Universität zu Köln WS 2019/2020
Institut für Mathematik
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To hand in: 18.12. during the exercise class

10. Exercise sheet Probability II

(Poisson process, Polya's statement)

Exercise 10.1 **(4 editing points)**

Let $\kappa : [0,\infty) \to [0,\infty)$ be a continuous and non-negative function. We say that a process $(X_t)_{t\geq 0}$ taking values in $\mathbb{N}_0$ is an *inhomogeneous Poisson process* (IPP) with rate κ if and only if

- $(X_t)_{t\geq 0}$ is càdlàg and has independent and non-negative increments,
- $\mathbb{P}(X_0 = 0) = 1$,
- for all $n \in \mathbb{N}_0$ and all $0 \leq s < t$ we have

$$\mathbb{P}(X_t - X_s = n) = \frac{\left(\int_{[s,t]} \kappa(r)\,\mathrm{d}r\right)^n}{n!} \exp\Big\{ - \int_{[s,t]} \kappa(r)\,\mathrm{d}r \Big\}.$$

a) Let $m(t) := \int_{[0,t]} \kappa(s)\,\mathrm{d}s$ and $(N_t)_{t\geq 0}$ be a Poisson process with rate 1. Define $\widetilde{N}_t := N_{m(t)}$, $t \geq 0$. Show that $(\widetilde{N}_t)_{t\geq 0}$ is an inhomogeneous Poisson process with rate κ.

b) Show that an inhomogeneous Poisson process with rate κ has stationary increments if and only if κ is constant.

c) Assume that κ is constant. Let μ_t, $t \in [0,\infty)$, be the Markov semigroup of transition kernels from Example 3.2.2.b), that is $\mu_t(n,B) = \mathbb{P}(\mathrm{Poi}(\kappa t) + n \in B)$ for all $n \in \mathbb{N}_0$ and $B \in 2^{\mathbb{N}_0}$. Show that, if $\mathbb{P}_n$, $n \in \mathbb{N}_0$, and $(Y_t)_{t\geq 0}$, describes a Markov process associated to μ_t, $t \in [0,\infty)$, then $(Y_t)_{t\geq 0}$ has the same distribution under $\mathbb{P}_0$ as any (inhomogeneous) Poisson process $(X_t)_{t\geq 0}$ with rate κ.

d) Show that a stochastic process $(X_t)_{t\geq 0}$ taking values in $\mathbb{N}_0$ is an inhomogeneous Poisson process with rate κ if and only if

   i) $(X_t)_{t\geq 0}$ is càdlàg and has independent and non-negative increments,
   ii) $\mathbb{P}(X_0 = 0) = 1$,
   iii) $\mathbb{P}(X_{t+h} - X_t = 1) = \kappa(t)h + o(h)$ as $h \downarrow 0$ for all $t \geq 0$ and
   iv) $\mathbb{P}(X_{t+h} - X_t \geq 2) = o(h)$.

   Hint: Show that in both cases for all $n \in \mathbb{N}_0$ and all $0 \leq s < t$ we have

$$\frac{\mathrm{d}\mathbb{P}(X_t - X_s = n)}{\mathrm{d}t} = \kappa(t)\big(\mathbb{P}(X_t - X_s = n-1) - \mathbb{P}(X_t - X_s = n)\big).$$

Exercise 10.2 **(4 editing points)**

Let $d \in \mathbb{N}$ and $L : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}^{\mathbb{Z}^d}$, be the generator of the discrete time d-dimensional simple random walk (SRW) $(X_n)_{n\in\mathbb{N}_0}$. For $A \subset \mathbb{Z}^d$ we define $\tau_A := \inf\{n \in \mathbb{N}_0 : X_n \in A\}$ the first entrance time in A.

a) Let $d \geq 3$ and for $\alpha \in (0, d-2)$ we define $V_\alpha(x) := \|x\|_2^{-\alpha}$. Show, applying the convergence theorem for a suitable supermartingale, that for some $R > 0$ big enough and all $x \in B(0,R)^c$ we have

$$\mathbb{P}_x\big(\tau_{B(0,R)} < \infty\big) \leq V_\alpha(x/R).$$

Hint: You can use without a proof that $(LV_\alpha)(x) = \frac{\alpha}{2d}\|x\|_2^{-\alpha-2}\big(\alpha + 2 - d + o(1)\big)$ as $\|x\|_2 \to \infty$.

b) Let $d \in \{1,2\}$ and define $V(x) := \ln\|x\|_2 - \frac{1}{\|x\|_2}$, $x \neq 0$. Show, applying the convergence theorem for a suitable supermartingale, that for some $r > 0$ large enough and all $x \in \mathbb{Z}^d$ we have

$$\mathbb{P}_x(\tau_{B(0,R)^c} < \tau_{B(0,r)}) \xrightarrow[R\to\infty]{} 0.$$

Hint: You can use without a proof that $(LV)(x) = \frac{d-2}{\|x\|_2^2} + \frac{d-3}{\|x\|_2^3} + o\Big(\frac{1}{\|x\|_2^3}\Big)$ as $\|x\|_2 \to \infty$.

c) Show that the SRW on $\mathbb{Z}^d$ is recurrent if and only if $d \in \{1,2\}$, i.e.

$$\{\text{for all } x \in \mathbb{Z}^d,\ \mathbb{P}_x(\tau_{\{0\}} < \infty) = 1\} \iff d \in \{1,2\}.$$

Hint: You can use without a proof, that if $A \subset \mathbb{Z}^d$ is a finite set such that $\mathbb{P}_x(\tau_A < \infty) = 1$ for all $x \in \mathbb{Z}^d$, then $\mathbb{P}_x(\tau_{\{y\}} < \infty) = 1$ for all $y \in A$ and $x \in \mathbb{Z}^d$.

Remark: Please write your name, Matrikel-number, group number and exercise number in the first row! If you need more than one paper, please staple all your sheets together!