Universität zu Köln WS 2019/2020
Institut für Mathematik
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To hand in: 04.12. during the exercise class

8. Exercise sheet Probability II

(Regular conditional distribution)

Exercise 8.1 **(2 editing points)**

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra, $(E, \mathcal{E})$ be a measurable space, $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (E, \mathcal{E})$ a random variable and μ a regular conditional distribution of X knowing $\mathcal{G}$. Furthermore, let $Y : (\Omega, \mathcal{G}) \to (E', \mathcal{E}')$ be $\mathcal{G} - \mathcal{E}'$-measurable and $f : (E \times E', \mathcal{E} \otimes \mathcal{E}') \to (\mathbb{R}, \mathcal{B})$ be measurable such that $\mathbb{E}[|f(X,Y)|] < \infty$.

a) Show that

$$\mathbb{E}\big[f(X,Y)\,|\,\mathcal{G}\big](\omega) = \int_E f(x, Y(\omega))\mu(\omega,\, \mathrm{d}x)$$

for $\mathbb{P}$-almost all $\omega \in \Omega$.

b) Let $(\Omega, \mathcal{F}, \mathbb{P}) = (\mathbb{D}, \mathcal{B}(\mathbb{D}), \mathbb{P})$ with $\mathbb{D}$ as in Exercise 7.4, $\mathbb{P}$ the uniform distribution on $\mathbb{D}$ and

$$\mathcal{G}_{\text{rot}} := \{B \in \mathcal{B}(\mathbb{D}) :\ \{(x,y) \in B \text{ and } x^2 + y^2 = u^2 + v^2\} \Longrightarrow \{(u,v) \in B\}\}.$$

the σ-algebra of measurable sets invariant by rotation. For $(x,y) \in \mathbb{D}$, let $Y(x,y) := x^2 + y^2$ and $X(x,y) := \arg((x,y)) \in [0, 2\pi)$ be the angle between the vectors $(\mathbf{0}, (x,y))$ and $(\mathbf{0}, (1,0))$.

Show that Y is $\mathcal{G}_{\text{rot}}$-measurable and that for every f, such that $\mathbb{E}[|f(X,Y)|] < \infty$ we have

$$\mathbb{E}\big[f(X,Y)\,|\,\mathcal{G}_{\text{rot}}\big](\omega) = \frac{1}{2\pi}\int_0^{2\pi} f(x, Y(\omega))\,\mathrm{d}x.$$

Exercise 8.2 **(4 editing points)**

Let $(S, \mathcal{O})$ be a Polish space, $X, Y : (\Omega, \mathcal{F}, \mathbb{P}) \mapsto (S, \mathcal{O})$ be random variables, and $\mathbb{P}_{(X,Y)}$ the law of (X,Y).

a) Let us define the projections $\widetilde{X}, \widetilde{Y} : (S^2, \mathcal{O}^{\otimes 2}, \mathbb{P}_{(X,Y)}) \to (S, \mathcal{O})$ by $\widetilde{X}(x,y) = x$ and $\widetilde{Y}(x,y) = y$ for all $x, y \in S$. Show that there exists a regular conditional distribution $\widetilde{\mu}$ of $\widetilde{Y}$ given $\widetilde{X}$ such that $\widetilde{\mu}((x,y), F) = \widetilde{\mu}((x,y'), F)$ for all $x, y, y' \in S$ and $F \in \mathcal{O}$.

b) Show that there exists $\mu : S \times \mathcal{B}(S) \to [0,1]$ such that

- $F \mapsto \mu(x, F)$ is a probability measure for all $x \in A_\mu$, with $\mathbb{P}_X(A_\mu) = 1$,
- $\omega \mapsto \mu(X(\omega), F)$ is a version of $\mathbb{P}(Y \in F \,|\, X)$ for all $F \in \mathcal{O}$.

Hint: You can assume without proof, that for some $\widetilde{\mu}$ as in a), there exists a set $\widetilde{A}_{\widetilde{\mu}} \in \sigma(\widetilde{X})$ such that $\mathbb{P}_{(X,Y)}(\widetilde{A}_{\widetilde{\mu}}) = 1$ and $\widetilde{\mu}((x,y),\cdot)$ is a probability for all $(x,y) \in \widetilde{A}_{\widetilde{\mu}}$.

Let $g : (S, \mathcal{O}) \mapsto (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a measurable function such that $g(Y) \in \mathcal{L}^1$. A function $F : S \mapsto \mathbb{R}$ is said to be a version of $\mathbb{E}[g(Y) \,|\, X = x]$ if $F(X)$ is a version of $\mathbb{E}[g(Y) \,|\, X]$.

c) Show that for all μ as in b),

$$F(x) = \mathbb{1}_{x \in A_\mu} \int g(y)\mu(x, \mathrm{d}y)$$

is a version of $\mathbb{E}[g(Y) \,|\, X = x]$.

d) Assume that $S = \mathbb{R}$, and that (X, Y) has density f w.r.t. $\nu_1 \otimes \nu_2$, for some measures ν_1, ν_2 on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and recall the notation from Exercise 7.4. Show that

$$F(x) = \mathbb{1}_{f_X(x) \in (0,\infty)} \int g(y) f_{Y|X}(y|x) \,\mathrm{d}\nu_2(y)$$

is a version of $\mathbb{E}[g(Y) \,|\, X = x]$.

Exercise 8.3 **(2 editing points)**

Let $(E_i, \mathcal{E}_i)$, $i \in \{0, \dots, 3\}$ be measurable spaces, and for each $i \in \{1, 2, 3\}$, let μ_i be a transition kernel from $(\times_{j=0}^{i-1} E_j, \otimes_{j=0}^{i-1} \mathcal{E}_j)$ to $(E_i, \mathcal{E}_i)$.

a) Show that for all measurable function $f : E_1 \times E_2 \to \mathbb{R}$ such that $f \in \mathcal{L}^1$

$$\int_{E_1 \times E_2} (\mu_1 \otimes \mu_2)(x_0, \mathrm{d}x) f(x) = \int_{E_1} \mu_1(x_0, \mathrm{d}x_1) \int_{E_2} \mu_2((x_0, x_1), \mathrm{d}x_2) f(x_1, x_2),$$

b) Show that $\mu_1 \otimes (\mu_2 \otimes \mu_3) = (\mu_1 \otimes \mu_2) \otimes \mu_3$.

c) Let $\mathbb{P}_1$ be a probability on $(E_1, \mathcal{E}_1)$, $\mathbb{P}_2$ be a probability on $(E_2, \mathcal{E}_2)$, and let $\mu(x_1, F) = \mathbb{P}_2(F)$ for all $x_1 \in E_1$ and $F \in \mathcal{E}_2$. Show that $\mathbb{P}_1 \otimes \mu = \mathbb{P}_1 \otimes \mathbb{P}_2$.
Hint: we recall that the product measure $\mathbb{P}_1 \otimes \mathbb{P}_2$ was defined in Proposition 2.4.3. in Probability I.

Remark: Please write your name, Matrikel-number, group number and exercise number in the first row! If you need more than one paper, please staple all your sheets together!