Maximum principles, a start.

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Contents

1 Preliminaries

Let Ω be an open connected set in $\mathbb{R}^n$ with boundary $\partial\Omega = \bar{\Omega} \cap (\mathbb{R}^n \backslash \Omega)$. Let L be the second order differential operator:

$$L = \sum_{i,j=1}^{n} a_{ij}(x) D_{ij} + \sum_{i=1}^{n} b_i(x) D_i + c(x) \tag{1}$$

with $a_{ij} \in L^\infty_{loc}(\Omega)$ and $b_i, c \in L^\infty(\Omega)$. Here we have used $D_i = \frac{\partial}{\partial x_i}$ and $D_{ij} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}$. Without loss of generality one assumes $a_{ij} = a_{ji}$.

Definition 1 *We will fix the following notions.*

- *The operator L is called elliptic on Ω if for every $x \in \Omega$ there is $\lambda(x) > 0$ such that*

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda(x)\,|\xi|^2 \text{ for all } \xi \in \mathbb{R}^n.$$

- *The operator L is called strictly elliptic on Ω if there is $\lambda > 0$ such that*

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda\,|\xi|^2 \text{ for all } \xi \in \mathbb{R}^n \text{ and } x \in \Omega.$$

- *The operator L is called uniformly elliptic on Ω if there are $\Lambda, \lambda > 0$ such that*

$$\lambda\,|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda\,|\xi|^2 \text{ for all } \xi \in \mathbb{R}^n \text{ and } x \in \Omega.$$

Remark 2 *These definitions are not uniform throughout the literature. However, if the a_{ij} are bounded on $\bar{\Omega}$ then strictly elliptic implies uniformly elliptic and most references then agree.*

Remark 3 *The assumption $a_{ij} \in L^{\infty}_{loc}(\Omega)$ is to weak to expect even solutions of $Lu = f \in C^{\infty}\left(\bar{\Omega}\right)$ to satisfy $u \in C^2(\Omega)$ and for that reason one usually assumes a_{ij} to be more regular. The maximum principle however does not need a_{ij} to be continuous.*

Some notations that we will use are as follows. For $r > 0$ and $y \in \mathbb{R}^n$ we will write an open ball by

$$B_r(y) = \{x \in \mathbb{R}^n ; |x - y| < r\}.$$

For a function u we will use u^+, u^- which are defined by

$$\begin{aligned} u^+(x) &= \max(0, u(x)), \\ u^-(x) &= \max(0, -u(x)). \end{aligned}$$

2 Classical Maximum Principles

Lemma 4 *Suppose that L is elliptic and that $c \leq 0$. If $u \in C^2(\Omega)$ and $Lu > 0$ in Ω, then u cannot attain a nonnegative maximum in Ω.*

Proof. Suppose that u has a nonnegative maximum in $x_0 \in \Omega$. Then

$$\sum_{i=1}^{n} b_i(x) D_i u(x_0) + c(x) u(x_0) \leq 0.$$

Moreover, we may diagonalize the symmetric matrix by $\mathsf{T}(a_{ij}(x_0))\mathsf{T}^* = \mathsf{D}$ with T^* the transpose of T. Notice that $d_{ii} \geq \lambda(x_0) > 0$. One finds with $y = \mathsf{T}x$ and $U(y) = u(x)$ that

$$\begin{aligned} \sum_{i,j=1}^{n} a_{ij}(x_0) D_{ij} u(x_0) &= \sum_{i,j=1}^{n} a_{ij}(x_0) \sum_{\ell=1}^{n} \mathsf{T}_{\ell i} \sum_{k=1}^{n} \mathsf{T}_{kj} \left(\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j} U\right)(\mathsf{T}x_0) \\ &= \sum_{i=1}^{n} d_{ii} \frac{\partial^2 U}{\partial y_i^2}(\mathsf{T}x_0). \end{aligned}$$

Since U has a maximum at $\mathsf{T}x_0$ we have $\frac{\partial^2}{\partial y_i^2} U(\mathsf{T}x_0) \leq 0$ for all i, implying

$$\sum_{i=1}^{n} d_{ii} \frac{\partial^2 U}{\partial y_i^2}(\mathsf{T}x_0) \leq 0.$$

Hence $Lu(x_0) \leq 0$, a contradiction. ■

Theorem 5 (Weak Maximum Principle) *Suppose that Ω is bounded and that L is strictly elliptic with $c \leq 0$. If $u \in C^2(\Omega) \cap C\left(\bar{\Omega}\right)$ and $Lu \geq 0$ in Ω, then a nonnegative maximum is attained at the boundary.*

Proof. Suppose that $\Omega \subset \{|x_1| < d\}$. Consider $w(x) = u(x) + \varepsilon\, e^{\alpha x_1}$ with $\varepsilon > 0$. Then

$$\begin{aligned} Lw &= Lu + \varepsilon \left(\alpha^2 a_{11}(x) + \alpha\, b_1(x) + c(x)\right) e^{\alpha x_1} \\ &\geq \varepsilon \left(\alpha^2 \lambda - \alpha \|b_1\|_\infty - \|c\|_\infty\right) e^{\alpha x_1}. \end{aligned}$$

One chooses α large enough to find $Lw > 0$. By the previous lemma w cannot have a nonnegative maximum in Ω. Hence

$$\sup_{\Omega} u \leq \sup_{\Omega} w \leq \sup_{\Omega} w^+ = \sup_{\partial\Omega} w^+ \leq \sup_{\partial\Omega} u^+ + \varepsilon\, e^{\alpha d}$$

if $\Omega \subset \{|x| < d\}$. The result follows for $\varepsilon \to 0$. ■

The proof of this maximum principle uses local arguments. If we skip the assumption that Ω is bounded we obtain:

Corollary 6 *Suppose that L is strictly elliptic with $c \leq 0$. If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and $Lu \geq 0$ in Ω, then u cannot attain a strict[1] nonnegative maximum in Ω.*

Before stating the strong maximum principle by E. Hopf (1927) for general L with coefficients which are solely uniformly bounded, let us recall from Protter and Weinberger's book ([5, page 156]) some historical dates.

1839, C.F. Gauss and S. Earnshaw (seperately), M.P. for (sub)harmonic functions.

1892, A. Paraf, M.P. for L with $c < 0$ in 2 dimensions.

1894, T. Moutard, M.P. for L with $c < 0$ in higher dimensions.

1905, E. Picard, M.P. for L with $c \leq 0$ in 2 dimensions.

1927, M. Picone, Generalized M.P. for L.

1927, E. Hopf, M.P. for L assuming just uniformly bounded coefficients.

1952, E. Hopf and O.A. Oleinik (seperately), M.P. including boundary point estimate.

Theorem 7 (Strong Maximum Principle) *Suppose that L is strictly elliptic and that $c \leq 0$. If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and $Lu \geq 0$ in Ω, then either $u \equiv \sup_\Omega u$ or u does not attain a nonnegative maximum in Ω.*

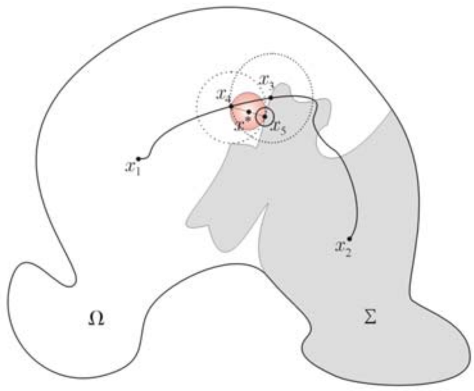
Proof. Let $m = \sup_\Omega u$ and set $\Sigma = \{x \in \Omega; u(x) = m\}$. We are done if $\Sigma \in \{\Omega, \emptyset\}$. Arguing by contradiction we assume that Σ and $\Omega \backslash \Sigma$ are non-empty.

The argument proceeds in three steps. First one fixes an appropriate open ball and in the next step an auxiliary function is defined that is positive on and only on this ball. For the sum of u and this auxiliary function one obtains a contradiction on a second ball by the weak maximum principle.

The first ball $B_r(x^*)$ needs to be away from $\partial\Omega$ and to 'touch' Σ in exactly one point. In other words, $B_r(x^*)$ having the following properties suffices:

1. $B_r(x^*) \subset \Omega \backslash \Sigma$;
2. $B_{2r}(x^*) \subset \Omega$;
3. $\overline{B_r(x^*)} \cap \Sigma$ contains a single point.

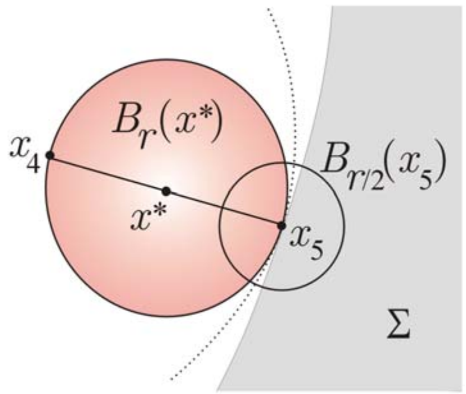
[1] We say that the function u has a strict maximum at x^* if there is a ball $B_r(x^*)$ such that $u(x) < u(x^*)$ for all $x \in B_r(x^*) \setminus \{x^*\}$.



I. Construction of $B_r(x^*)$**.** Since u is continuous $\Omega\backslash\Sigma$ is open and hence contains a open ball. Taking the center x_1 of this ball and a point $x_2 \in \Sigma$ there is an arc in Ω connecting x_1 with x_2. We continue in shorthand.
- Denote by x_3 the first point on this arc in Σ.
- Set $s = d(x_3, \partial\Omega) = \inf\{|x_3 - x|\,; x \in \partial\Omega\}$.
- Take x_4 on the arc between x_1 and x_3 with $|x_4 - x_3| < \frac{1}{2}s$.
- Let $B_{r_1}(x_4)$ denote the largest ball around x_4 that is contained in $\Omega\backslash\Sigma$. One finds $0 < r_1 < \frac{1}{2}s$.
- Take $x_5 \in \partial B_{r_1}(x_4) \cap \Sigma$.
- Finally set $x^* = \frac{1}{2}x_4 + \frac{1}{2}x_5$ and $r = \frac{1}{2}r_1$.

