and

(4) $$\theta_{m,\mu}(\tau,z) \;:=\; \sum_{\substack{r\in\mathbb{Z}\\ r\equiv\mu(\mathrm{mod}\ 2m)}} q^{r^2/4m}\,\zeta^r \quad .$$

(The $\theta_{m,\mu}$ are independent of the function ϕ.) Then

$$\phi(\tau,z) \;=\; \sum_{\mu(\mathrm{mod}\ 2m)} \;\sum_{\substack{r\in\mathbb{Z}\\ r\equiv\mu(2m)}} \;\sum_{n\geq r^2/4m} c_\mu(4nm-r^2)q^n\zeta^r$$

(5) $$= \sum_{\mu(\mathrm{mod}\ 2m)} \;\sum_{r\equiv\mu(2m)} \;\sum_{N\geq 0} c_\mu(N)\; q^{\frac{N+r^2}{4m}}\,\zeta^r$$

$$= \sum_{\mu(\mathrm{mod}\ 2m)} h_\mu(\tau)\;\theta_{m,\mu}(\tau,z) \quad .$$

Thus knowing the (2m)-tuple $(h_\mu)_{\mu(\mathrm{mod}\ 2m)}$ of functions of one variable is equivalent to knowing ϕ. Reversing the above calculation, we see that given any functions h_μ as in (3) with $c_\mu(N)=0$ for $N \not\equiv -\mu^2(\mathrm{mod}\ 4m)$, equation (5) defines a function ϕ (with Fourier coefficients as in (1)) which transforms like a Jacobi form with respect to $z \mapsto z+\lambda\tau+\mu$ $(\lambda,\mu\in\mathbb{Z})$ and satisfies the right conditions at infinity. In order for ϕ to be a Jacobi form, we still need a transformation law with respect to $SL_2(\mathbb{Z})$. Since the theta-series (4) have weight $\frac{1}{2}$ and index m, while ϕ has weight k and index m, we see from (5) that the h_μ must be modular forms of weight $k-\frac{1}{2}$. To specify their precise transformation law, it suffices to consider the generators $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ of Γ_1. For the first we have

(6) $$\theta_{m,\mu}(\tau+1,z) \;=\; e_{4m}(\mu^2)\;\theta_{m,\mu}(\tau,z)$$

and

(7) $$h_\mu(\tau+1) \;=\; e_{4m}(-\mu^2)\;h_\mu(\tau) \quad ,$$

as one sees either from the invariance of the sum (5) under $\tau \mapsto \tau+1$ or from the congruence $N\equiv -\mu^2(\mathrm{mod}\ 4m)$ in (3). For the second we have as an easy consequence of the Poisson summation formula the identity

(8) $$\theta_{m,\mu}\Big(-\frac{1}{\tau},\frac{z}{\tau}\Big) \;=\; \sqrt{\tau/2mi}\; e^{2\pi imz^2/\tau} \sum_{\nu(\mathrm{mod}\ 2m)} e_{2m}(-\mu\nu)\theta_{m,\nu}(\tau,z) \;,$$

so (5) and the transformation law of ϕ under $(\tau,z)\mapsto\Big(-\frac{1}{\tau},\frac{z}{\tau}\Big)$ give

(9) $$h_\mu(-\tfrac{1}{\tau}) \;=\; \frac{\tau^k}{\sqrt{2m\tau/i}} \sum_{\nu(\mathrm{mod}\ 2m)} e_{2m}(\mu\nu)\; h_\nu(\tau) \quad .$$

We have proved

<u>THEOREM 5.1</u>. *Equation (5) gives an isomorphism between* $J_{k,m}$ *and the space of vector valued modular forms* $(h_\mu)_{\mu(\mathrm{mod}\ 2m)}$ *on* $SL_2(\mathbb{Z})$ *satisfying the transformation laws (7) and (9) and bounded as* $\mathrm{Im}(\tau)\to\infty$.

