Universität zu Köln WS 2019/2020
Institut für Mathematik
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To hand in: 15.1. during the exercise class

12. Exercise sheet Probability II

(Ergodicity, subbaditive ergodic theorem)

Throughout the exercise sheet, for a measure-preserving transformation θ, you can replace the definition of the σ-algebra of θ-invariant sets $\mathcal{I}$ from Definition 4.0.7. by

$$\mathcal{I}^* := \{A \in \mathcal{F} : \ \mathbb{P}(A \triangle \theta^{-1}(A)) = 0\},$$

where Δ denotes the symmetric difference, and use any result from the lecture, up to $\mathbb{P}$-a.s. modifications, with $\mathcal{I}^*$ instead of $\mathcal{I}$ without justification.

Exercise 12.1 (2 editing points)

Let θ be an ergodic transformation and let $A \in \mathcal{F}$. Define the $\mathcal{F} - 2^{\mathbb{N}\cup\{\infty\}}$-measurable random variable

$$\eta_A(\omega) := \inf\{n \geq 1 : \ \theta^n(\omega) \in A\},$$

where $\inf(\emptyset) := \infty$. Show that

a) $\{\eta_A = \infty\} \subset \theta^{-1}(\{\eta_A = \infty\})$;

b) $\eta_A < \infty$ $\mathbb{P}$-a.s. for all $A \in \mathcal{F}$ such that $\mathbb{P}(A) > 0$;

c) for all $A \in \mathcal{F}$ and $n \in \mathbb{N}$, taking $A_n := A \cap \{\eta_A = n\}$ and $B_n := A^c \cap \{\eta_A = n\}$, we have
$$\mathbb{P}(B_n) = \mathbb{P}(A_{n+1}) + \mathbb{P}(B_{n+1}).$$

d) for all $A \in \mathcal{F}$ such that $\mathbb{P}(A) > 0$ we have
$$\mathbb{E}[\eta_A \mid A] = \frac{1}{\mathbb{P}(A)}.$$

Exercise 12.2 (2 editing points)

Let $(Y_n)_{n\in\mathbb{N}_0} \subset \mathcal{L}^1$ be a sequence of integrable random variables and θ be a measure-preserving transformation such that for all $m, n \in \mathbb{N}_0$ with $0 \leq m \leq n$ we have

$$Y_n \leq Y_m + Y_{n-m} \circ \theta^m \quad \text{and} \quad \inf_{n\in\mathbb{N}} \frac{\mathbb{E}[Y_n]}{n} > -\infty.$$

Show that

a) there exists a random variable Y_∞ such that
$$\frac{Y_n}{n} \underset{n\to\infty}{\longrightarrow} Y_\infty \quad \mathbb{P}\text{-a.s. and in } L^1 \text{ and } \mathbb{E}[Y_\infty] = \lim_{n\to\infty} \frac{\mathbb{E}[Y_n]}{n};$$

b) Y_∞ is $\mathcal{I}^* - \mathcal{B}(\mathbb{R})$-measurable, and, if θ is ergodic, then Y_∞ is constant $\mathbb{P}$-a.s.

Exercise 12.3 **(4 editing points)**

Let us fix some $d \geq 2$, and for some $x \in \mathbb{Z}^d$ we denote by x_i, $i \in \{1, \dots, d\}$, the i-th coordinate of x, by $e_i = (0, \dots 0, 1, 0, \dots, 0)$ the i-th coordinate vector, and we simply write $(i, 0)$ for the vector with first coordinate equal to i, and all other coordinates equal to 0. Let $E = \{\{x, y\} : x = y + e_p \text{ for some } p \in \{1, \dots, d\}\}$ be the set of edges of $\mathbb{Z}^d$, $(\Omega, \mathcal{F}) = ([0, \infty)^E, \mathcal{B}([0, \infty))^{\otimes E})$, and $X_e : (\Omega, \mathcal{F}) \to ([0, \infty), \mathcal{B}([0, \infty))$ the projection on the e coordinate, $e \in E$. We also define for all $i \in \mathbb{Z}$ the set of edges E_i on the right and hitting the hyperplane $\{x_1 = i\}$ by

$$E_i = \{\{x, y\} \in E : x_1 = i, y = x + e_p \text{ for some } p \in \{1, \dots, d\}\},$$

and $\theta : \Omega \to \Omega$ the transformation such that $X_e \circ \theta = X_{e+e_1}$ for all $e \in E$, where we take $\{x, y\} + e_1 = \{x + e_1, y + e_1\}$. Let $\mathbb{P}$ be a probability on $(\Omega, \mathcal{F})$, such that $X_e \in \mathcal{L}^1$ for all $e \in E$, and the sequence of columns $(X_e)_{e \in E_i}$, $i \in \mathbb{Z}$, is stationary. For all $n \in \mathbb{N}$, we define

$$Y_n = \inf_{\pi:(0,0) \to (n,0)} \sum_{e \in \pi} X_e,$$

where the infimum is taken over all finite and connected path π of edges starting in $(0, 0)$ and ending in $(n, 0)$. In other words, Y_n is the length of the shortest path between $(0, 0)$ and $(n, 0)$, when each edge e has length X_e.

a) Show that θ is measure-preserving and that for all $0 \leq m \leq n$,

$$Y_{n-m} \circ \theta^m = \inf_{\pi:(m,0) \to (n,0)} \sum_{e \in \pi} X_e.$$

b) Show that there exists a random variable Y_∞ such that

$$\frac{Y_n}{n} \underset{n \to \infty}{\longrightarrow} Y_\infty \quad \mathbb{P}\text{-a.s. and in } L^1.$$

From now on, we assume that the sequence $(X_e)_{e \in E_i}$, $i \in \mathbb{Z}$, is i.i.d, and we say that an event $A \in \mathcal{F}$ is finite-dimensional if there exists a finite set $K \subset \mathbb{Z}^d$ such that $A \in \sigma(X_e, e \in K)$.

c) Show that for all $A \in \mathcal{F}$ and $n \in \mathbb{N}$, there exists a finite-dimensional event $A_n \in \mathcal{F}$ such that $\mathbb{P}(A \Delta A_n) \leq \frac{1}{n}$.
Hint: You can use without a proof that for all events $A_i, B_i \in \mathcal{F}$, $i \in \mathbb{N}$,

$$\Big(\bigcup_{i \in \mathbb{N}} A_i\Big) \Delta \Big(\bigcup_{i \in \mathbb{N}} B_i\Big) \subset \bigcup_{i \in \mathbb{N}} (A_i \Delta B_i).$$

d) Show that for any event $A \in \mathcal{F}$ with $\theta^{-1}(A) = A$, A_n as in c) and $p \in \mathbb{N}$

$$|\mathbb{P}\big(A_n \cap \theta^{-p}(A_n)\big) - \mathbb{P}(A)| \leq \frac{2}{n}.$$

e) Show that θ is ergodic, and so

$$Y_\infty = \lim_{n \to \infty} \frac{\mathbb{E}[Y_n]}{n} \ \mathbb{P}\text{-a.s.}.$$

Remark: Please write your name, Matrikel-number, group number and exercise number in the first row! If you need more than one paper, please staple all your sheets together!