Universität zu Köln WS 2019/2020
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Solution for
0. Exercise sheet Probability II

(To work on during the second week)

Exercise 0.1

a) Compute the characteristic function of a random variable X with distribution $\mathrm{Bin}(n, p)$, for $n \in \mathbb{N}$ and $p \in (0, 1)$.

b) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables, such that the law of X_n is $\mathrm{Uni}(1, \ldots, n)$ for all $n \in \mathbb{N}$. What is the probability that $X_n = 1$ infinitely often?

c) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables, such that the law of X_n is $\mathrm{Geo}(\frac{\lambda}{n})$ for all $n \in \mathbb{N}$, $\lambda > 0$. Show that

$$\frac{X_n}{n} \quad \Longrightarrow \quad X \sim \mathrm{Exp}(\lambda).$$

d) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables, so that the density of X_n is $f_n(x) = 2xn^2e^{-x^2n^2}\mathbb{1}_{(0,\infty)}(x)$ for all $n \in \mathbb{N}$, $\lambda > 0$. Show that

$$X_n \xrightarrow{\mathbb{P}} 0.$$

Lösung.

a) We have

$$\begin{aligned}
\varphi_X(t) = \mathbb{E}[\exp(iXt)] &= \sum_{k=0}^{n} \binom{n}{k} p^k(1-p)^{n-k}\exp(ikt) \\
&= \sum_{k=0}^{n} \binom{n}{k} \big(\exp(it)p\big)^k (1-p)^{n-k} \\
&= \big(pe^{it} + 1 - p\big)^n.
\end{aligned}$$

b) For all $n \in \mathbb{N}$ we have

$$\mathbb{P}(X_n = 1) = \text{Uni}_{\{1,\dots,n\}}(\{1\}) = \frac{1}{n}$$

and thus

$$\sum_{n\in\mathbb{N}} \mathbb{P}(X_n = 1) = \sum_{n\in\mathbb{N}} \frac{1}{n} = \infty.$$

Since $(X_n)_{n\in\mathbb{N}}$ is an independent family, we have by Borel-Cantelli lemma

$$\mathbb{P}(X_n = 1 \text{ infinitely often}) = \mathbb{P}\left(\limsup_{n\to\infty}\{X_n = 1\}\right) = 1.$$

c) We have for all $x > 0$

$$\begin{aligned}\mathbb{P}\left(\frac{1}{n}X_n \leq x\right) &= 1 - \mathbb{P}(X_n > nx) \\ &= 1 - \mathbb{P}(X_n > \lfloor nx \rfloor) \\ &= 1 - \left(1 - \frac{\lambda}{n}\right)^{\lfloor nx \rfloor} = 1 - \left((1 - \frac{\lambda}{n})^n\right)^{\frac{\lfloor nx \rfloor}{n}} \\ &\underset{n\to\infty}{\longrightarrow} 1 - \left(e^{-\lambda}\right)^x = 1 - e^{-x\lambda},\end{aligned}$$

since $\frac{xn-1}{n} = x - \frac{1}{n} < \frac{\lfloor nx \rfloor}{n} \leq x$. For $x \leq 0$ we have $\mathbb{P}(\frac{1}{n}X_n \leq x) = 0$. Thus the claim follows.

d) Let $\varepsilon > 0$, we have

$$\begin{aligned}\mathbb{P}(|X_n - 0| \geq \varepsilon) = 1 - \mathbb{P}(X_n \in [0,\varepsilon)) &= 1 - \int_0^\varepsilon 2tn^2 e^{-t^2n^2}\mathrm{d}t \\ &= 1 + \left[e^{-t^2n^2}\right]_0^\varepsilon = e^{-\varepsilon^2 n^2} \underset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

Exercise 0.2

Let $(X_n)_{n\in\mathbb{N}}$ and X be non-negative random variables in $\mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$, such that $X_n \underset{n\to\infty}{\longrightarrow} X$ $\mathbb{P}$-a.s. In the first two questions, we assume that $p \in \mathbb{N}$.

a) Show that if $\|X_n - X\|_p \underset{n\to\infty}{\longrightarrow} 0$, then $\|X_n\|_p \underset{n\to\infty}{\longrightarrow} \|X\|_p$.

b) Show that

$$\int \left((X_n - X)^p\right)^- \,\mathrm{d}\mathbb{P} \underset{n\to\infty}{\longrightarrow} 0.$$

c) Show that, if $p = 1$,

$$\|X_n\|_1 \underset{n\to\infty}{\longrightarrow} \|X\|_1 \qquad \Longleftrightarrow \qquad \|X_n - X\|_1 \underset{n\to\infty}{\longrightarrow} 0.$$

d) If $p = \infty$, does the following equivalence always hold:

$$\|X_n\|_\infty \xrightarrow[n\to\infty]{} \|X\|_\infty \qquad \Longleftrightarrow \qquad \|X_n - X\|_\infty \xrightarrow[n\to\infty]{} 0 \ ?$$

Lösung.

a) By Minkowski Inequality we have

$$\|X_n\|_p = \|X_n - X + X\|_p \le \|X_n - X\|_p + \|X\|_p.$$

Similarly we have $\|X\|_p \le \|X_n - X\|_p + \|X_n\|_p$, and thus $|\|X_n\|_p - \|X\|_p| \le \|X_n - X\|_p$, and we can conclude.