Since $\overline{B_r(x^*)} \subset B_{r_1}(x_4) \cup \{x_5\}$ it follows that $\overline{B_r(x^*)} \cap \Sigma$ contains a single point, namely x_5. Since $B_r(x^*) \subset B_{r_1}(x_4) \subset B_{\frac{1}{2}s}(x_3)$ and $d(x_3, \partial\Omega) = s$ one finds $B_{2r}(x^*) \subset \Omega$.

The second ball mentioned above will be $B_{\frac{1}{2}r}(x_5)$.



II. The auxiliary function. Set

$$h(x) = \frac{e^{-\frac{\alpha}{2}|x-x^*|^2} - e^{-\frac{\alpha}{2}r^2}}{1 - e^{-\frac{\alpha}{2}r^2}}, \tag{2}$$

with α to be fixed later. Notice that $h(x^*) = \max_\Omega h = 1$ and

$$\begin{cases} h(x) > 0 \text{ if } x \in B_r(x^*), \\ h(x) < 0 \text{ if } x \notin B_r(x^*). \end{cases}$$

Moreover, for $x \in B_{\frac{1}{2}r}(x_5)$ we have $|x - x^*| \in \left(\frac{1}{2}r, \frac{3}{2}r\right)$ and

$$\begin{aligned} Lh &= \tfrac{\alpha^2 \sum_{i,j=1}^n a_{ij}(x_i - x_i^*)(x_j - x_j^*) - \alpha \sum_{i=1}^n (a_{ii} + b_i(x_i - x_i^*)) + c}{1 - e^{-\frac{\alpha}{2}r^2}} e^{-\frac{\alpha}{2}|x-x^*|^2} - c\tfrac{1}{1-e^{-\frac{\alpha}{2}r^2}} e^{-\frac{\alpha}{2}r^2} \\ &\geq \left(4\alpha^2 \lambda \left(\tfrac{1}{2}r\right)^2 - 2\alpha \sum_{i=1}^n \left(a_{ii} + |b_i| \tfrac{3}{2}r\right) + c\right) \frac{e^{-\frac{\alpha}{2}|x-x^*|^2}}{1 - e^{-\frac{\alpha}{2}r^2}}. \end{aligned}$$

Hence one may take α large such that $Lh > 0$ on $B_{\frac{1}{2}r}(x_5)$.

III. The contradiction. Finally one considers $w = u + \varepsilon h$ and chooses $\varepsilon > 0$ by

$$\varepsilon = \frac{1}{2}\left(m - \max\left\{u(x); x \in \overline{B_r(x^*)} \cap \partial B_{\frac{1}{2}r}(x_5)\right\}\right). \tag{3}$$

Since u is continuous the maximum on this compact set is attained and since that set is disjoint from Σ one finds $\varepsilon > 0$. Consider the boundary value problem for w on $B_{\frac{1}{2}r}(x_5)$.

- For $x \in \partial B_{\frac{1}{2}r}(x_5) \cap \overline{B_r(x^*)}$ one has

$$w(x) = u(x) + \varepsilon h(x) \leq m - \varepsilon + \frac{1}{2}\varepsilon < m.$$

- For $x \in \partial B_{\frac{1}{2}r}(x_5) \backslash \overline{B_r(x^*)}$ one has

$$w(x) = u(x) + \varepsilon h(x) \leq m + \varepsilon h(x) < m.$$

One also has $w(x_5) = u(x_5) + \varepsilon h(x_5) = m + 0$. Since $Lw > 0$ one obtains a contradiction with the weak M.P. or even with Lemma 2.1. ■

Corollary 2.5 (Positivity Preserving Property) *Let Ω be bounded and suppose that L is strictly elliptic with $c \leq 0$. If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies*

$$\begin{cases} -Lu \geq 0 & \text{in } \Omega, \\ u \geq 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

then either $u(x) > 0$ for $x \in \Omega$ or $u \equiv 0$.

Remark 2.6 *Let $f \in C(\bar{\Omega})$ and $\varphi \in C(\partial\Omega)$. A function $w \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying*

$$\begin{cases} -Lw \geq f & \text{in } \Omega, \\ w \geq \varphi & \text{on } \partial\Omega, \end{cases} \tag{5}$$

is called a supersolution *for*

$$\begin{cases} -Lu = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{6}$$

If the maximum principle holds then one finds that a supersolution for (6) lies above a solution for (6); $w \geq u$. In particular, since solutions are also supersolutions, if there are two solutions u_1 and u_2 then both $u_1 \geq u_2$ and $u_2 \geq u_1$ hold true. In other words, (6) has at most one solution in $C^2(\Omega) \cap C(\bar{\Omega})$.

Assuming more for $\partial\Omega$ one obtains an even stronger conclusion, that is, E. Hopf's result in 1952.

Theorem 2.7 (Hopf's boundary point Lemma) *Suppose that Ω satisfies the* interior sphere condition[2] *at $x_0 \in \partial\Omega$. Let L be strictly elliptic with $c \leq 0$. If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu \geq 0$ and $\max_{\bar{\Omega}} u(x) = u(x_0)$. Then either $u \equiv u(x_0)$ on Ω or*

$$\liminf_{t \downarrow 0} \frac{u(x_0) - u(x_0 + t\nu)}{t} > 0 \text{ (possibly } +\infty)$$

for every direction ν pointing into an interior sphere.
If $u \in C^1(\Omega \cup \{x_0\})$ then $\frac{\partial u(x_0)}{\partial \nu} < 0$.

[2] There is a ball $B \subset \Omega$ with $x_0 \in \partial B$.

Proof. By the S.M.P. we have $u(x) < m := u(x_0)$ for all $x \in \Omega$. Let $B = B_{r_1}(x_1) \subset \Omega$ be such that $x_0 \in \partial B$. We proceed as in the proof of the S.M.P. with $B_r(x^*) = B_{\frac{1}{2}r_1}\left(\frac{1}{2}x_0 + \frac{1}{2}x_1\right)$ and $B_{\frac{1}{2}r}(x_5) = B_{\frac{1}{2}r}(x_0)$. One defines h and ε as in (2) and (3) to find

$$\begin{aligned} L(u+\varepsilon h) &> 0 \text{ in } \Omega \cap B_{\frac{1}{2}r}(x_0), \\ u+\varepsilon h &\leq m \text{ on } \partial\Omega \backslash \{x_0\}, \\ u+\varepsilon h &< m \text{ on } \Omega \cap \partial B_{\frac{1}{2}r}(x_0). \end{aligned}$$

Hence $u + \varepsilon h < m$ in $\Omega \cap B_{\frac{1}{2}r}(x_0)$ implying that for some $c_h > 0$

$$u(x_0 + t\nu) - u(x_0) \leq -\varepsilon\, h(x_0 + t\nu) \leq -\varepsilon\, c_h\, t \text{ for } t \in \left(0, \frac{1}{2}r\right).$$

■

There are two directions in order to weaken the restriction on c. Skipping the sign condition for c but adding one for u one obtains the next result.

Theorem 2.8 (Maximum Principle for nonpositive functions) *Let Ω be bounded. Suppose that L is strictly elliptic (no sign assumption on c). If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu \geq 0$ in Ω and $u \leq 0$ on $\bar{\Omega}$, then either $u(x) < 0$ for all $x \in \Omega$, or $u \equiv 0$.*
Moreover, if Ω satisfies an interior sphere condition at $x_0 \in \partial\Omega$ and $u \in C^1(\Omega \cup \{x_0\})$ with $u < u(x_0) = 0$ in Ω, then $\frac{\partial u(x_0)}{\partial \nu} < 0$ for every direction ν pointing into an interior sphere.

Proof. Writing $c(x) = c^+(x) - c^-(x)$ with $c^+, c^- \geq 0$ one finds that $L - c^+$ satisfies the condition of the S.M.P. and moreover from $u \leq 0$ it follows that

$$\left(L - c^+\right) u \geq -c^+ u \geq 0.$$

The conclusion for the derivative follows from Theorem 2.7. ■

Theorem 2.9 (Maximum Principle when a positive supersolution exists) *Let Ω be bounded. Suppose that L is strictly elliptic (no sign assumption on c) and that there exists $w \in C^2(\bar{\Omega})$ with $w > 0$ and $-Lw \geq 0$ on $\bar{\Omega}$. If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu \geq 0$ in Ω, then either there exists a constant $t \in \mathbb{R}$ such that $u \equiv t\,w$, or u/w does not attain a nonnegative maximum in Ω.*

Remark 2.10 *One may rephrase this for supersolutions as follows. If there exists one function $w \in C^2(\bar{\Omega})$ with $-Lw \geq 0$ and $w > 0$ on $\bar{\Omega}$ then all functions $v \in C^2(\Omega) \cap C(\bar{\Omega})$ such that*

$$\begin{cases} -Lv \geq 0 & \text{in } \Omega, \\ v \geq 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

satisfy either $v \equiv 0$ or $v > 0$ in Ω. One swallow makes summer. Apply the theorem to $-v$.

Remark 2.11 *If $\bar{\Omega} \subset \Omega_1$ and if L is defined on Ω_1 and happens to have a positive eigenfunction φ with eigenvalue λ_{Ω_1} for the Dirichlet problem on Ω_1 :*

$$\begin{cases} -L\varphi = \lambda_{\Omega_1}\varphi & \text{in } \Omega_1, \\ \varphi = 0 & \text{on } \partial\Omega_1, \\ \varphi > 0 & \text{in } \Omega_1, \end{cases} \tag{8}$$

then this $w = \varphi$ may serve in the theorem above. It shows that a maximum principle holds on Ω for $c < \lambda_{\Omega_1}$.