When we spak of "vector-valued" forms in Theorem 5.1, we mean that the vector $\vec{h}(\tau) = (h_\mu)_{\mu(\mathrm{mod}\ 2m)}$ satisfies

(10) $$\vec{h}(M\tau) \;=\; (c\tau+d)^{k-\frac{1}{2}}\,U(M)\vec{h}(\tau) \qquad \Big(M=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_1\Big)$$

where $U(M)=(U_{\mu\nu}(M))$ is a certain $2m\times 2m$ matrix (the map $U\colon \Gamma_1\to GL_{2m}(\mathbb{C})$ is not quite a homomorphism because of the ambiguities arising from the choice of square-root in (10); to get a homomorphism one must replace Γ_1 by a double cover). The result 5.1 would be more pleasing if we could identify $J_{k,m}$ with a space of ordinary (i.e. scalar) modular forms of weight $k-\frac{1}{2}$ on some congruence subgroup of Γ_1. We will do this below in the cases $m=1$ and m prime, k even, and also discuss the general case a little. First, however, we look at some immediate consequences of Theorem 5.1.

4.2 Jacobi'sche Thetatransformationsformel (C. G. J. Jacobi, 1828). *Für $(z,w)\in\mathbb{H}\times\mathbb{C}$ gilt die Formel*

$$\boxed{\sqrt{\frac{z}{\mathrm{i}}}\sum_{n=-\infty}^{\infty} e^{\pi\mathrm{i}(n+w)^2 z} = \sum_{n=-\infty}^{\infty} e^{\pi\mathrm{i}n^2(-1/z)+2\pi\mathrm{i}nw}.}$$

Dabei ist die Quadratwurzel aus z/i durch den Hauptzweig des Logarithmus definiert.

Beweis. Die Funktion

$$f(w) := \sum_{n=-\infty}^{\infty} e^{\pi\mathrm{i}z(n+w)^2} \quad (z \text{ fest})$$

hat offenbar die Periode 1 und gestattet daher eine Fourierentwicklung

$$f(w) = \sum_{m=-\infty}^{\infty} a_m e^{2\pi\mathrm{i}mw}$$

mit

$$a_m = \int_0^1 \sum_{n=-\infty}^{\infty} e^{\pi\mathrm{i}z(n+w)^2-2\pi\mathrm{i}mw}\,du.$$

Dabei sei $w=u+\mathrm{i}v$. Der Imaginärteil v von w kann dabei beliebig gewählt werden. Wir werden über ihn noch geeignet verfügen. Wegen der lokal gleichmäßigen Konvergenz darf man Summe und Integral vertauschen. Anschließende Substitution $u\mapsto u-n$ zeigt

$$a_m = \int_{-\infty}^{\infty} e^{\pi\mathrm{i}(zw^2-2mw)}\,du.$$

Durch quadratische Ergänzung erhält man

$$zw^2-2mw = z\Big(w-\frac{m}{z}\Big)^2 - z^{-1}m^2,$$

also

$$a_m = e^{-\pi\mathrm{i}m^2z^{-1}}\int_{-\infty}^{\infty} e^{\pi\mathrm{i}z(w-m/z)^2}\,du.$$

Nun wählen wir den Imaginärteil v von w so, daß $w-m/z$ reell wird. Nach einer Translation von u erhält man dann

$$a_m = e^{\pi\mathrm{i}m^2(-1/z)}\int_{-\infty}^{\infty} e^{\pi\mathrm{i}zu^2}\,du.$$

Es bleibt das Integral zu berechnen. Wir müssen die Formel

$$\int_{-\infty}^{\infty} e^{\pi \mathrm{i} z u^2}\, du = \sqrt{\frac{z}{\mathrm{i}}}^{\,-1}$$

beweisen. Da beide Seiten analytische Funktionen in z darstellen, genügt es, sie für rein imaginäre $z = \mathrm{i}y$ zu beweisen. Die Substitution

$$t = u \cdot \sqrt{y}$$

führt die Berechnung auf das bekannte Integral

$$\int_{-\infty}^{\infty} e^{-\pi t^2} dt = 1$$

zurück. □

Spezialisiert man die Jacobi'sche Thetatransformationsformel, so erhält man

4.3 Satz. *Die Funktion*

$$\vartheta(z) = \sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i} n^2 z}$$

stellt eine analytische Funktion dar. Sie genügt den Transformationsformeln

a) $$\vartheta(z+2) = \vartheta(z) \quad \textit{und}$$

b) $$\vartheta\Big(-\frac{1}{z}\Big) = \sqrt{\frac{z}{\mathrm{i}}}\,\vartheta(z).$$

