This function does not appear in the "Lost" Notebook; however $\beta(q)$ is related to $\mu(q)$ and Mordell integrals through identities such as

$$(1.9)\qquad q^{-1/8}\sqrt{\frac{\pi}{2\alpha}}\ \mu(q) = \frac{2\pi}{\alpha}\ q_1^{1/2}\ \beta(q_1) + \int_{-\infty}^{\infty} \frac{e^{-2\alpha x^2+\alpha x}}{1+e^{2\alpha x}}\ dx,$$

where $q = e^{-\alpha}$, $q_1 = e^{-\beta}$, $\alpha\beta = \pi^2$.

The main body of related results that Ramanujan presents in the "Lost" Notebook entails Eulerian series (or q-series) identities for these functions and will be treated in Section 3. The developments in Section 3 will center on a single analytic result (Lemma 2) that can be utilized to prove a very large number of Ramanujan's results. The transformations of Section 2 together with the resulting discovery of $M_3(q)$ allows us to find a number of results that apparently eluded Ramanujan. Section 4 will provide this extension of some of Ramanujan's results. As in [7], [8], and [9] we shall use the subscript "R" on each equation that appears in Ramanujan's "Lost" Notebook.

2. <u>The Mordell integrals</u>. Here we derive the basic modular type transformations for $M_1(q)$, $M_2(q)$ and $M_3(q)$.

<u>THEOREM 1</u>. For $q = e^{-\alpha}$, $q_1 = e^{-\pi^2/\alpha}$, Re $\alpha > 0$,

$$(2.1)\qquad M_1(q) = \frac{2\pi}{\alpha}\, q_1^{1/2}\, M_3(q_1) + \Big(\sum_{k=-\infty}^{\infty} (-1)^k q_1^{2k^2}\Big)\int_{-\infty}^{\infty} \frac{e^{-2\alpha x^2+\alpha x}}{e^{2\alpha x}+1}\ dx.$$

<u>REMARK</u>. To simplify our proof we shall assume that α is real with $0 < \alpha < 1$. The full theorem then follows by analytic continuation.

<u>PROOF</u>. We apply the Poisson summation formula [11; pp. 7-9]:

$$(2.2)\qquad M_1(e^{-\alpha}) = \sum_{n=-\infty}^{\infty} \frac{e^{-2\alpha n^2}}{e^{-\alpha n}+e^{\alpha n}}$$

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty} \frac{e^{-2\alpha n^2}}{\cosh \alpha n}$$

$$= \frac{1}{2} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-2\alpha x^2+2\pi ikx}}{\cosh \alpha x} \, dx.$$

$$= \sum_{k=0}^{\infty}{}^{*} \int_{-\infty}^{\infty} \frac{e^{-2\alpha x^2+2\pi ikx}}{\cosh \alpha x} \, dx,$$

where $\sum^* a_k = \frac{1}{2} a_0 + a_1 + a_2 + a_3 + \dots$.

We now shift the line of integration from the real x-axis to the line passing through the stationary point of the function

$$e^{-2\alpha x^2+2\pi ikx}$$

following the method of steepest descent.

Hence with $x_k = \frac{i\pi k}{2\alpha}$,

$$\text{(2.3)} \qquad M_1(e^{-\alpha}) = \Big(\sum_{k=0}^{\infty}{}^{*} P \int_{-\infty+x_k}^{\infty+x_k} \frac{e^{-2\alpha x^2+2\pi ikx}}{\cosh \alpha x} \, dx\Big)$$

$$+ \Big(\sum_{k=0}^{\infty}{}^{*} 2\pi i \sum_{0 \leqq j \leqq (k-1)/2}^{**} \lambda_{k,j}\Big)$$

$$= I + R,$$

where P denotes the "principal value" of the integral, $\lambda_{k,j}$ is the residue of the integrand at $x = \frac{\pi i}{\alpha}(j+1/2)$, and $\sum^{**}$ means that the factor $\frac{1}{2}$ is introduced for the term $j = \frac{k-1}{2}$ (which occurs only when k is odd).