Proof. Set $v = u/w$. Then with $\tilde{b}_i = b_i + \sum_{j=1}^{n} \frac{2a_{ij}}{w} D_j w$ and $\tilde{c} = \frac{Lw}{w}$ one has

$$Lu = L(vw) = \left(\sum_{i,j=1}^{n} a_{ij} D_{ij} v + \sum_{i=1}^{n} \tilde{b}_i D_i v + \frac{Lw}{w} v \right) w. \tag{9}$$

Then $Lu \leq 0$ implies $\tilde{L}v \leq 0$ with $\tilde{L}$ as in (9) and since this $\tilde{L}$ does satisfy the conditions of the Strong Maximum Principle, in particular the sign condition for $\tilde{c} = \frac{Lw}{w}$, one finds that either v is constant or that v does not attain a nonnegative maximum in Ω. ∎

Theorem 2.12 (Maximum Principle for narrow domains) *Suppose that L is strictly elliptic (no sign assumption on c). Then there is $d > 0$ such that if $S \subset \{x \in \Omega; |x_1| < d\}$ and $u \in C^2(S) \cap C(\bar{S})$ satisfies $Lu \geq 0$ in S, then there exists $w \in C^2(\bar{S})$ with $w > 0$ and $-Lw \geq 0$ on $\bar{S}$.*

Proof. For $w(x) = \cos(\alpha x_1)$ and $|x_1| \leq \frac{\pi}{4\alpha}$ we have $w(x) > 0$ and

$$\begin{aligned} -Lw &= \left(\alpha^2 a_{11} - c\right) \cos(\alpha x_1) + \alpha b_1 \sin(\alpha x_1) \\ &\geq \left(\alpha^2 \lambda - \alpha \|b_1\|_\infty - \|c\|_\infty\right) \tfrac{1}{2}\sqrt{2}. \end{aligned}$$

The claim follows by taking α large enough and defining $d = \frac{\pi}{4\alpha}$. ∎

3 A priori estimates based on the maximum principle

First we will derive an L^∞-estimate for solutions of

$$\begin{cases} -Lu = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{10}$$

with $f \in C(\bar{\Omega})$ and $\varphi \in C(\partial\Omega)$.

Proposition 3.1 *Assume that Ω is bounded. Let L be uniformly elliptic with $c \leq 0$ and suppose that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies (10). Then*

$$u(x) \leq \max_{\partial\Omega} \varphi^+ + C \max_{\bar{\Omega}} f^+,$$

where $C = C(\operatorname{diam}(\Omega), \lambda, \|b\|_\infty / \lambda, \|c\|_\infty / \lambda)$.

Remark 3.2 *The result applied to $-u$ immediately yields $u(x) \geq -(\max_{\partial\Omega} \varphi^- + C \max_{\bar{\Omega}} f^-)$ and hence*

$$|u(x)| \leq \max_{\partial\Omega} |\varphi| + C \max_{\bar{\Omega}} |f|.$$

Proof. Assume that $\Omega \subset \{x \in \mathbb{R}^n; |x_1| < r\}$ and set

$$w(x) = \max_{\partial\Omega} \varphi^+ + (\cosh \alpha r - \cosh \alpha x_1) \max_{\bar{\Omega}} f^+,$$

with α chosen large enough such that $L(\cosh \alpha r - \cosh \alpha x_1) \leq -1$. Indeed, this estimate holds true by

$$\begin{aligned} L(\cosh \alpha r - \cosh \alpha x_1) &= \left(-\alpha^2 a_{11} - c\right) \cosh \alpha x_1 - \alpha b_1 \sinh \alpha x_1 + c \cosh \alpha r \\ &\leq -\left(\alpha^2 \lambda - \alpha \|b\|_\infty - \|c\|_\infty\right) \cosh \alpha x_1 \end{aligned}$$

and taking $\alpha = 1 + \lambda^{-1} + \lambda^{-1}\|b\|_\infty + \lambda^{-1}\|c\|_\infty$. Notice that for $|x_1| < r$ it holds that $0 \le \cosh \alpha r - \cosh \alpha x_1 \le \cosh \alpha r$. We have

$$-L(w-u) \ge \left(\max_{\bar\Omega} f^+\right) - f \ge 0 \text{ in } \Omega$$

and

$$w - u = \left(\max_{\partial\Omega} \varphi^+\right) - \varphi \ge 0 \text{ on } \partial\Omega.$$

By Corollary 2.5 one finds $w - u \ge 0$ in Ω and hence

$$u(x) \le w(x) \le \max_{\partial\Omega} \varphi^+ + C \max_{\bar\Omega} f^+$$

when we take $C \ge \cosh \alpha r$. Since we may assume $r = \frac{1}{2}\operatorname{diam}(\Omega)$ we set

$$C = C\left(\operatorname{diam}(\Omega), \lambda, \|b\|_\infty/\lambda, \|c\|_\infty/\lambda\right) = e^{\operatorname{diam}(\Omega)\left(1+\lambda^{-1}+\lambda^{-1}\|b\|_\infty+\lambda^{-1}\|c\|_\infty\right)}.$$

■

Next we will derive an L^∞-estimate for the first derivatives of functions satisfying

$$-Lu = f \in C^1\left(\bar\Omega\right) \tag{11}$$

by $\sup_{\partial\Omega}|\nabla u|$ plus the C^1-norm of f,

$$\|f\|_{C^1(\bar\Omega)} = \|f\|_\infty + \|\nabla f\|_\infty.$$

Whenever $L = \Delta := \sum_{i=1}^n D_{ii}$ such an estimate for C^3-functions u follows from $D_i \Delta u = \Delta D_i u$ and applying the previous theorem. For non-constant coefficients this does not go through that simple.

Proposition 3.3 *Assume that Ω is bounded. Let L be uniformly elliptic with $a_{ij}, b_i, c \in C^1\left(\bar\Omega\right)$. Suppose that $u \in C^3(\Omega) \cap C^1\left(\bar\Omega\right)$ satisfies (11). Then*

$$|\nabla u(x)| \le \sup_{\partial\Omega}|\nabla u| + C\left(1 + \|f\|_{C^1}\right) \text{ for all } x \in \Omega,$$

where $C = C\left(\operatorname{diam}(\Omega), \|u\|_\infty, \lambda, \|a_{ij}\|_{C^1}, \|b_i\|_{C^1}, \|c\|_{C^1}\right)$.

Proof. Instead for a single derivative we will use a maximum principle for $|\nabla u|^2$. Set $L_0 = L - c$. Denoting the Hessian by $\nabla^2 = (D_{ij})$ with $\left|\nabla^2 u\right|^2 = \sum_{i,j=1}^n (D_{ij}u)^2$, it follows from

$$D_i\left(|\nabla u|^2\right) = 2\sum_{\ell=1}^n D_\ell u\, D_{i\ell} u \text{ and } D_{ij}\left(|\nabla u|^2\right) = 2\sum_{\ell=1}^n \left(D_{i\ell}u\, D_{j\ell}u + D_\ell u\, D_{ij\ell}u\right)$$

that, using strict ellipticity,

$$\begin{aligned} \sum_{\ell=1}^n (L_0 D_\ell u)\, D_\ell u &= \tfrac{1}{2} L_0\left(|\nabla u|^2\right) - \sum_{\ell=1}^n \sum_{i,j=1}^n a_{ij} D_{i\ell}u\, D_{j\ell}u \\ &\le \tfrac{1}{2} L_0\left(|\nabla u|^2\right) - \lambda\left|\nabla^2 u\right|^2. \end{aligned} \tag{12}$$

Moreover, differentiating (11) with respect to x_ℓ, multiplying by $\partial_\ell u$ we have:

$$\begin{aligned} - &\quad D_\ell f\, D_\ell u = (D_\ell L u)\, D_\ell u = \\ = &\quad (L D_\ell u)\, D_\ell u + \left(\sum_{i,j=1}^n D_\ell a_{ij}\, D_{ij}u + \sum_{i=1}^n D_\ell b_i\, D_i u + D_\ell c\, u\right)\partial_\ell u \\ \le &\quad (L_0 D_\ell u)\, D_\ell u - c\,(D_\ell u)^2 + C.\left(\left|\nabla^2 u\right| + |\nabla u| + |u|\right)|\nabla u| \end{aligned} \tag{13}$$

with $C_0 = \sum_{i,j=1}^{n} \|a_{ij}\|_{C^1} + \sum_{i=1}^{n} \|b_i\|_{C^1} + \|c\|_{C^1}$. Summing (13) and combining with (12) one obtains

$$\begin{aligned} L_0\left(|\nabla u|^2\right) &\geq 2\sum_{\ell=1}^{n}\left(L_0 D_\ell u\right) D_\ell u + 2\lambda\left|\nabla^2 u\right|^2 \\ \geq\quad & -2\nabla f\cdot\nabla u + 2c\left|\nabla u\right|^2 + 2\lambda\left|\nabla^2 u\right|^2 - 2n\,C_0.\left(\left|\nabla^2 u\right| + |\nabla u| + |u|\right)|\nabla u|\,, \end{aligned}$$

and by using Cauchy-Schwarz's inequality it follows with $C_1 = 1 + \frac{n^2 C_0^2}{\lambda} + 3n\,C_0 + 2\,\|c\|_\infty$ that

$$L_0\left(|\nabla u|^2\right) \geq \lambda\left|\nabla^2 u\right|^2 - |\nabla f|^2 - C_1\,|\nabla u|^2 - n\,C_0\;|u|^2\,. \tag{14}$$

One also has

$$\begin{aligned} L_0\left(u^2\right) &= 2\sum_{i,j=1}^{n} a_{ij} D_i u\, D_j u + 2u\,Lu - 2c\,u^2 \\ &\geq 2\lambda\,|\nabla u|^2 + 2u\,f - 2c\,u^2 \\ &\geq 2\lambda\,|\nabla u|^2 -\; f^2 - (2c+1)\;u^2. \end{aligned} \tag{15}$$

Taking $\alpha = 1 + \lambda^{-1} C_1$ one obtains from (14) and (15) that

$$L_0\left(|\nabla u|^2 + \alpha u^2\right) \geq \lambda\left|\nabla^2 u\right|^2 + \lambda\,|\nabla u|^2 - |\nabla f|^2 - \alpha\,|f|^2 - C_2 \tag{16}$$

for $C_2 = C_2\left(\|u\|_\infty\,, \lambda, \|a_{ij}\|_{C^1}\,, \|b_i\|_{C^1}\,, \|c\|_{C^1}\right)$.