Die Thetareihe $\vartheta(z)$ hat nur die Periode 2. Um zu einer Modulform zu gelangen, betrachten wir neben ϑ auch $\widetilde{\vartheta}(z) = \vartheta(z+1)$,

$$\widetilde{\vartheta}(z) = \sum_{n=-\infty}^{\infty} (-1)^n \exp \pi \mathrm{i} n^2 z.$$

Die Funktion $\widetilde{\vartheta}$ ist ein spezieller Wert der Jacobi'schen Thetafunktion $\vartheta(z,w)$, nämlich

$$\widetilde{\vartheta}(z) = \vartheta(z, 1/2).$$

Man erhält aus 4.2 eine Transformatinsformel für $\widetilde{\vartheta}$, nämlich

$$\widetilde{\vartheta}\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{\mathrm{i}}}\,\widetilde{\widetilde{\vartheta}}(z)$$

Chapter 5

Quadratic Reciprocity

If p is a prime, the discussion of the congruence $x^2 \equiv a\ (p)$ is fairly easy. It is solvable iff $a^{(p-1)/2} \equiv 1\ (p)$. With this fact in hand a complete analysis is a simple matter. However, if the question is turned around, the problem is much more difficult. Suppose that a is an integer. For which primes p is the congruence $x^2 \equiv a\ (p)$ solvable? The answer is provided by the law of quadratic reciprocity. This law was formulated by Euler and A. M. Legendre but Gauss was the first to provide a complete proof. Gauss was extremely proud of this result. He called it the Theorema Aureum, *the golden theorem.*

§1 Quadratic Residues

If $(a, m) = 1$, a is called a quadratic residue mod m if the congruence $x^2 \equiv a\ (m)$ has a solution. Otherwise a is called a quadratic nonresidue mod m.

For example, 2 is a quadratic residue mod 7, but 3 is not. In fact, $1^2, 2^2, 3^2, 4^2, 5^2$, and 6^2 are congruent to 1, 4, 2, 2, 4, and 1, respectively. Thus 1, 2, and 4 are quadratic residues, and 3, 5, and 6 are not.

Given any fixed positive integer m it is possible to determine the quadratic residues by simply listing the positive integers less than and prime to m, squaring them, and reducing mod m. This is what we have just done for $m = 7$.

The following proposition gives a less tedious way of deciding when a given integer is a quadratic residue mod m.

Proposition 5.1.1. *Let $m = 2^e p_1^{e_1} \cdots p_l^{e_l}$ be the prime decomposition of m, and suppose that $(a, m) = 1$. Then $x^2 \equiv a\ (m)$ is solvable iff the following conditions are satisfied:*

(a) *If $e = 2$, then $a \equiv 1\ (4)$.*
If $e \geq 3$, then $a \equiv 1\ (8)$.
(b) *For each i we have $a^{(p_i-1)/2} \equiv 1\ (p_i)$.*

PROOF. By the Chinese Remainder Theorem the congruence $x^2 \equiv a\ (m)$ is equivalent to the system $x^2 \equiv a\ (2^e)$, $x^2 \equiv a\ (p_1^{e_1}), \ldots, x^2 \equiv a\ (p_l^{a_l})$.

Consider $x^2 \equiv a\ (2^e)$. 1 is the only quadratic residue mod 4, and 1 is the only quadratic residue mod 8. Thus we have solvability iff $a \equiv 1\ (4)$ if $e = 2$ and $a \equiv 1\ (8)$ if $e = 3$. A direct application of Proposition 4.2.4 shows that $x^2 \equiv a\ (8)$ is solvable iff $x^2 \equiv a\ (2^e)$ is solvable for all $e \geq 3$.

