Universität zu Köln WS 2019/2020

Institut für Mathematik
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To hand in: 8.1. during the exercise class

11. Exercise sheet Probability II

(Doob's h-transform, measure-preserving and ergodic transformations)

Exercise 11.1 **(4 editing points)**

Let $h(x) = x$ for all $x \in \mathbb{Z}$, P be the transition matrix associated to the one-dimensional simple random walk on $\mathbb{Z}$, and P^h the Doob's h-transform of P. We also define under some probability $\mathbb{P}_y$, $y \in \mathbb{Z}$, the Markov process associated to the simple random walk on $\mathbb{Z}$, and under some probabilities $\mathbb{P}^h_y$, $y \in \mathbb{Z}$, the Markov process $(X^h_n)_{n\in\mathbb{N}}$ associated to the transition matrix P^h. For each $x \in \mathbb{N}$, let $\tau_x = \inf\{n \in \mathbb{N}_0 : X^h_n = x\}$, with the convention $\inf \varnothing = \infty$.

Show that

a) for all $M > 0$ and $x, y \in \{1, \dots, M\}$, $\mathbb{P}^h_x(X^h_1 = y) = \mathbb{P}_x(X_1 = y \,|\, \tau_M < \tau_0)$ (we say that $(X_n)_{n\in\mathbb{N}}$ is a simple random walk conditioned on not hitting 0),

b) for all $x, y, z \in \mathbb{N}$, $x < y < z$, $\mathbb{P}^h_y(\tau_x < \tau_z) = \frac{x(z-y)}{y(z-x)}$,

c) for all $x, y \in \mathbb{N}$, $\mathbb{P}^h_y(\tau_x < \infty) = \frac{x}{y} \wedge 1$,

d) for all $x \in \mathbb{N}$, $\mathbb{P}^h_x(\forall n \in \mathbb{N}, X_n \neq x) = \frac{1}{2x}$.

Please turn over.

Christmas Bonus Exercise

Exercise 11.2 **(4 bonus points)**

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\theta : \Omega \to \Omega$ be a measurable function and $p \in \mathbb{N} \setminus \{1\}$. Define two other measurable function $T, S : \Omega \to \Omega$, which have the property

$$T = \theta^p \qquad \text{and} \qquad \theta = S^p.$$

We also define I_θ, I_S and I_T as the σ-algebras of θ, S or T invariant sets, and we take $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Prove or refute the following statements

a) if S is measure-preserving, then θ is measure-preserving;

b) if T is measure-preserving, then θ is measure-preserving;

c) if S is mixing, then θ is mixing;

d) if θ is measure-preserving and T is mixing, then θ is mixing;

e) if S is ergodic, then θ is ergodic;

f) if θ is measure-preserving and T is ergodic, then θ is ergodic;

g) if T is measure-preserving, then $\mathbb{P}$-a.s.

$$\frac{1}{n}\sum_{k=0}^{n-1} X \circ \theta^k \underset{n\to\infty}{\longrightarrow} \frac{1}{p}\sum_{k=0}^{p-1} \mathbb{E}\big[X \circ \theta^k \,|\, \mathcal{I}_T\big];$$

h) if S is ergodic, then $\mathbb{P}$-a.s.

$$\mathbb{E}[X] = \frac{1}{p}\sum_{k=0}^{p-1} \mathbb{E}\big[X \circ S^k \,|\, \mathcal{I}_\theta\big].$$

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Merry Christmas !

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Remark: Please write your name, Matrikel-number, group number and exercise number in the first row! If you need more than one paper, please staple all your sheets together!