To remove the remaining negative terms on the right hand side we add another term, namely $\left(C_2 + \alpha\,\|f\|_{C^1}\right)\cosh\left(\beta x_1\right)$ with $\beta = \lambda^{-1/2} + \lambda^{-1}\,\|b\|_\infty$. Such β is large enough for $\beta^2\lambda - \beta\,|b| \geq 1$ to hold and

$$\begin{aligned} L_0 \cosh\left(\beta x_1\right) &= a_{11}\beta^2 \cosh\left(\beta x_1\right) + b_1 \beta \sinh\left(\beta x_1\right) \\ &\geq \left(\beta^2\lambda - \beta\,|b|\right)\cosh\left(\beta x_1\right) \geq 1. \end{aligned}$$

One gets

$$L_0\left(|\nabla u|^2 + \alpha u^2 + \left(C_2 + \alpha\,\|f\|_{C^1}\right)\cosh\left(\beta x_1\right)\right) \geq 0,$$

and the maximum principle implies

$$\begin{aligned} \sup_{\Omega} |\nabla u|^2 &\leq \sup_{\Omega}\left(|\nabla u|^2 + \alpha u^2 + \left(C_2 + \left(\alpha + \frac{1}{2}\right)\|f\|_{C^1}\right)\cosh\left(\beta x_1\right)\right) \\ &= \sup_{\partial\Omega}\left(|\nabla u|^2 + \alpha u^2 + \left(C_2 + \left(\alpha + \frac{1}{2}\right)\|f\|_{C^1}\right)\cosh\left(\beta x_1\right)\right) \end{aligned}$$

and since we may assume $\cosh\left(\beta x_1\right) \leq \cosh\left(\beta\,\mathrm{diam}\left(\Omega\right)\right)$ it holds that

$$\sup_{\Omega} |\nabla u| \leq \sup_{\partial\Omega} |\nabla u| + C_3\left(1 + \|f\|_{C^1}\right)$$

with $C_3 = C_3\left(\mathrm{diam}\left(\Omega\right), \|u\|_\infty\,, \lambda, \|a_{ij}\|_{C^1}\,, \|b_i\|_{C^1}\,, \|c\|_{C^1}\right)$. ■

A similar estimate holds for the first derivatives of functions satisfying the semilinear equation

$$-Lu = f\left(x, u\right)\;\text{ for } x \in \Omega \tag{17}$$

with $f \in C^1\left(\bar{\Omega} \times \mathbb{R}\right)$.

Proposition 3.4 *Assume that Ω is bounded. Let L be uniformly elliptic with $a_{ij}, b_i, c \in C^1(\bar{\Omega})$. Suppose that $u \in C^3(\Omega) \cap C^1(\bar{\Omega})$ satisfies (17). Then*

$$|\nabla u(x)| \leq \sup_{\partial\Omega} |\nabla u| + C \text{ for all } x \in \Omega,$$

where $C = C\left(\operatorname{diam}(\Omega), \|f\|_{C^1(\bar{\Omega}\times\mathbb{R})}, \|u\|_\infty, \lambda, \|a_{ij}\|_{C^1}, \|b_i\|_{C^1}, \|c\|_{C^1}\right)$.

Proof. Instead of (14) and (15) we have

$$\begin{aligned} L_0\left(|\nabla u|^2\right) &\geq \lambda \left|\nabla^2 u\right|^2 - 2|\nabla_x f|^2 - \left(C_1 + 2|f_u|^2\right)|\nabla u|^2 - n\,C_0\,|u|^2, && (18) \\ L_0\left(u^2\right) &\geq 2\lambda|\nabla u|^2 - f^2 - (2c+1)\,u^2. && (19) \end{aligned}$$

Taking $\alpha = 1 + \lambda^{-1}\left(C_1 + 2|f_u|^2\right)$ we continue by

$$L_0\left(|\nabla u|^2 + \alpha u^2\right) \geq -C_5$$

with $C_5 = C_5\left(\|u\|_\infty, \lambda, \|a_{ij}\|_{C^1}, \|b_i\|_{C^1}, \|c\|_{C^1}, \|f\|_{C^1(\bar{\Omega}\times\mathbb{R})}\right)$. And with an added $C_5 \cosh(\beta x_1)$ we find

$$L_0\left(|\nabla u|^2 + \alpha u^2 + C_5 \cosh(\beta x_1)\right) \geq 0$$

and proceed as before. ■

4 Comparison principles

For linear problems a comparison principle is the maximum principle used for the difference of two functions. See for example Remark 2.6. Only for nonlinear problems one should make a distinction between those two types of principles. If the nonlinearity does not appear in the differential operator, such as in

$$\begin{cases} -Lu = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{20}$$

the system is usually called semilinear.

Proposition 4.1 (semilinear comparison principle) *Let L be strictly elliptic with $c = 0$ and $f \in C^1(\bar{\Omega}\times\mathbb{R})$. If $u_1 \leq u_2$ are both solutions of (20), then either $u_1 \equiv u_2$ or $u_1(x) < u_2(x)$ for all $x \in \Omega$.*

Proof. Define $g \in L^\infty(\Omega)$ by

$$g(x) = \begin{cases} \frac{f(x,u_2(x)) - f(x,u_1(x))}{u_2(x)-u_1(x)} & \text{if } u_2(x) \neq u_1(x), \\ \frac{\partial f}{\partial u}(x, u_2(x)) & \text{if } u_2(x) = u_1(x). \end{cases} \tag{21}$$

Then $-(L - g(x))(u_2 - u_1) = 0$ in Ω and $u_2 - u_1 = 0$ on $\partial\Omega$. One proceeds by Theorem 2.8. ■

The comparison principle shows a semilinear equivalent of Corollary 2.5:

Corollary 4.2 *Let L be strictly elliptic with $c = 0$ and $f \in C^1(\bar{\Omega}\times\mathbb{R})$ with $f(x,0) \geq 0$. If $u \geq 0$ is a solution of (20), then either $u \equiv 0$ or $u(x) > 0$ for all $x \in \Omega$.*

Proof. Define $g \in L^{\infty}(\Omega)$ by

$$g(x) = \begin{cases} \frac{f(x,u(x))-f(x,0)}{u(x)} & \text{if } u(x) > 0, \\ \frac{\partial f}{\partial u}(x,0) & \text{if } u(x) = 0. \end{cases} \tag{22}$$

Then $-(L - g(x))u = f(x,0) \geq 0$ and one concludes by Theorem 2.8. ■

For

$$\begin{cases} -\Delta u &= f(u) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \tag{23}$$

with $f \in C^1(\mathbb{R})$, one conjectures that $f(0) \geq 0$ is not necessary in Corollary 4.2. The problem is still open in general domains!

An application of Hopf's boundary point Lemma for solutions of (23) is the following.

Proposition 4.3 *Let $f \in C^1(\mathbb{R})$ with $f(0) \geq 0$ and suppose that Ω has an entrant corner*[3]*. If $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ is a nonnegative solution of (23), then $u \equiv 0$ and $f(0) = 0$.*

Proof. If $x_0 \in \partial\Omega$ denotes the corner and $B_r(x_1)$ and $B_r(x_2)$ are two inner balls touching $\partial\Omega$ in x_0, then there exists $\nu \in \mathbb{R}^n$ such that $x_0 + \nu \in B_r(x_1)$ and $x_0 - \nu \in B_r(x_2)$. Note that Ω satisfies an interior sphere condition at x_0. Set g as in (22) to find

$$-\Delta u - g(x)u = f(0).$$

Since $f(0) \geq 0$ we find $-\Delta u + g^- u = g^+ u + f(0) \geq 0$ and by the strong maximum principle either $u \equiv 0$ or $u > 0$. Supposing that $u > 0$ it follows by Hopf's boundary point Lemma, and using $u \in C^1(\bar{\Omega})$, that

$$0 > \limsup_{t \downarrow 0} \frac{u(x_0 - t\nu) - u(x_0)}{t|\nu|} = \frac{\partial u}{\partial \nu}(x_0) = \liminf_{t \downarrow 0} \frac{u(x_0) - u(x_0 + t\nu)}{t|\nu|} > 0.$$

Hence $u \equiv 0$ and consequently $f(0) = 0$. ■

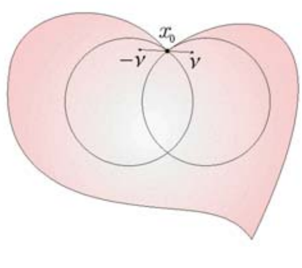
5 Alexandrov's maximum principle

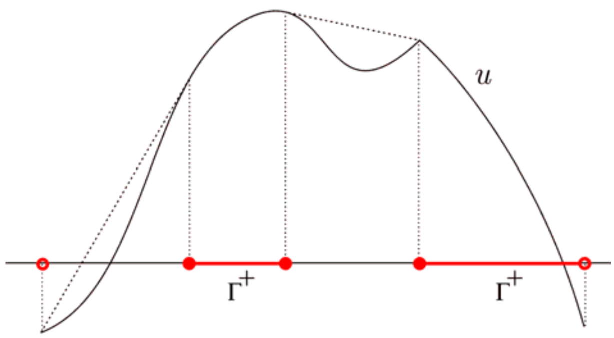
One needs to define the notion of *upper contact set* of a function u, which is roughly said the set of points in Ω that have a tangent plane above u.

Definition 5.1 *For $u \in C(\bar{\Omega})$ the upper contact set Γ^+ is defined by*

$$\Gamma^+ = \{y \in \Omega;\ \exists\, p_y \in \mathbb{R}^n \text{ such that } \forall x \in \Omega:\ u(x) \leq u(y) + p_y \cdot (x - y)\}.$$

[3] We say that Ω has an entrant corner at $x_0 \in \partial\Omega$ if there are two different balls $B_r(x_1), B_r(x_2) \subset \Omega$ (same r) with $x_0 \in \partial B_r(x_1) \cap \partial B_r(x_2)$.





If Ω is bounded the set Γ^+ is relatively closed in Ω.

If $u \in C^1(\Omega)$ and $y \in \Gamma^+$ one takes $p_y = \nabla u(y)$. Moreover, if $u \in C^2(\Omega)$ then the Hessian matrix $(D_{ij}u)$ is nonpositive on Γ^+.

First a lemma with this upper contact set.

