-a)C_b + (b-r)C_a}{(b-a)(1+r)}.$$

Hint: Negative values of A_1 and V_1 are permitted and mean that the respective shares are borrowed (without interest).

Exercise 7.2 **(2 editing points)**

Let (X_n) be a $\mathbb{Z}$-valued process.

a) Show that $\mathbb{P}(X_{n+1} - X_n \in \{-1, 1\}) = 1$ for all $n \in \mathbb{N}$ if and only if for all $f : \mathbb{Z} \to \mathbb{R}$ and $n \in \mathbb{N}$

$$f(X_n) = (f'(X) \bullet X)_n + f(X_0) + \frac{1}{2}\sum_{k=0}^{n-1} f''(X_k) \text{ } \mathbb{P}\text{-a.s.}$$

b) Assume that $(X_n)_{n\in\mathbb{N}}$ is a martingale with $\mathbb{P}(X_{n+1} - X_n \in \{-1, 1\}) = 1$ for all $n \in \mathbb{N}$, and that $f : \mathbb{Z} \to \mathbb{R}$ is such that $(f(X_n))$ is a submartingale. Show that for all $n \in \mathbb{N}$

$$f(X_n) \geq (f'(X) \bullet X)_n + f(X_0) \text{ } \mathbb{P}\text{-a.s.}$$

Exercise 7.3 **(2 editing points)**

Let $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}^1([0,1]), \lambda^1([0,1]))$ and $\sup(F) = \inf\{y \in \Omega : y \geq x \text{ for all } x \in F\}$ is the usual supremum for $F \in \mathcal{F}$, with the convention $\inf \emptyset = 0$. Let

$$\tilde{\mu} : \Omega \times \mathcal{F} \to \mathbb{R},\ \tilde{\mu}(\omega, F) := \mathbb{1}_{F \cup \{\sup(F)\}}(\omega).$$

Show that

a) for every $F \in \mathcal{F}$ the mapping $\omega \mapsto \tilde{\mu}(\omega, F)$ defines a version of the conditional probability $\mathbb{P}(F \mid \mathcal{F})$,

b) $\tilde{\mu}$ is not a regular conditional probability given $\mathcal{F}$,

c) a regular conditional probability given $\mathcal{F}$ exists and give an explicit expression for it.

Exercise 7.4 **(2 editing points)**

Let ν_1 and ν_2 be two measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, X, Y be real random variables, such that the random vector (X, Y) has probability density f w.r.t. $\nu_1 \otimes \nu_2$. For all $x, y \in \mathbb{R}$, let

$$f_X(x) = \int f(x,y)\, \mathrm{d}\nu_2(y) \quad \text{and} \quad f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)},$$

and define for all $A \in \mathcal{B}(\mathbb{R})$

$$\mu(x, A) := \begin{cases} \int_A f_{Y|X}(y|x)\ \mathrm{d}\nu_2(y), & \text{if } f_X(x) \in (0, \infty), \\ 0, & \text{otherwise.} \end{cases}$$

In this exercise, you can already use the result from Theorem 3.1.8.

a) Show that $\mu(X, \cdot)$ is a regular conditional distribution of Y given X, and that for every measurable function $g : \mathbb{R} \to \mathbb{R}$

$$\mathbb{E}[g(Y) \mid X] = \mathbb{1}_{f_X(X) \in (0,\infty)} \int g(y) f_{Y|X}(y|X)\, \mathrm{d}\nu_2(y)\ \mathbb{P}\text{-a.s.}$$

b) Let (X, Y) be uniformly distributed on the disc

$$\mathbb{D} := \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\right\}.$$

Compute the regular conditional distribution of Y given X and $\mathbb{E}[Y^2 \mid X]$.

Remark: Please write your name, Matrikel-number, group number and exercise number in the first row! If you need more than one paper, please staple all your sheets together!