Now

$$\text{(2.4)} \qquad I = \sum_{k=0}^{\infty}{}^{*} P \int_{-\infty}^{\infty} \frac{e^{-2\alpha(x+x_k)^2+2\pi ik(x+x_k)}}{\cosh \alpha(x+x_k)} \, dx$$

$$= \sum_{k=0}^{\infty}{}^{*} e^{\frac{-\pi^2 k^2}{2\alpha}} P \int_{-\infty}^{\infty} \frac{e^{-2\alpha x^2}}{\cosh \alpha(x+x_k)} \, dx$$

$$= \sum_{k=0}^{\infty}{}^{*} e^{\frac{-2\pi^2 k^2}{\alpha}} (-1)^k \int_{-\infty}^{\infty} \frac{e^{-2\alpha x^2}}{\cosh \alpha x} dx$$

$$-i \sum_{k=0}^{\infty} e^{\frac{-\pi^2(2k+1)^2}{2\alpha}} (-1)^k P \int_{-\infty}^{\infty} \frac{e^{-2\alpha x^2}}{\sinh \alpha x} dx$$

$$= \frac{1}{2} \left(\sum_{k=-\infty}^{\infty} (-1)^k e^{-\frac{2\pi^2 k^2}{\alpha}} \right) \int_{-\infty}^{\infty} \frac{e^{-2\alpha x^2}}{\cosh \alpha x} dx.$$

On the other hand,

$$\text{(2.5)} \qquad R = 2\pi i \sum_{k=0}^{\infty}{}^{*} \sum_{0 \leqq j \leqq (k-1)/2}^{**} \lambda_{k,j}$$

$$= 2\pi i \sum_{j=0}^{\infty} \left(\frac{1}{2} \lambda_{2j+1,j} + \lambda_{2j+2,j} + \lambda_{2j+3,j} + \ldots\right).$$

Since $\lambda_{k,j}$ is the residue of the integrand at $\rho_j = \pi i(j+1/2)/\alpha$, we see that

$$\text{(2.6)} \qquad \lambda_{k,j} = \lim_{x \to \rho_j} (x-\rho_j) \frac{e^{-2\alpha x^2+2\pi ikx}}{\cosh \alpha x}$$

$$= \frac{e^{-2\alpha\rho_j^2+2\pi ik\rho_j}}{i\alpha(-1)^j},$$

and so

$$\text{(2.7)} \qquad \lambda_{2j+1+\ell,j} = \lambda_{2j+1,j} e^{2\pi i\ell\rho_j}.$$

Therefore

$$\text{(2.8)} \qquad R = 2\pi i \sum_{j=0}^{\infty} \lambda_{2j+1,j} \frac{1+e^{2\pi i\rho_j}}{1-e^{2\pi i\rho_j}}$$

$$= \frac{2\pi}{\alpha} \sum_{j=0}^{\infty} (-1)^j e^{-2(\frac{\pi^2}{\alpha})(j+1/2)^2} \frac{1+e^{-(\frac{\pi^2}{\alpha})(2j+1)}}{1-e^{-(\frac{\pi^2}{\alpha})(2j+1)}} .$$

Substituting (2.4) and (2.8) into (2.3), we obtain (2.1). Thus Theorem 1 is proved.

The three remaining theorems of this section are proved similarly so we omit many of the details.

THEOREM 2. For $q = e^{-\alpha}$, $q_1 = e^{-\pi^2/\alpha}$, $\mathrm{Re}\ \alpha > 0$,

$$(2.9)\quad M_2(q) = -\frac{2\pi}{\alpha} q_1^{1/2} M_3(-q_1) + \left(\sum_{k=-\infty}^{\infty} (-1)^k q_1^{2k^2}\right) \int_{\infty}^{\infty} \frac{e^{-2\alpha x^2+\alpha x}}{e^{2\alpha x}+1}\, dx.$$

PROOF. As in Theorem 1,

$$M_2(q) = \frac{1}{2} \sum_{k=-\infty}^{\infty} (-1)^k \int_{-\infty}^{\infty} \frac{e^{-2\alpha x^2+2\pi i k x}}{\cosh x\alpha}\, dx.$$

Thus the only change from Theorem 1 is the introduction of $(-1)^k$, and if we trace its effect throughout the argument we wind up with (2.9).