Lemma 5.2 *Suppose that Ω is bounded. Let $g \in C(\mathbb{R}^n)$ be nonnegative and $u \in C(\bar{\Omega}) \cap C^2(\Omega)$. Set*

$$M = \frac{\sup_\Omega u - \sup_{\partial\Omega} u}{\operatorname{diam}(\Omega)}.$$

Then

$$\int_{B_M(0)} g(z)\, dz \le \int_{\Gamma^+} g(\nabla u(x)) \left|\det(D_{ij}u(x))\right| dx. \tag{24}$$

Proof. Let Σ denote the images of Γ^+ under the mapping ∇u. If this mapping is a bijection, then by a change of variables,

$$\int_\Sigma g(z)\, dz = \int_{\Gamma^+} g(\nabla u(x)) \left|\det(D_{ij}u(x))\right| dx.$$

Since the mapping is only onto and since $g \ge 0$ we find

$$\int_\Sigma g(z)\, dz \le \int_{\Gamma^+} g(\nabla u(x)) \left|\det(D_{ij}u(x))\right| dx.$$

Hence it is sufficient to show that $B_M(0) \subset \Sigma$. In other words show that for every $a \in \mathbb{R}^n$ with $|a| < M$ there exists $y \in \Gamma^+$ such that $a = \nabla u(y)$.

Set $L_a(t) = \min_{x\in\bar{\Omega}}(t + a\cdot x - u(x))$. This function L_a is continuous, positive for t large enough and negative for $-t$ large. Let t_a denote its largest root. It follows that $t_a + a\cdot x - u(x) \ge 0$ for all $x \in \Omega$ and $t_a + a\cdot y - u(y) = 0$ for some $y \in \bar{\Omega}$. Hence

$$u(y) \ge u(x) + a\cdot(y - x).$$

Taking x_0 such that $u(x_0) = \sup_\Omega u$, one finds

$$u(y) \ge \sup_{\partial\Omega} u + M \operatorname{diam}(\Omega) + a\cdot(y - x_0) > \sup_{\partial\Omega} u,$$

and hence that $y \notin \partial\Omega$. For $y \in \Omega$ the assumption $u \in C^1(\Omega)$ implies that $a = \nabla u(y)$. By construction we have $y \in \Gamma^+$. ■

For an elliptic operator L as in (1) one defines

$$\mathscr{D}^*(x) = (\det(a_{ij}(x)))^{1/n}.$$

Indeed, the definition of ellipticity implies that $\mathscr{D}^*$ is well-defined and even that $\mathscr{D}^*(x) \ge \lambda(x)$. In case of strict ellipticity $\mathscr{D}^*(x) \ge \lambda$ and assuming uniform ellipticity also $\mathscr{D}^*(x) \le \Lambda$ holds; $\mathscr{D}^*(x)$ is the geometric average of the eigenvalues of $(a_{ij}(x))$.

Corollary 5.3 *Under the conditions of Lemma 5.2 one obtains for $g \equiv 1$:*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + \frac{\operatorname{diam}\Omega}{\omega_n^{1/n}} \left(\int_{\Gamma^+} \left(-\frac{\sum_{i,j=1}^n a_{ij}(x) D_{ij}u(x)}{n\,\mathscr{D}^*} \right)^n dx \right)^{1/n}$$

where $\omega_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)}$ is the volume of the unit ball in $\mathbb{R}^n$.

Remark 5.4 *Considering $\Omega^+ = \{x \in \Omega; u(x) > 0\}$ one obtains*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + \frac{\operatorname{diam}\Omega}{\omega_n^{1/n}} \left(\int_{\Gamma^+ \cap \Omega^+} \left(\frac{-\sum_{i,j=1}^n a_{ij}(x) D_{ij}u(x)}{n\,\mathscr{D}^*} \right)^n dx \right)^{1/n}.$$

Proof. Note that $\sum_{i,j=1}^n a_{ij}(x) D_{ij}u(x) = \operatorname{tr}(\mathsf{AD})$ with $\mathsf{A} = (a_{ij}(x))$ and $\mathsf{D} = (D_{ij}u(x))$. Remember that both the determinant and the trace do not change under orthogonal transformations. On Γ^+ the matrix AD is nonpositive. Hence, using the fact that the geometric mean of nonnegative numbers is less than the arithmetic mean, we find, denoting the eigenvalues of $-\mathsf{AD}$ by λ_i, that

$$\mathscr{D}^* (\det(-\mathsf{D}))^{1/n} = (\det(-\mathsf{AD}))^{1/n} = \sqrt[n]{\lambda_1 \lambda_2 \ldots \lambda_n} \leq \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_n}{n} = \frac{\operatorname{tr}(-\mathsf{AD})}{n}, \tag{25}$$

or in other words

$$\det(D_{ij}u(x)) \leq \left(\frac{-\sum_{i,j=1}^n a_{ij}(x) D_{ij}u(x)}{n\,\mathscr{D}^*} \right)^n. \tag{26}$$

Using (26) for the right hand side of (24) and

$$\int_{B_M(0)} dz = M^n \omega_n = \left(\frac{\sup_{\Omega} u - \sup_{\partial\Omega} u}{\operatorname{diam}\Omega} \right)^n \omega_n$$

for the left hand side, the claim follows. ∎

Theorem 5.5 (Alexandrov's Maximum Principle) *Let Ω be bounded and L elliptic with $c \leq 0$. Suppose that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu \geq f$ with*

$$\frac{|b|}{\mathscr{D}^*}, \frac{f}{\mathscr{D}^*} \in L^n(\Omega),$$

and let Γ^+ denote the upper contact set of u. Then one has

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C.\operatorname{diam}\Omega \,.\left\| \frac{f^-}{\mathscr{D}^*} \right\|_{L^n(\Gamma^+)},$$

with

$$C = C\left(n, \left\| \frac{|b|}{\mathscr{D}^*} \right\|_{L^n(\Gamma^+)} \right).$$

Proof. In Corollary 5.3 the term on the right hand side contains the leading order derivatives of Lu. In case that $b_i = c = 0$ the proof is complete since $f \leq Lu \leq 0$ on Γ^+.

For b_i or c not equal 0 one proceeds as follows for $x \in \Omega^+$, hence $c(x)u(x) \leq 0$, and for $\mu \in \mathbb{R}^+$,

$$\begin{aligned} -\sum_{i,j=1}^n a_{ij} D_{ij} u &\leq \sum_{i=1}^n b_i D_i u + cu - f \\ &\leq \sum_{i=1}^n b_i D_i u + f^- \leq \left(|b|, \mu^{-1} f^- \right) \cdot (|\nabla u|, \mu) \\ &\leq \left(|b|^n + \left(\mu^{-1} f^- \right)^n \right)^{\frac{1}{n}} (|\nabla u|^n + \mu^n)^{\frac{1}{n}} (1+1)^{\frac{n-2}{n}} \end{aligned}$$

by Cauchy-Schwarz and Hölder's inequality.

We will use Lemma 5.2 with Ω replaced by Ω^+, that is

$$\int_{B_{\tilde{M}}(0)} g(z)\,dz \leq \int_{\Gamma^+\cap\Omega^+} g(\nabla u(x))\,|\det(D_{ij}u(x))|\,dx, \tag{27}$$

with

$$\tilde{M} = \frac{\sup_\Omega u - \sup_{\partial\Omega} u^+}{\operatorname{diam}(\Omega)}.$$

Choosing in Lemma 5.2 $g(z) = (|z|^n + \mu^n)^{-1}$ we obtain for the left side, with $\sigma_n = n\omega_n$ the surface area of the unit ball in $\mathbb{R}^n$,

$$\int_{B_{\tilde{M}}(0)} g(z)\,dz = \sigma_n \int_{r=0}^{\tilde{M}} (r^n + \mu^n)^{-1} r^{n-1} dr = \sigma_n \log\left(\left(\tilde{M}/\mu\right)^n + 1\right).$$

For the right hand side, as in the proof of Corollary 5.3, see (26),

$$\begin{aligned}
&\int_{\Gamma^+} g(\nabla u(x))\,|\det(D_{ij}u(x))|\,dx \\
\leq\;& \int_{\Gamma^+\cap\Omega^+} \frac{1}{|\nabla u|^n + \mu^n}\left(-\frac{\sum_{i,j=1}^n a_{ij}(x)\,D_{ij}u(x)}{n\,\mathscr{D}^*}\right)^n dx \\
\leq\;& \int_{\Gamma^+\cap\Omega^+} \frac{1}{|\nabla u|^n + \mu^n}\,\frac{\left(|b|^n + \left(\mu^{-1}f^-\right)^n\right)\left(|\nabla u|^n + \mu^n\right)2^{n-2}}{n^n\left(\mathscr{D}^*\right)^n} dx \\
=\;& \frac{2^{n-2}}{n^n}\int_{\Gamma^+\cap\Omega^+} \frac{|b|^n + \left(\mu^{-1}f^-\right)^n}{\left(\mathscr{D}^*\right)^n} dx.
\end{aligned}$$

Now we choose $\mu = \left\|f^-/\,\mathscr{D}^*\right\|_{L^n(\Gamma^+\cap\Omega^+)}$ and find

$$\sigma_n \log\left(\left(\frac{\tilde{M}}{\left\|f^-/\,\mathscr{D}^*\right\|_{L^n(\Gamma^+\cap\Omega^+)}}\right)^n + 1\right) \leq \frac{2^{n-2}}{n^n}\left(\int_{\Gamma^+\cap\Omega^+} \frac{|b|^n}{\left(\mathscr{D}^*\right)^n} dx + 1\right).$$

Defining

$$C = \exp\left(\frac{2^{n-2}}{\sigma_n n^n}\left(\left\|\frac{|b|}{\mathscr{D}^*}\right\|_{L^n(\Gamma^+\cap\Omega^+)} + 1\right)\right) \tag{28}$$

the claim follows. ■

Without the sign condition for c there remains a maximum principle for narrow domains. The Alexandrov's Maximum Principle implies a similar result for small domains.

Theorem 5.6 (Maximum Principle for small domains) *Suppose that Ω is bounded and that L is strictly elliptic (without sign condition for c). Then there exists a constant δ, with $\delta = \delta\left(n, \operatorname{diam}\Omega, \lambda, \|b\|_{L^n(\Omega)}, \|c^+\|_\infty\right)$, such that the following holds.*
If $|\Omega| < \delta$ and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu \geq 0$ in Ω and $u \leq 0$ on $\partial\Omega$, then $u \leq 0$ in Ω.