Now consider $x^2 \equiv a\ (p_i^{e_i})$. Since $(2, p_i) = 1$ it follows from Proposition 4.2.3 that this congruence is solvable iff $x^2 \equiv a\ (p_i)$ is solvable. To this congruence apply Proposition 4.2.1 with $n = 2$, $m = p$, and $d = (n, \phi(m)) = (2, p - 1) = 2$. We obtain that $x^2 \equiv a\ (p_i)$ is solvable iff $a^{(p_i-1)/2} \equiv 1\ (p_i)$. □

This result reduces questions about quadratic residues to the corresponding questions for prime moduli. In what follows p will denote an odd prime.

Definition. The symbol (a/p) will have the value 1 if a is a quadratic residue mod p, -1 if a is a quadratic nonresidue mod p, and zero if $p \mid a$. (a/p) is called the *Legendre symbol.*

The Legendre symbol is an extremely convenient device for discussing quadratic residues. We shall list some of its properties.

Proposition 5.1.2.
(a) $a^{(p-1)/2} \equiv (a/p)\ (p)$.
(b) $(ab/p) = (a/p)(b/p)$.
(c) *If $a \equiv b\ (p)$, then $(a/p) = (b/p)$.*

PROOF. If p divides a or b, all three assertions are trivial. Assume that $p \nmid a$ and that $p \nmid b$.

We know that $a^{p-1} \equiv 1\ (p)$; thus $(a^{(p-1)/2} + 1)(a^{(p-1)/2} - 1) = a^{p-1} - 1 \equiv 0\ (p)$. It follows that $a^{(p-1)/2} \equiv \pm 1\ (p)$. By Proposition 5.1.1, $a^{(p-1)/2} \equiv 1\ (p)$ iff a is a quadratic residue mod p. This proves part (a).

To prove part (b) we apply part (a). $(ab)^{(p-1)/2} \equiv (ab/p)\ (p)$ and $(ab)^{(p-1)/2} = a^{(p-1)/2}b^{(p-1)/2} \equiv (a/p)(b/p)\ (p)$. Thus $(ab/p) \equiv (a/p)(b/p)\ (p)$, which implies that $(ab/p) = (a/p)(b/p)$.

Part (c) is obvious from the definition. □

Corollary 1. *There are as many residues as nonresidues* mod p.*

PROOF. $a^{(p-1)/2} \equiv 1\ (p)$ has $(p - 1)/2$ solutions. Thus there are $(p - 1)/2$ residues and $p - 1 - ((p - 1)/2) = (p - 1)/2$ nonresidues. □

Corollary 2. *The product of two residues is a residue, the product of two nonresidues is a residue, and the product of a residue and a nonresidue is a nonresidue.*

PROOF. This all follows easily from part (b). □

* In the remainder of this chapter "residues" and "nonresidues" refer to quadratic residues and quadratic nonresidues.

1.2.3. Dimension formulas.

Proposition 1.25 provides formulas for the dimensions of M_k and S_k. Using the Riemann-Roch Theorem, Cohen and Oesterlé **[COe]** explicitly computed further dimension formulas which we record here because of their utility. To state these formulas, suppose that k is an integer, and that χ is a Dirichlet character modulo N for which $\chi(-1) = (-1)^k$. If $p \mid N$ is prime, then let r_p (resp. s_p) denote the power of p dividing N (resp. the conductor of χ). Define the integer $\lambda(r_p, s_p, p)$ by

$$\lambda(r_p, s_p, p) := \begin{cases} p^{r'} + p^{r'-1} & \text{if } 2s_p \leq r_p = 2r', \\ 2p^{r'} & \text{if } 2s_p \leq r_p = 2r'+1, \\ 2p^{r_p - s_p} & \text{if } 2s_p > r_p. \end{cases} \tag{1.11}$$

In addition, define rational numbers ν_k and μ_k by

$$\begin{aligned} \nu_k &:= \begin{cases} 0 & \text{if } k \text{ is odd}, \\ -1/4 & \text{if } k \equiv 2 \pmod 4, \\ 1/4 & \text{if } k \equiv 0 \pmod 4, \end{cases} \\ \mu_k &:= \begin{cases} 0 & \text{if } k \equiv 1 \pmod 3, \\ -1/3 & \text{if } k \equiv 2 \pmod 3, \\ 1/3 & \text{if } k \equiv 0 \pmod 3. \end{cases} \end{aligned} \tag{1.12}$$

In this notation, we have the following dimension formulas.