It is now a simple matter to obtain the corresponding results for $M_3(q)$ and $M_3(-q)$. The main tools required are the transformation formulas for the theta functions $\theta_2(0,q)$ and $\theta_4(0,q)$ as well as the following simple result:

LEMMA 1. Let $\mathrm{Re}\ \alpha > 0$, $\beta = \pi^2/\alpha$. Then

$$\int_0^{\infty} \frac{e^{-2\alpha x^2}}{\cosh \alpha x}\, dx = \frac{1}{2}\left(\frac{2\beta}{\pi}\right)^{3/2} \int_0^{\infty} \frac{e^{-2\beta x^2}}{\cosh 2\beta x}\, dx.$$

PROOF. An equivalent form of this result was actually given by Ramanujan in an issue of the Journal of the Indian Mathematical Society [18; Question 295, pp. 324-325].

THEOREM 3. For $q = e^{-\alpha}$, $q_1 = e^{-\pi^2/\alpha} = e^{-\beta}$, $\mathrm{Re}\ \alpha > 0$,

$$= -\frac{(q;q^2)_\infty}{(-bq)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2}}{(-q^2b^{-1};q^2)_m}$$

$$+ (1+b) \sum_{m=0}^{\infty} \frac{(-q/b;q^2)_m(-b)^m}{(-b^{-1}q^2;q^2)_m}$$

$$(3.31) \qquad = -\frac{(q;q^2)_\infty}{(-bq)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2}}{(-q^2b^{-1};q^2)_m}$$

$$+ \sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2}(q;q^2)_m}{(-bq^2;q^2)_m(-b^{-1}q^2;q^2)_m}$$

(by [18; p. 174, eq. (10.1)] with q replaced by q^2, then $a = q$, $b = q/\tau$, $c = q^2$, $e = -q^2/b$, $f = -bq^2$, $\tau \to 0$),

and this is just (3.13) with a replaced by b.

Now that we have seen the intricacy of these results of Ramanujan we proceed to view them in terms of their interrelationship with the transformations presented in Section 2.

4. Extensions and modular transformations of Ramanujan's functions.

While the approach of the last section provides uniformity, it fails to place these results in the basic hypergeometric hierarchy. In fact many of the results of Section 3 may be viewed as specializations of Watson's q-analog of Whipple's theorem [21; p. 100, eq. (3.4.1.5)]: (N is a nonnegative integer)

$$(4.1) \qquad {}_8\phi_7 \left[\begin{matrix} \alpha, & q\sqrt{\alpha}, & -q\sqrt{\alpha}, & b, & c, & d, & e, & q^{-N} & ; q, \dfrac{\alpha^2 q^{N+2}}{bcde} \\ & \sqrt{\alpha}, & -\sqrt{\alpha}, & \dfrac{\alpha q}{b}, & \dfrac{\alpha q}{c}, & \dfrac{\alpha q}{d}, & \dfrac{\alpha q}{e}, & \alpha q^{N+1} & \end{matrix} \right]$$

$$= \frac{(\alpha q)_N (\frac{\alpha q}{de})_N}{(\frac{\alpha q}{e})_N (\frac{\alpha q}{d})_N} \; {}_4\phi_3 \left[\begin{matrix} \frac{\alpha q}{bc}, & d, & e, & q^{-N} & ; q,q \\ & \frac{\alpha q}{b}, & \frac{\alpha q}{c}, & \frac{deq^{-N}}{\alpha} & \end{matrix} \right]$$

where

(4.2) $${}_r\phi_s \left[\begin{matrix} a_1, a_2, \ldots, & a_r; \; q,t \\ b_1, b_2, \ldots, b_s & \end{matrix} \right]$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_r)_n t^n}{(q)_n (b_1)_n \ldots (b_s)_n} .$$

If in (4.1) we replace q by q^2 then set $\alpha = 1$, $b = a$, $c = a^{-1}$, $d = q$, $e \to \infty$, $N \to \infty$, we obtain a result easily seen to be equivalent to (3.4).