Proof. The operator $L - c^+$ satisfies the condition of Theorem 5.5 and from $-Lu \leq 0$ it follows that

$$\left(L - c^+\right) u \geq -c^+ u \geq -c^+ u^+$$

and hence, since $\sup_{\partial\Omega} u^+ = 0$,

$$\begin{aligned}\sup_\Omega u &\leq \frac{C \operatorname{diam}\Omega}{\lambda} \left\| c^+ u^+ \right\|_{L^n(\Omega)} \\ &\leq \frac{C \operatorname{diam}\Omega}{\lambda} \left\| c^+ \right\|_\infty |\Omega|^{\frac{1}{n}} \sup_\Omega u^+ .\end{aligned}$$

If $\delta = \left(\frac{C \operatorname{diam}\Omega}{\lambda} \|c^+\|_\infty\right)^{-n}$ one finds $\sup_\Omega u \leq 0$. Notice that the C used here, defined in (28), depends on $\|b\|_{L^n(\Omega)}$. ■

6 Maximum principle and continuous perturbations

Using comparison principles for connected families of sub- and supersolutions one obtains a very powerful tool in deriving a priori estimates. One such result is the moving plane method used by Gidas, Ni and Nirenberg to prove symmetry of positive solutions to

$$\left\{ \begin{array}{rcll} -\Delta u &=& f(u) & \text{in } \Omega, \\ u &=& 0 & \text{on } \partial\Omega, \end{array} \right. \tag{29}$$

on domains Ω satisfying some symmetry conditions. Another one is McNabb's sweeping principle used to its full extend by Serrin. First let us fix a notion of supersolution.

Definition 6.1 *A function $v \in C\left(\bar{\Omega}\right) \cap C^2(\Omega)$ is called a supersolution for*

$$\left\{ \begin{array}{rcll} -\Delta u &=& f(u) & \textit{in } \Omega, \\ u &=& h & \textit{on } \partial\Omega, \end{array} \right. \tag{30}$$

if $v \geq h$ on $\partial\Omega$ and $-\Delta v \geq f(v)$.

Remark 6.2 *A much more useful concept of supersolution assumes $v \in C\left(\bar{\Omega}\right)$ and replaces $-\Delta v \geq f(v)$ by $\int_\Omega ((-\Delta\phi)\, v - \phi\, f(v))\, dx \geq 0$ for all $\phi \in C_0^\infty(\Omega)$ with $\phi \geq 0$.*

The argument combining the strong maximum principle and continuous perturbations goes as follows. One needs:

- a continuous family of supersolutions v_t, say $t \in [0,1]$, possibly on a subdomain or on an appropriate family of subdomains Ω_t;
- a strong maximum principle on each of the subdomains.

Roughly spoken the conclusion is that if $v_0 > u$ on Ω_t, then either $v_t > u$ for all $t \in [0,1]$ or there is a $t_1 \in [0,1]$ such that $v_{t_1} \equiv u$. For a precise statement we have to refine the notion of positivity.

For a fixed domain such a result is known as a 'sweeping principle'. A first reference to this result is a paper of McNabb from 1961. In the following version we assume that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, with Γ_1, Γ_2 closed and disjoint, possibly empty. We let $e \in C\left(\bar{\Omega}\right) \cap C^1(\Omega \cup \Gamma_2)$ be such that $e > 0$ on $\Omega \cup \Gamma_1$ and $\frac{\partial}{\partial n} e < 0 = e$ on Γ_2, where n denotes the outward normal. Moreover, we define

$$\begin{aligned} C_e\left(\bar{\Omega}\right) &= \left\{ w \in C\left(\bar{\Omega}\right); |w| < c\, e \text{ for some } c > 0 \right\}, \\ \|w\|_e &= \left\| \frac{w}{e} \right\|_\infty . \end{aligned}$$

Theorem 6.3 (Sweeping Principle) *Let Ω be bounded with $\partial\Omega \in C^2$ and $\partial\Omega = \Gamma_1 \cup \Gamma_2$ as above, $f \in C^1(\mathbb{R})$ and let $u \in C\left(\bar{\Omega}\right) \cap C^1(\Omega \cup \Gamma_2) \cap C^2(\Omega)$ be a solution of (30). Suppose $\{v_t; t \in [0,1]\}$ is a family of supersolutions in $C\left(\bar{\Omega}\right) \cap C^1(\Omega \cup \Gamma_2) \cap C^2(\Omega)$ for (30) such that:*

1. $t \to (v_t - v_0) \in C_e\left(\bar{\Omega}\right)$ *is continuous (with respect to the* $\|\cdot\|_e$*-norm);*

2. $v_t = u$ *on* Γ_2 *and* $v_t > u$ *on* Γ_1;

3. $v_0 \geq u$ *in* $\bar{\Omega}$;

   Then either $v_t \equiv u$ *for some* $t \in [0,1]$, *or there exists* $c > 0$ *such that* $v_t \geq u + c\,e$ *on* $\bar{\Omega}$ *for all* $t \in [0,1]$.

Proof. Set $I = \left\{t \in [0,1]\,; v_t \geq u \text{ in } \bar{\Omega}\right\}$ and assume that $v_t \not\equiv u$ for all $t \in [0,1]$. By the way, notice that $v_t \equiv u$ for some $t \in [0,1]$ can only occur when Γ_1 is empty. The set I is nonempty by assumption 3 and closed by assumption 1. We will show that I is open. Indeed, if t is such that $v_t \geq u$ in $\bar{\Omega}$ then $-\Delta(v_t - u) = f(v_t) - f(u)$ and setting g as in (21) one obtains

$$\begin{cases} -\Delta(v_t - u) + g^-(v_t - u) \geq g^+(v_t - u) \geq 0 \text{ in } \Omega, \\ v_t - u \geq 0 \text{ on } \partial\Omega, \end{cases}$$

and hence by the strong maximum principle $v_t - u > 0$ in Ω or $v_t \equiv u$. Moreover, for the first case Hopf's boundary point Lemma and the assumption that $u, v_t \in C^1(\Omega \cup \Gamma_2)$ imply that $-\frac{\partial}{\partial n}(v_t - u) > 0$ on Γ_2. Hence there is $c_t > 0$ such that $v_t - u \geq c_t\, e$ on $\bar{\Omega}$. By assumption 1 it follows that t lies in the interior of I. Hence I is open, implying $I = [0,1]$. Moreover, since I is compact a uniform $c > 0$ exists. ∎

Example 6.4 *Suppose that* $f \in C^1$ *is such that* $f(u) > 0$ *for* $u < 1$ *and* $f(u) < 0$ *for* $u > 1$. *Using the above version of the sweeping principle one may show that for* $\lambda \gg 1$ *every positive solution* u_λ *of*

$$\begin{cases} -\Delta u &= \lambda f(u) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \tag{31}$$

is near 1 *in the interior of* Ω. *Indeed, let* φ_1, μ_1 *denote the first eigenfunction/eigenvalue of* $-\Delta\varphi = \mu\varphi$ *in* B_1 *and* $\varphi = 0$ *on* ∂B_1 *where* B_1 *is the unit ball in* $\mathbb{R}^n$. *This first eigenfunction is radially symmetric and we assume it to be normalized such that* $\varphi(0) = 1$. *Now let* $\varepsilon > 0$ *and take* $\delta_\varepsilon > 0$ *such that*

$$\delta_\varepsilon\, u \leq f(u) \;\; \textit{for } u \in [0, 1-\varepsilon].$$

Set $v_t(x) = t\,\varphi_1\left(\frac{\sqrt{\lambda\delta_\varepsilon}}{\sqrt{\mu}}(x - x^*)\right)$ *for any* $x^* \in \Omega$ *with distance to the boundary* $d(x^*, \partial\Omega) > \frac{\sqrt{\mu}}{\sqrt{\lambda\delta_\varepsilon}} =: r_0$. *On* $B_{r_0}(x^*)$ *one finds for* $t \in [0, 1-\varepsilon]$ *that* $-\Delta v_t = \lambda\delta_\varepsilon v_t \leq \lambda f(v_t)$. *Since* $v_t = 0 < u_\lambda$ *on* $\partial B_{r_0}(x^*)$ *and* $v_0 = 0 < u_\lambda$ *on* $\overline{B_{r_0}(x^*)}$ *it is an appropriate family of subsolutions. From* $v_{1-\varepsilon} < u_\lambda$ *in* $B_{r_0}(x^*)$ *one concludes that* $1 - \varepsilon = v_{1-\varepsilon}(x^*) < u_\lambda(x^*)$. *Since the strong maximum principle implies that* $\max u_\lambda < 1$ *we may summarize:*

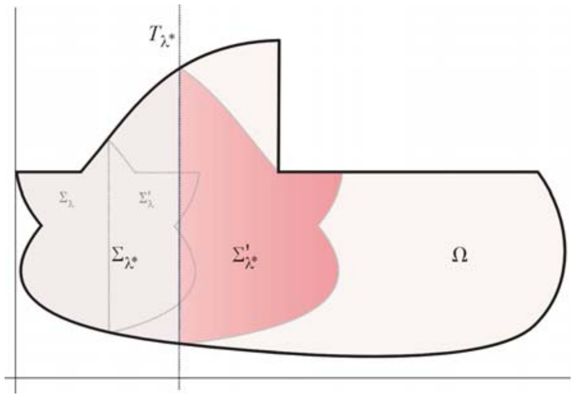
$$1 - \varepsilon < u_\lambda(x) < 1 \textit{ for } x \in \Omega \textit{ with } d(x^*, \partial\Omega) > \frac{\sqrt{\mu}}{\sqrt{\lambda\delta_\varepsilon}}.$$

More intricate results not only consider a family of supersolutions but also simultaneously modify the domain. One such result is the result by Gidas, Ni and Nirenberg ([3] or [1]). The idea was used earlier by Serrin in [10].

Before stating the result let us first fix some notations. We will be moving planes in the x_1-direction but of course also any other direction will do.

Some sets that will be used are:

the moving plane:	$T_\lambda = \{x \in \mathbb{R}^n; x_1 = \lambda\}$,
the subdomain:	$\Sigma_\lambda = \{x \in \Omega; x_1 < \lambda\}$,
the reflected point:	$x_\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$,
the reflected subdomain:	$\Sigma'_\lambda = \{x_\lambda; x \in \Sigma_\lambda\}$,
the starting value for λ:	$\lambda_0 = \inf\{x_1; x \in \Omega\}$,
the maximal value for λ:	$\lambda^* = \sup\left\{\lambda; \Sigma'_\mu \subset \Omega \text{ for all } \mu < \lambda\right\}$.