THEOREM 1.34. *If k is an integer and χ is a Dirichlet character modulo N for which $\chi(-1) = (-1)^k$, then*

$$\begin{aligned} &\dim_{\mathbb{C}}(S_k(\Gamma_0(N), \chi)) - \dim_{\mathbb{C}}(M_{2-k}(\Gamma_0(N), \chi)) \\ &= \frac{(k-1)N}{12} \cdot \prod_{p \mid N}(1 + p^{-1}) \\ &\quad - \frac{1}{2}\prod_{p \mid N} \lambda(r_p, s_p, p) + \nu_k \sum_{\substack{x \pmod N, \\ x^2+1 \equiv 0 \pmod N}} \chi(x) + \mu_k \sum_{\substack{x \pmod N, \\ x^2+x+1 \equiv 0 \pmod N}} \chi(x), \end{aligned}$$

where p denotes a prime divisor of N (note. empty products are taken to be 1).

REMARK 1.35. If $k > 2$, then $\dim_{\mathbb{C}}(M_{2-k}(\Gamma_0(N), \chi)) = 0$. Hence the left hand of side of Theorem 1.34 reduces to $\dim_{\mathbb{C}}(S_k(\Gamma_0(N), \chi))$. A similar argument applies when $k = 2$, and the result depends on whether χ is trivial. If $k \leq 0$, then $\dim_{\mathbb{C}}(S_k(\Gamma_0(N), \chi)) = 0$. In these cases, the left hand side of Theorem 1.34 reduces to $-\dim_{\mathbb{C}}(M_{2-k}(\Gamma_0(N), \chi))$.

1.3. Half-integral weight modular forms

Although the study of half-integral weight modular forms has its origins in the classic works of Euler, Gauss and Jacobi (among others), many of their most important and fundamental properties require results on half-integral weight Hecke operators in Shimura's 1973 *Annals of Mathematics* paper [**Shi2**][1]. This important paper provided a general framework for studying half-integral weight modular forms by introducing the so-called "Shimura correspondence", a family of maps which relate the Fourier expansions of half-integral weight modular forms to those of integer weight forms. Here we briefly recall basic facts about half-integral weight forms. For background information, one may consult [**Kob2, SSt, Shi2**].

To define these forms, we first define $\left(\frac{c}{d}\right)$ and ϵ_d. If d is an odd prime, then let $\left(\frac{c}{d}\right)$ be the usual Legendre symbol. For positive odd d, define $\left(\frac{c}{d}\right)$ by multiplicativity. For negative odd d, we let

$$\left(\frac{c}{d}\right) := \begin{cases} \left(\frac{c}{|d|}\right) & \text{if } d < 0 \text{ and } c > 0, \\ -\left(\frac{c}{|d|}\right) & \text{if } d < 0 \text{ and } c < 0. \end{cases} \tag{1.13}$$

Also let $\left(\frac{0}{\pm 1}\right) = 1$. Define ϵ_d, for odd d, by

$$\epsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \mod 4, \\ i & \text{if } d \equiv 3 \mod 4. \end{cases} \tag{1.14}$$

Throughout, we let $\sqrt{z}$ be the branch of the square root having argument in $(-\pi/2, \pi/2]$. Hence, $\sqrt{z}$ is a holomorphic function on the complex plane with the negative real axis removed.

