Universität zu Köln
Institut für Mathematik
Dozent: Dr. P. Gracar
Assistenten: A. Prévost, L. Schmitz

WS 2019/2020

To hand in: 11.12. during the exercise class

9. Exercise sheet Probability II

(Markov processes, indistinguishable modifications, Ornstein-Uhlenbeck process)

Exercise 9.1 **(2 editing points)**

Let T be a subgroup of $\mathbb{R}$, and $(X_t)_{t\in T}$ be a stochastic process taking values on a Polish space $(S, \mathcal{B}(S))$. Denote $\mathcal{F}_t = \sigma(X_s : 0 \leq s \leq t)$ and $\mathcal{G}_t := \sigma(X_u : u \geq t)$ the σ-algebra of information generated by the process *until* time t and *after* time t, respectively. Show that the following conditions are equivalent:

i) $(X_t)_{t\in T}$ has the *Markov property*, i.e. for all $s, t \in T$, $s \leq t$, and $C \in \mathcal{B}(S)$

$$\mathbb{P}(X_t \in C \mid \mathcal{F}_s) = \mathbb{P}(X_t \in C \mid X_s) \quad \mathbb{P}\text{-a.s.}$$

ii) For all $t \in T$ and $B \in \mathcal{G}_t$

$$\mathbb{P}(B \mid \mathcal{F}_t) = \mathbb{P}(B \mid X_t) \quad \mathbb{P}\text{-a.s.}$$

iii) For all $t \in T$, $f \in \mathcal{L}^1$ $\mathcal{F}_t$-measurable and $g \in \mathcal{L}^1$ $\mathcal{G}_t$-measurable

$$\mathbb{E}[fg \mid X_t] = \mathbb{E}[f \mid X_t]\,\mathbb{E}[g \mid X_t] \quad \mathbb{P}\text{-a.s.}$$

Hint: You can use that $\mathcal{G}_t$ is generated by the set

$$\mathcal{A}_t := \left\{ X_{u_1}^{-1}(C_1) \cap \ldots \cap X_{u_n}^{-1}(C_n) : \ t \leq u_1 \leq \ldots \leq u_n, \ n \in \mathbb{N}, \ C_i \in \mathcal{B}(S) \right\}.$$

Exercise 9.2 **(2 editing points)**

Let $(X_t)_{t\in\mathbb{R}}$ and $(Y_t)_{t\in\mathbb{R}}$ be stochastic processes on $(\Omega, \mathcal{F}, \mathbb{P})$ with state space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and assume that $(X_t)_{t\in\mathbb{R}}$ and $(Y_t)_{t\in\mathbb{R}}$ are modifications with right-continuous sample paths, i.e. for $\mathbb{P}$-almost all $\omega \in \Omega$

$$t \mapsto X_t(\omega) \quad \text{and} \quad t \mapsto Y_t(\omega)$$

are right-continuous functions.
Show that $(X_t)_{t\in\mathbb{R}}$ and $(Y_t)_{t\in\mathbb{R}}$ are indistinguishable, i.e. there exists an event $A \in \mathcal{F}$ such that $A \subset \{(X_t)_{t\in\mathbb{R}} = (Y_t)_{t\in\mathbb{R}}\}$ and $\mathbb{P}(A) = 1$.

Exercise 9.3 **(4 editing points)**

Let us fix $\theta, \sigma \in (0,\infty)$. An $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$-valued process $(X_t)_{t\geq 0}$ is called *Ornstein–Uhlenbeck process* starting at $x \in \mathbb{R}$ and with parameters (θ, σ) if

i) $X_0 = x$ with probability 1,

ii) for any $0 \leq s < t$, $X_t - X_s e^{-\theta(t-s)}$ is independent of $\mathcal{F}_s = \sigma(X_u, u \leq s)$,

iii) for any $0 \leq s < t$, $X_t - X_s e^{-\theta(t-s)}$ is $\mathcal{N}\big(0, \frac{\sigma^2}{2\theta}(1 - e^{-2\theta(t-s)})\big)$-distributed,

iv) $\mathbb{P}$-a.s, the mapping $t \mapsto X_t$ is continuous.

a) Let $\mu_t : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0,1]$, $t \in [0,\infty)$, be defined by $\mu_0(x,\cdot) := \delta_x$ and

$$\mu_t(x,B) := \mathbb{P}\Big(\mathcal{N}\big(xe^{-\theta t}, \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t})\big) \in B\Big), \quad x \in \mathbb{R},\ B \in \mathcal{B}(\mathbb{R}),\ t > 0.$$

Show that $(\mu_t)_{t\geq 0}$ is a Markov semigroup of transition kernels from $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

b) Show that there exists a family of probability measures $(\mathbb{P}_x)_{x\in\mathbb{R}}$ on a measurable space $(\Omega, \mathcal{F})$ and a Markov process $(X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P}_x)$, $x \in \mathbb{R}$, such that for all $x \in \mathbb{R}$, under $\mathbb{P}_x$, the properties i), ii) and iii) of an Ornstein-Uhlenbeck process starting at x and with parameters (θ, σ) are verified.
Hint: Use Exercise 8.1.a)

c) Show that for all $x \in \mathbb{R}$, there exists a modification $(\tilde{X}_t)_{t\geq 0}$ from the process $(X_t)_{t\geq 0}$ from b), such that, under $\mathbb{P}_x$, $(\tilde{X}_t)_{t\geq 0}$ is an Ornstein-Uhlenbeck process starting at $x \in \mathbb{R}$ and with parameters (θ, σ), and such that, $\mathbb{P}_x$-a.s, $(\tilde{X}_t)_{t\geq 0}$ is locally γ-Hölder continuous for all $\gamma \in (0, \frac{1}{2})$.

d) Show that if $(Y_t)_{t\geq 0}$ is an Ornstein-Uhlenbeck process starting at $x \in \mathbb{R}$ and with parameters (θ, σ) under some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, it has the same law as $(X_t)_{t\geq 0}$ under $\mathbb{P}_x$, where $(X_t)_{t\geq 0}$ is as in b).

Ankündigung!

Am Montag, den 9.12.2019, um 19:30 Uhr
treffen wir uns am Glühweinstand
auf dem Weihnachtsmarkt am Rudolfplatz
(neben dem Hahnentor).

Wir freuen uns über rege Teilnahme :-)

Remark: Please write your name, Matrikel-number, group number and exercise number in the first row! If you need more than one paper, please staple all your sheets together!