Next in (4.1), we may replace q by q^2 then set $\alpha = q^2$, $b = qa$, $c = qa^{-1}$, $d = -q$, $e \to \infty$, $N \to \infty$ we obtain an extension of (3.2) which, in fact, explicitly specializes to (3.7) when $a = 1$.

Most noteworthy in all these observations however is the fact that neither $M_3(q)$ nor a generalization appears in any of Ramanujan's work. This is of course easily overcome once the role of (4.1) in this work has been established. Actually we find a generalization of $M_3(q)$ if in (4.1) we replace q by q^2 and then set $\alpha = q^2$, $b = aq$, $c = a^{-1}q$, $d = -q^2$, $e \to \infty$, $N \to \infty$. The case $a = 1$ is thus

(4.3) $$M_3(q) = \frac{(q^2;q^2)_\infty}{(-q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n^2+n}}{(q)_{2n+1}(q;q^2)_{n+1}}$$

$$\equiv \frac{(q^2;q^2)_\infty}{(-q^2;q^2)_\infty} \beta(q),$$

where $\beta(q)$ was originally given by (1.8).

Once we have found $\beta(q)$ from (4.3), we see immediately that the four *modular transformations* (2.1), (2.9), (2.10) and (2.12) are now easily translated into transformations connecting $\mu(q)$, $\beta(q)$ and $\alpha(q)$. Indeed the transformations from Section 2 yield the following *four results* immediately once we recall that for

$$(4.4) \qquad \theta_4(0,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q)_\infty}{(-q)_\infty} ,$$

and

$$(4.5) \qquad \frac{1}{2} q^{-1/8} \theta_2(0,\sqrt{q}) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} ,$$

we have [22; p. 263, eq. (8)]

$$(4.6) \qquad \theta_4(0,q_1^2) = \sqrt{\frac{\alpha}{2\pi}}\, \theta_2(0,\sqrt{q}) = q^{1/8} \sqrt{\frac{2\alpha}{\pi}} \sum_{n=0}^{\infty} q^{n(n+1)/2} ,$$

where $q = e^{-\alpha}$, $q_1 = e^{-\beta}$, $\alpha\beta = \pi^2$.

Hence from (2.1) we see that

$$(4.7) \qquad q^{-1/8}\mu(q) = 2\sqrt{\frac{2\pi}{\alpha}}\, q^{1/2}\, \beta(q_1) + \sqrt{\frac{2\alpha}{\pi}} \int_{-\infty}^{\infty} \frac{e^{-2\alpha x^2+\alpha x}}{e^{2\alpha x}+1}\, dx,$$

essentially a restatement of (1.9).

From (2.9), we obtain that

$$(4.8) \qquad q^{-1/8}\alpha(q) = \sqrt{\frac{2\pi}{\alpha}}\, q_1^{1/2}\, \beta(-q_1) - \sqrt{\frac{\alpha}{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-2\alpha x^2+\alpha x}}{e^{2\alpha x}+1}\, dx.$$

From (2.10), it follows that

$$(4.9) \qquad q^{1/2}\beta(q) = \frac{1}{2}\sqrt{\frac{\pi}{2\alpha}}\, q_1^{-1/8}\, \mu(q_1) - 2\sqrt{\frac{\alpha}{2\pi}} \int_{0}^{\infty} \frac{e^{-2\alpha x^2+2\alpha x}}{e^{4\alpha x}+1}\, dx.$$