Definition 1.36. Suppose that λ is a nonnegative integer and that N is a positive integer. Furthermore, suppose that χ is a Dirichlet character modulo $4N$. A meromorphic function $g(z)$ on $\mathcal{H}$ is called a *meromorphic half-integral weight modular form with Nebentypus* χ and weight $\lambda + \frac{1}{2}$ if it is meromorphic at the cusps of Γ, and if

$$g\left(\frac{az+b}{cz+d}\right) = \chi(d)\left(\frac{c}{d}\right)^{2\lambda+1} \epsilon_d^{-1-2\lambda}(cz+d)^{\lambda+\frac{1}{2}} g(z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$. If $g(z)$ is holomorphic on $\mathcal{H}$ and at the cusps of $\Gamma_0(4N)$, then $g(z)$ is referred to as a *holomorphic half-integral weight modular form.* If $g(z)$ is a holomorphic modular form which vanishes at the cusps of $\Gamma_0(4N)$, then $g(z)$ is known as a *cusp form.* If $g(z)$ is a meromorphic form whose poles (if there are any) are supported at the cusps of $\Gamma_0(4N)$, then $g(z)$ is known as a *weakly holomorphic modular form.*

Remark 1.37. The cusp conditions in Definition 1.36 are determined in natural way which is analogous to the integer weight case (see Definition 1.8 and Remark 1.10).

Remark 1.38. As in the integer weight case, we refer to a *holomorphic half-integral weight modular form* as a *half-integral weight modular form,* and we continue to use the terminology *meromorphic* (resp. *weakly holomorphic*) *half-integral weight modular form.*

[1] In 1977 Shimura was awarded the Frank Nelson Cole Prize by the American Mathematical Society for two of his research papers; one of these was [**Shi2**].

REMARK 1.39. Since $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma_0(4N)$, it follows that there are no nonzero meromorphic half-integral weight modular forms with odd Nebentypus character χ (i.e. with $\chi(-1) = -1$).

As in the integer weight case, these forms constitute $\mathbb{C}$-vector spaces. We denote the $\mathbb{C}$-vector space of weight $\lambda + \frac{1}{2}$ modular (resp. cusp) forms on $\Gamma_0(4N)$ with Nebentypus χ by

$$M_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi) \qquad (\text{resp. } S_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi)). \tag{1.15}$$

If $\chi = \chi_0$ is the trivial character modulo $4N$, then we use the notation

$$M_{\lambda+\frac{1}{2}}(\Gamma_0(4N)) \qquad (\text{resp. } S_{\lambda+\frac{1}{2}}(\Gamma_0(4N))). \tag{1.16}$$

1.3.1. Theta-functions. Theta-functions provide the first examples of half-integral weight modular forms. We begin by defining the prototypical form.

DEFINITION 1.40. The *theta-function* $\theta_0(z)$ is given by the Fourier series

$$\theta_0(z) := \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots .$$

PROPOSITION 1.41. *We have that*

$$\theta_0(z) \in M_{\frac{1}{2}}(\Gamma_0(4)).$$

More generally, we have the following two families of theta-functions.

DEFINITION 1.42. Suppose that ψ is a Dirichlet character.

(1) If ψ is even, then define $\theta(\psi, 0, z)$ by

$$\theta(\psi, 0, z) := \sum_{n=-\infty}^{\infty} \psi(n) q^{n^2}.$$

(2) If ψ is odd, then define $\theta(\psi, 1, z)$ by

$$\theta(\psi, 1, z) := \sum_{n=1}^{\infty} \psi(n) n q^{n^2}.$$

By convention, we agree that

$$\theta(\chi_0, 0, z) := \theta_0(z).$$

REMARK 1.43. We shall refer to these theta-functions as *single variable theta-functions.*

As modular forms, we have the following elegant fact.

THEOREM 1.44. *Suppose that ψ is a primitive Dirichlet character with conductor $r(\psi)$.*

(1) *If ψ is even, then $\theta(\psi, 0, z) \in M_{\frac{1}{2}}(\Gamma_0(4 \cdot r(\psi)^2), \psi)$.*

(2) *If ψ is odd, then $\theta(\psi, 1, z) \in S_{\frac{3}{2}}(\Gamma_0(4 \cdot r(\psi)^2), \psi\chi_{-4})$, where χ_{-4} is the nontrivial Dirichlet character modulo 4.*

Serre and Stark [**SSt**] proved that every modular form of weight $1/2$ is a linear combination of theta-functions. In particular, they obtained the following complete description of the spaces of weight $1/2$ modular forms.