Theorem 6.5 (Moving plane argument) *Assume that f is Lipschitz, that Ω is bounded and that λ_0, λ^* and Σ_λ are as above. If $u \in C\left(\bar{\Omega}\right) \cap C^2(\Omega)$ satisfies (29) and $u > 0$ in Ω, then*

$$\begin{array}{rcll} u(x) & < & u(x_\lambda) & \text{for all } \lambda < \lambda^* \text{ and } x \in \Sigma_\lambda, \\ \frac{\partial u}{\partial x_1}(x) & > & 0 & \text{for all } x \in \Sigma_\lambda. \end{array}$$

Remark 6.6 *In case that $f(0) \geq 0$ one might replace $u > 0$ in Ω by $0 \not\equiv u \geq 0$. Although it seems likely that $f(0) \geq 0$ is not a genuine restriction, see [2], the general case is still open.*

Proof. First remark that if $u \equiv 0$ on some open set in Ω, then $f(0) = 0$. If $f(0) = 0$ then u satisfies $-\Delta u = c(x)\, u(x)$ with $c(x) = f(u(x))/u(x)$ a bounded function. By using the Maximum Principle for nonpositive functions it follows that $u \equiv 0$ in Ω. Hence $u \not\equiv 0$ on any open set in Ω.

We will consider $w_\lambda(x) = u(x_\lambda) - u(x)$ for $x \in \Sigma_\lambda$. Defining g_λ similar as before in (21), namely

$$g_\lambda(x) = \begin{cases} \frac{f(u(x_\lambda)) - f(u(x))}{u(x_\lambda) - u(x)} & \text{if } u(x_\lambda) \neq u(x), \\ 0 & \text{if } u(x_\lambda) \neq u(x), \end{cases}$$

we find

$$\begin{aligned} -\Delta w_\lambda(x) &= -\Delta(u(x_\lambda)) + \Delta u(x) = -(\Delta u)(x_\lambda) + \Delta u(x) \\ &= f(u(x_\lambda)) - f(u(x)) = g_\lambda(x)\, w_\lambda(x) \text{ in } \Sigma_\lambda. \end{aligned} \tag{32}$$

Moreover, for $\lambda < \lambda^*$ we have $u(x_\lambda) \geq 0 = u(x)$ on $\partial\Omega \cap \partial\Sigma_\lambda$ and $u(x_\lambda) = u(x)$ on $\Omega \cap \partial\Sigma_\lambda$. Hence

$$w_\lambda(x) \geq 0 \text{ on } \partial\Sigma_\lambda. \tag{33}$$

Also note that since u is bounded and f is Lipschitz the function g_λ is uniformly bounded on $\bar{\Omega}$ and hence that we may use the maximum principles for linear equations with a uniform bound for $\|c^+\|_\infty$.

The two basic ingredients in the proof are the maximum principle for small domains (Theorem 5.6) and again the strong maximum principle for nonpositive functions (Theorem 2.8).

Set $\lambda_* = \sup\{\lambda \in [\lambda_0, \lambda^*]\,; w_\mu(x) \geq 0 \text{ for all } x \in \Sigma_\mu \text{ and } \mu \in [0, \lambda]\}$. We will suppose that $\lambda_* < \lambda^*$ and arrive at a contradiction.

By the maximum principle for small domains (or even the one for narrow domains) one finds from (32) and (33) that there is $\lambda_1 > \lambda_0$ such that $w_\lambda(x) \geq 0$ on $\bar{\Sigma}_\lambda$ for all $\lambda \in (\lambda_0, \lambda_1]$. Hence $\lambda_* > 0$. By the strong maximum principle either $w_\lambda > 0$ in Σ_λ for all $\lambda \in (\lambda_0, \lambda_1]$ or $w_\lambda \equiv 0$ on $\bar{\Sigma}_\lambda$ for some $\lambda \in (\lambda_0, \lambda_1]$. We have found that $\lambda_* \geq \lambda_1$.

Next we will show that $w_\lambda \equiv 0$ on $\bar{\Sigma}_\lambda$ does not occur for $\lambda \in (\lambda_0, \lambda_1)$. If $w_\lambda \equiv 0$ on $\bar{\Sigma}_\lambda$ for some $\lambda \in (\lambda_0, \lambda^*)$ then, $u(x) = 0$ for $x \in \partial\left(\Sigma'_\lambda \cup \bar{\Sigma}_\lambda\right)$ and moreover, since part of this boundary, $\Omega \cap \partial\left(\Sigma'_\lambda \cup \bar{\Sigma}_\lambda\right)$ lies inside Ω it contradicts $u > 0$ in Ω.

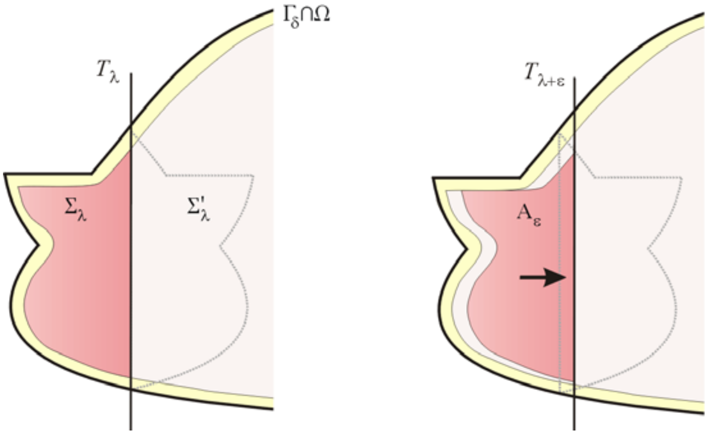
For $\lambda = \lambda_*$ we have that $w_{\lambda_*}(x) \geq 0$ on $\bar{\Sigma}_{\lambda_*}$ and since $\bar{\Sigma}'_{\lambda_*} \cup \bar{\Sigma}_{\lambda_*} \neq \Omega$ we just found that $w_{\lambda_*}(x) \not\equiv 0$ on $\bar{\Sigma}_{\lambda_*}$. By the strong maximum principle for signed functions (Theorem 2.8) it follows that $w_{\lambda_*}(x) > 0$ in Σ_{λ_*} and even that $\frac{\partial}{\partial x_1} w_{\lambda_*}(x) > 0$ for every $x \in \partial\Sigma_{\lambda_*} \cap \Omega$. In order to find a strict bound for w_{λ_*} away from 0 we restrict ourselves to a compact set as follows. Let δ be as in Theorem 5.6 and let Γ_δ be an open neighborhood of $\partial\Omega \cap \partial\Sigma_{\lambda_*}$ such that $|\Omega \cap \Gamma_\delta| < \frac{1}{2}\delta$. By the previous estimate we have for some $c > 0$ that

$$w_{\lambda_*}(x) > c(\lambda_* - x_1) \text{ for } x \in \Sigma_{\lambda_*} \backslash \Gamma_\delta.$$

By continuity we may increase λ somewhat without loosing the positivity. Indeed, by the fact that $w_{\lambda_*} \in C^1(\Sigma_{\lambda^*})$ there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in [0, \varepsilon_0]$ one finds

$$w_{\lambda_* + \varepsilon}(x) > 0 \text{ for } x \in A_\varepsilon,$$

where $A_\varepsilon = \left\{x \in \Omega; (x_1 + \varepsilon, x_2, \ldots, x_n) \in \bar{\Sigma}_{\lambda_*} \backslash \Gamma_\delta\right\} \subset \Sigma_{\lambda_* + \varepsilon}$ is a shifted Σ_{λ_*}.



Since $\lambda \to |\Sigma_\lambda|$ is continuous and $A_\varepsilon \subset \Sigma_{\lambda+\varepsilon}$ we may take $\varepsilon_1 \in (0, \varepsilon_0)$ such that

$$\left|\bar{\Sigma}_{\lambda_* + \varepsilon} \backslash (\Gamma_\delta \cap A_\varepsilon)\right| < \frac{1}{2}\delta \text{ for } \varepsilon \in [0, \varepsilon_1].$$

Finally we consider the maximum principle for small domains on the remaining subset of $\Sigma_{\lambda+\varepsilon}$ defined by $R_\varepsilon = (\Sigma_{\lambda_*+\varepsilon} \cap \Gamma_\delta) \cup (\Sigma_{\lambda_*+\varepsilon} \backslash A_\varepsilon)$ with $\varepsilon \in (0, \varepsilon_1)$. Since $w_{\lambda_*+\varepsilon} > 0$ on ∂R_ε and $-\Delta w_{\lambda_*+\varepsilon}(x) = g_{\lambda_*+\varepsilon}(x)\, w_{\lambda_*+\varepsilon}(x)$ in R_ε we find that $w_{\lambda_*+\varepsilon} \geq 0$ in R_ε and hence $w_\lambda \geq 0$ on $\bar{\Sigma}_\lambda$ for all $\lambda \in [\lambda_0, \lambda_* + \varepsilon_1]$, a contradiction.

The conclusion that $\frac{\partial}{\partial x_1} u(x) > 0$ for $x \in \Sigma_{\lambda^*}$ follows from the fact that $w_\lambda(x) > 0$ on Σ_λ for all $\lambda \in (\lambda_0, \lambda^*)$ and by Hopf's boundary point Lemma $\frac{\partial}{\partial x_1} u(x) = \frac{1}{2} \frac{\partial}{\partial x_1} w_\lambda(x) > 0$ on $T_\lambda \cap \Omega$. ■

Corollary 6.7 *Let $B_R(0) \subset \mathbb{R}^n$. If f is Lipschitz and if $u \in C^2(B) \cap C(\bar{B})$ satisfies*

$$\left\{ \begin{array}{rcll} -\Delta u & = & f(u) & \text{in } B, \\ u & = & 0 & \text{on } \partial B, \\ u & > & 0 & \text{in } B, \end{array} \right. \tag{34}$$

then u is radially symmetric and $\frac{\partial}{\partial |x|} u(x) < 0$ for $0 < |x| < R$.

