Universität zu Köln WS 2019/2020
Institut für Mathematik
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To hand in: 22.1. during the exercise class

13. Exercise sheet Probability II

(Gaussian vectors, Donsker's theorem)

(Last sheet)

Exercise 13.1 (2 editing points)

Recall that for $d \in \mathbb{N}$ the random variable $(X_k)_{k\in\{1,\dots,d\}}$ is a d-dimensional Gaussian random vector (GRV) if and only if there exist $\boldsymbol{\mu} \in \mathbb{R}^d$ and a positive definite matrix $\Sigma \in \mathbb{R}^{d\times d}$ such that for all $\boldsymbol{a} = (a_1, \dots, a_d)^\intercal \in \mathbb{R}^d$ we have $\mathbb{E}[e^{i\sum_{i=1}^d a_i X_i}] = e^{i\boldsymbol{a}^\intercal\boldsymbol{\mu} - \frac{1}{2}\boldsymbol{a}^\intercal\Sigma\boldsymbol{a}}$.

a) Show that $(X_k)_{k\in\{1,\dots,d\}}$ is a d-dimensional GRV if and only if for all $(a_1, \dots, a_d)^\intercal \in \mathbb{R}^d \setminus \{0\}$ the sum $\sum_{k=1}^d a_k X_k$ is a (non-zero) Gaussian random variable (or 1-dimensional GRV).

b) Let $(Y_k)_{k\in\{1,\dots,d\}}$ and $(Z_k)_{k\in\{1,\dots,d\}}$ be independent d-dimensional GRVs and define $(X_k)_{k\in\{1,\dots,2d\}}$ by

$$X_{2k-1} := \frac{1}{2}Y_k + Z_k,\ k \in \{1, \dots, d\},$$
$$X_{2k} := \frac{1}{2}Y_k - Z_k,\ k \in \{1, \dots, d\}.$$

Show that $(X_k)_{k\in\{1,\dots,2d\}}$ is a $2d$-dimensional GRV.

c) As in (5.1.3) in the script we recall $D_n := \{k2^{-n} :\ k \in \{0, \dots, 2^n\}\}$, $n \in \mathbb{N}_0$, $D := \bigcup_n D_n$, $B_0 := 0$, $B_1 := X_1$ and for all $n \in \mathbb{N}$ and $t \in D_n \setminus D_{n-1}$

$$B_t := \frac{B_{t-2^{-n}} + B_{t+2^{-n}}}{2} + 2^{-\frac{n+1}{2}} \cdot X_t = \frac{B_{t+2^{-n}} - B_{t-2^{-n}}}{2} + B_{t-2^{-n}} + 2^{-\frac{n+1}{2}} \cdot X_t,$$

where $(X_t)_{t\in D}$ is an i.i.d. family of standard Gaussian random variables, $X_t \sim \mathcal{N}(0,1)$, $t \in D$. Show that for all $n \in \mathbb{N}$ the vector of the increments $(B_{t+2^{-n}} - B_t)_{t\in D_n\setminus\{1\}}$ is a GRV.

Exercise 13.2 (2 editing points)

Let $(X_i)_{i\in\mathbb{N}}$ be an i.i.d. family of centered random variables with variance $\sigma^2 \in (0,\infty)$, let $S_n = \sum_{i=1}^n X_i$ be the associated random walk, and $(B_t)_{t\geq 0}$ be a 1-dimensional Brownian motion.

a) Show that for all $0 \leq s < t$

$$\frac{S_{\lfloor nt\rfloor - \lfloor ns\rfloor}}{\sigma\sqrt{n}} \Longrightarrow B_t - B_s.$$

b) Let $p \in \mathbb{N}$, and, for each $n \in \mathbb{N} \cup \{\infty\}$, $(X_1^{(n)}, \dots, X_p^{(n)})$ be a vector of random variables, such that $X_i^{(n)} - X_{i-1}^{(n)}$, $i \in \{1, \dots, p\}$, are independent, where $X_0^{(n)} = 0$. Show that

$$(X_1^{(n)}, \dots, X_p^{(n)}) \Longrightarrow (X_1^{(\infty)}, \dots, X_p^{(\infty)})$$
$$\text{if and only if}$$
$$X_i^{(n)} - X_{i-1}^{(n)} \Longrightarrow X_i^{(\infty)} - X_{i-1}^{(\infty)} \text{ for all } i \in \{1, \dots, p\}.$$

c) Show that for all $p \in \mathbb{N}$ and $t_1 < \dots < t_p$

$$\left(\frac{S_{\lfloor nt_1 \rfloor}}{\sigma\sqrt{n}}, \dots, \frac{S_{\lfloor nt_p \rfloor}}{\sigma\sqrt{n}} \right) \Longrightarrow (B_{t_1}, \dots, B_{t_p}).$$

Exercise 13.3 **(4 editing points)**

Let $(B_t)_{t\geq 0}$ be a 1-dimensional Brownian motion and for each $p \in \mathbb{N}$ let us define $\tau_0^{(p)} = 0$ and $S_0^{(p)} = 0$, and, recursively on $n \in \mathbb{N}$,

$$\tau_n^{(p)} = \inf \left\{ t > \tau_{n-1}^{(p)} : B_t \in \frac{1}{p}\mathbb{Z} \setminus \{S_{n-1}^{(p)}\} \right\} \text{ and } S_n^{(p)} = B_{\tau_n^{(p)}},$$

the print of $(B_t)_{t\geq 0}$ on $\frac{1}{p}\mathbb{Z}$, where $\frac{1}{p}\mathbb{Z} = \{\frac{k}{p}, k \in \mathbb{Z}\}$. You can use without a proof the strong Markov property for Brownian motion: for all $p \in \mathbb{N}$ and $n \in \mathbb{N}$, if $\mathbb{P}(\tau_n^{(p)} < \infty) = 1$, then the process $(B_t^{(n,p)})_{t\geq 0} := (B_{t+\tau_n^{(p)}} - S_n^{(p)})_{t\geq 0}$ is a Brownian motion independent of $(\tau_1^{(p)}, \dots, \tau_n^{(p)})$ and $(S_1^{(p)}, \dots, S_n^{(p)})$.

a) For each $p \in \mathbb{N}$, show that $\mathbb{P}(\tau_n^{(p)} < \infty) = 1$ for all $n \in \mathbb{N}$, and that $(\tilde{S}_n^{(p)})_{n\in\mathbb{N}} := (pS_n^{(p)})_{n\in\mathbb{N}}$ is a simple random walk on $\mathbb{Z}$.

b) Show that $\mathbb{P}$-a.s, for all $a \in \mathbb{R}$, there exists $t > 0$ such that $B_t = a$.

c) Show that for all $p \in \mathbb{N}$, $p^2(\tau_i^{(p)} - \tau_{i-1}^{(p)})$, $i \in \mathbb{N}$, is a sequence of i.i.d. random variable with the same law as $\tau_1^{(1)}$.

d) Show that $\mathbb{P}$-a.s.

$$\frac{\tilde{S}_{\lfloor p^2 t \rfloor}^{(p)}}{p} \underset{p\to\infty}{\longrightarrow} B_t \text{ for all } t \geq 0.$$

Hint: you can use without a proof that $\mathbb{E}[\tau_1^{(1)}] = 1$.

Remark: Please write your name, Matrikel-number, group number and exercise number in the first row! If you need more than one paper, please staple all your sheets together!