b) Since $\big((X_n - X)^p\big)^- \le |X_n - X|^p \mathbb{1}_{X > X_n} \le X^p$ and $X \in \mathcal{L}^p$, we have have by the dominated convergence theorem

$$\int \big((X_n - X)^p\big)^- \, \mathrm{d}\mathbb{P} \to 0.$$

c) We have

$$\begin{aligned}\int (X_n - X)^+ \, \mathrm{d}\mathbb{P} &= \int_{X_n \ge X} (X_n - X) \, \mathrm{d}\mathbb{P} \\ &= \int X_n \, \mathrm{d}\mathbb{P} - \int X \, \mathrm{d}\mathbb{P} - \int_{X_n < X} (X_n - X) \, \mathrm{d}\mathbb{P}.\end{aligned}$$

But

$$\left| \int_{X_n < X} (X_n - X) \, \mathrm{d}\mathbb{P} \right| \le \left| \int (X_n - X)^- \, \mathrm{d}\mathbb{P} \right| \to 0,$$

and thus $\lim_{n\to\infty} \int X_n \, \mathrm{d}\mathbb{P} = \int X \, \mathrm{d}\mathbb{P}$ and b) give

$$\int (X_n - X)^+ \, \mathrm{d}\mathbb{P} \to 0.$$

Combining this with a) and b), we can conclude.

d) No: By definition we have $\|X\|_\infty = \inf\{M \in \mathbb{R} : \mathbb{P}(X \ge M) = 0\}$. We take $\Omega = \mathbb{N}$, $\mathcal{F} = 2^{\mathbb{N}}$, $\mathbb{P} = \mathrm{Poi}(1)$, $X_n = \mathbb{1}_{\{1,n\}}$ and $X = \mathbb{1}_{\{1\}}$. Then we have $X_n \xrightarrow[n\to\infty]{} X$ $\mathbb{P}$-a.s. But $\mathbb{P}(X \ge M) = 0$ for all $M > 1$ and $\mathbb{P}(X \ge 1) = \mathrm{Poi}(1)(\{1\}) > 0$, so $\|X\|_\infty = 1$. Analogously, $\|X_n\|_\infty = \|X\|_\infty = 1$. But we have $\|X_n - X\|_\infty = \|\mathbb{1}_n\|_\infty = 1$.

Exercise 0.3

Let $X : (\Omega, \mathcal{F}) \to (E, \mathcal{E})$ and $Y : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be two random variables. The goal of this exercise is to show that:

Y is $\sigma(X)$-measurable $\Leftrightarrow \exists$ a $\mathcal{E} - \mathcal{B}(\mathbb{R})$-measurable function $f : E \to \mathbb{R}$ such that $Y = f \circ X$.

a) Show $\Leftarrow$.

b) Show $\Rightarrow$ when Y is a simple non-negative function.

c) Show $\Rightarrow$ when Y is non-negative.

d) Show $\Rightarrow$.

Lösung.

a) We have for all $B \in \mathcal{B}(\mathbb{R})$

$$Y^{-1}(B) = (f \circ X)^{-1}(B) = X^{-1}(\underbrace{f^{-1}(B)}_{\in \mathcal{E}}) \in \sigma(X),$$

and thus Y is $\sigma(X)$-measurable.

b) Let us assume that $Y = \sum_{i=1} a_i \mathbb{1}_{A_i}$, where $A_i \in \sigma(X)$, and let $B_i \in \mathcal{E}$ be such that $X^{-1}(B_i) = A_i$. Defining

$$f = \sum_{i=1}^{n} a_i \mathbb{1}_{B_i},$$

we have that f is $\mathcal{E} - \mathcal{B}(\mathbb{R})$-measurable and

$$f \circ X = \sum_{i=1}^{n} a_i \mathbb{1}_{X \in B_i} = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i} = Y.$$

c) We first assume that X is non-negative. By the lecture "Wahrscheinlichkeitstheory I", there exists a sequence of $\sigma(X) - \mathcal{B}(\mathbb{R})$-measurable simple functions $(Y_n)_{n \in \mathbb{N}}$, increasing to Y, that we can explicitly write as

$$Y_n := \sum_{k=0}^{n2^n - 1} \frac{k}{2^n} 1_{\{\frac{k}{2^n} \leq Y < \frac{k+1}{2^n}\}} + n 1_{Y \geq n}.$$

By b), there exists for all $n \in \mathbb{N}$ a $\mathcal{E} - \mathcal{B}(\mathbb{R})$-measurable function f_n such that $Y_n = f_n \circ X$. Let us define

$$f(x) := \mathbb{1}_{\limsup_{n \to \infty} f_n(x) < \infty} \limsup_{n \to \infty} f_n(x).$$

Since the limsup of measurable function is measurable, we clearly have that f is $\mathcal{E} - \mathcal{B}(\mathbb{R})$-measurable. Since $f_n \circ X = Y_n \underset{n \to \infty}{\longrightarrow} Y$, we also have $Y = f(X)$.

d) We write $Y = Y^+ - Y^-$, then there exists two $\sigma(X) - \mathcal{B}(R)$-measurable functions f^+ and f^- such that

$$Y^+ = f^+ \circ X \quad \text{und} \quad Y^- = f^- \circ X.$$

Taking $f = f^+ - f^-$, we clear have that f is $\sigma(X) - \mathcal{B}(R)$-measurable and that $Y = f \circ X$.