Proof. From the previous theorem we find that $x_1 \frac{\partial}{\partial x_1} u(x) < 0$ for $x \in B_R(0)$ with $x_1 \neq 0$. Hence $\frac{\partial}{\partial x_1} u(x) = 0$ for $x_1 = 0$. Since Δ is radially invariant this holds for every direction and we find $\frac{\partial}{\partial \tau} u(x) = 0$ in $B_R(0)$ for any tangential direction. In other words, u is radially symmetric. Since $\frac{\partial}{\partial |x|} u(x) = \frac{\partial}{\partial r} u(r, 0, \ldots, 0)$ for $0 < r = |x| < R$ the second claim follows from $\frac{\partial u}{\partial x_1}(x) < 0$ for $x \in B_R(0)$ with $x_1 > 0$. ■

Corollary 6.8 *Let Ω be a bounded domain in $\mathbb{R}^n$. For $\nu \in \mathbb{R}^n$ with $|\nu| = 1$ we set*

$$\begin{array}{rcl} \Sigma_\lambda(\nu) & = & \{x \in \Omega; x \cdot \nu > \lambda\}, \\ \Sigma'_\lambda(\nu) & = & \{x + 2(\lambda - x \cdot \nu)\nu \in \Sigma_\lambda(\nu)\}, \\ \lambda^*(\nu) & = & \inf\{\lambda; \Sigma'_\mu(\nu) \subset \Omega \text{ for all } \mu > \lambda\}. \end{array}$$

Define $\Omega^\star = \Omega \backslash \cup \left\{\Sigma'_{\lambda^*(\nu)}(\nu); \nu \in \mathbb{R}^n \text{ with } |\nu| = 1\right\}$. If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a solution of

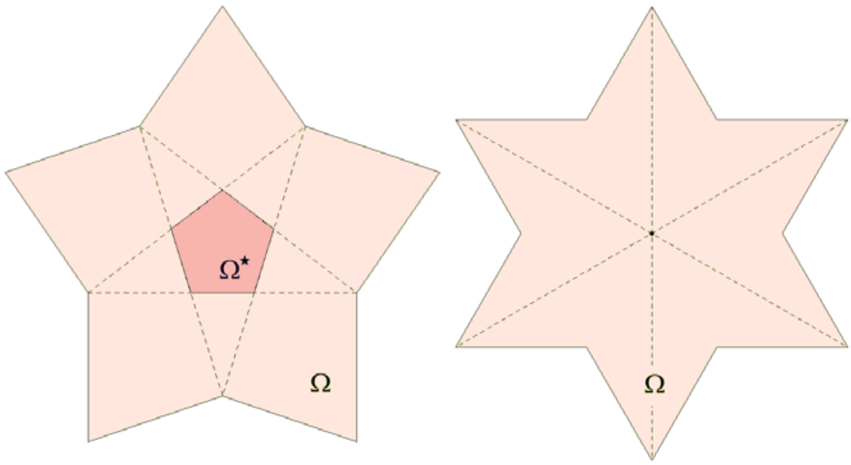
$$\left\{\begin{array}{rcll} -\Delta u & = & f(u) & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \\ u & > & 0 & \text{in } \Omega, \end{array}\right. \tag{35}$$

then

$$u < \max_{\Omega} u \text{ on } \Omega \backslash \Omega^\star.$$

Proof. The result is a direct consequence of using a moving plane argument with a plane perpendicular to ν and applying the previous theorem. ■

Example 6.9 *Consider solutions of (35) for the following two two-dimensional domains, respectively a star with five and one with six points, both having some parallel boundary segments:*



Some optimal positions for the moving plane are drawn by the dashed lines. Note that viewing on a screen might result in some distortion.

For the six-star one finds by the moving plane argument that the maximum lies in the center and even that the solution is symmetric with respect to the three axes that coincide with optimal planes. For the five-star the moving plane argument shows that the maximum of a solution lies in the central pentagon but the argument does not yield symmetry.

The moving plane argument has recently been extended to a moving sphere argument by Reichel and Zou ([9]). It can be used to obtain a result closely related with a famous one by Pohožaev ([7]).

Theorem 6.10 *Let $\Omega \subset \mathbb{R}^n$ with $n \geq 3$ be bounded and* starshaped[4] *and let f be Lipschitz and supercritical:*

$$u \longmapsto u^{-\frac{n+2}{n-2}} f(u) \text{ is nondecreasing.} \tag{36}$$

Then (35) has no nonzero $C^2(\Omega) \cap C(\bar{\Omega})$-solution.

[4] A set Ω is called starshaped with respect to x_0 if for every $x \in \Omega$ the segment $[x_0, x]$ lies in Ω :

$$\{\theta x + (1-\theta) x_0; 0 \leq \theta \leq 1\} \subset \Omega.$$

Remark 6.11 *A famous result of Pohožaev in [7] states that (35), with $f \in C^{0,\alpha}$ ($\alpha > 0$) and Ω as above, has no positive solution in $W^{2,2}(\Omega) \cap C(\bar{\Omega})$ whenever*

$$\frac{n-2}{2n} u\, f(u) \geq \int_0^u f(t)\, dt \text{ for all } u \geq 0. \tag{37}$$

Condition (36) implies (37):

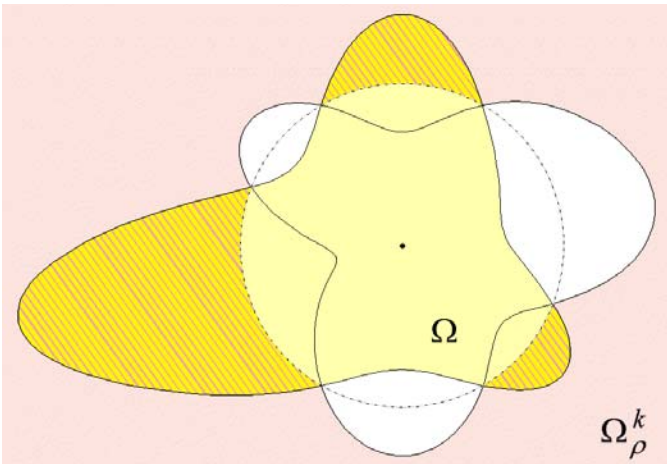
$$\int_0^u f(t)\, dt = \int_0^u t^{\frac{n+2}{n-2}} t^{-\frac{n+2}{n-2}} f(t)\, dt \leq u^{-\frac{n+2}{n-2}} f(u) \int_0^u t^{\frac{n+2}{n-2}} dt = \frac{n-2}{2n} u\, f(u).$$

The conditions are equal for pure powers $f(u) = u^p$.

Proof. Without loss of generality we may suppose that Ω is starshaped with respect to the origin. Suppose that u is a nonzero $C^2(\Omega) \cap C(\bar{\Omega})$-solution of (35). Instead of a standard reflection one uses a weighted reflection with respect to circles. The 'Kelvin'-transformed u becomes

$$u_\rho(x) = \left(\frac{\rho}{|x|}\right)^{n-2} u\left(\frac{\rho^2}{|x|^2} x\right)$$

and is well-defined on $\Omega_\rho^k = \left\{x \in \mathbb{R}^n; \frac{\rho^2}{|x|^2} x \in \Omega\right\}$.



This u_ρ satisfies, writing $r = |x|$, $\omega = \frac{x}{|x|}$ and Δ_ω the Laplace-Beltrami operator on ∂B_1,

$$\begin{aligned}
\Delta u_\rho(r,\omega) &= \left(r^{1-n}\frac{\partial}{\partial r} r^{n-1}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_\omega\right) u_\rho(r,\omega) \\
&= \left(r^{1-n}\frac{\partial}{\partial r} r^{n-1}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_\omega\right)\left(\left(\frac{\rho}{r}\right)^{n-2} u\left(\tfrac{\rho^2}{r},\omega\right)\right) \\
&= \rho^{n-2} r^{1-n}\frac{\partial}{\partial r} r^{n-1}\frac{\partial}{\partial r} r^{2-n} u\left(\tfrac{\rho^2}{r},\omega\right) + \left(\frac{\rho}{r}\right)^{n+2}\frac{1}{(\rho^2/r)^2}\Delta_\omega u\left(\tfrac{\rho^2}{r},\omega\right) \\
&= \left(\frac{\rho}{r}\right)^{n+2}\left((u_{rr})\left(\tfrac{\rho^2}{r},\omega\right) + \frac{n-1}{\rho^2/r}(u_r)\left(\tfrac{\rho^2}{r},\omega\right) + \frac{1}{(\rho^2/r)^2}\Delta_\omega u\left(\tfrac{\rho^2}{r},\omega\right)\right) \\
&= \left(\frac{\rho}{r}\right)^{n+2}(\Delta u)\left(\tfrac{\rho^2}{r},\omega\right)
\end{aligned}$$

One uses $u \longmapsto u^{-\frac{n+2}{n-2}} f(u)$ is nondecreasing to find that for $\rho \leq r$

$$\begin{aligned}
-\Delta u_\rho(r,\omega) &= \left(\frac{\rho}{r}\right)^{n+2} f\left(u\left(\tfrac{\rho^2}{r},\omega\right)\right) \\
&= \left(\frac{\rho}{r}\right)^{n+2} f\left(\left(\tfrac{r}{\rho}\right)^{n-2} u_\rho(r,\omega)\right) \geq f(u_\rho(r,\omega)).
\end{aligned}$$

Denoting $\Sigma_\rho = \{x \in \Omega; |x| > \rho\}$ the starshapedness of Ω implies that u_ρ is defined on Σ_ρ (whenever it is nonempty) and one proceeds as in the moving plane argument with

$$w_\rho(x) = u_\rho(x) - u(x).$$

One finds on $\partial\Sigma_\rho \cap \Omega$ that $w_\rho = 0$ and on $\partial\Sigma_\rho \cap \partial\Omega$ that $w_\rho = u_\rho - u = u_\rho \geq 0$. Moreover, for $x \in \Sigma_\rho$ one has

$$\begin{aligned} -\Delta w_\rho &= -\Delta u_\rho + \Delta u \\ &= f(u_\rho) - f(u) + (-\Delta u_\rho - f(u_\rho)) \geq g_\rho(x)\, w_\rho, \end{aligned}$$

with $g_\rho(x)$ defined similar as before using u_ρ and u. One finds

$$u_\rho(x) > u(x) \text{ for } x \in \Omega \text{ with } |x| > \rho.$$

By decreasing ρ one obtains for $x \neq 0$ that

$$u(x) \leq \lim_{\rho \downarrow 0} u_\rho(x) \leq \lim_{\rho \downarrow 0} \left(\frac{\rho}{|x|}\right)^{n-2} \max_\Omega u = 0,$$

a contradiction. ■

7 References

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