Wireless Network Simplification: the Gaussian N-Relay Diamond Network

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***Abstract*—We consider the Gaussian N-relay diamond network, where a source wants to communicate to a destination node through a layer of N-relay nodes. We investigate the following question: what fraction of the capacity can we maintain by using only k out of the N available relays? We show that independent of the channel configurations and the operating SNR, we can always find a subset of k relays which alone provide a rate $\frac{k}{k+1}\bar{C} - G$, where $\bar{C}$ is the information theoretic cutset upper bound on the capacity of the whole network and G is independent of the channel coefficients and the SNR and depends only on N and k (logarithmic in N and linear in k). In particular, for $k = 1$, this means that half of the capacity of any N-relay diamond network can be approximately achieved by routing information over a single relay. We also show that this fraction is tight: there are configurations of the N-relay diamond network where every subset of k relays alone can at most provide approximately a fraction $\frac{k}{k+1}$ of the total capacity. These high-capacity k-relay subnetworks can be also discovered efficiently. We propose an algorithm that computes a constant gap approximation to the capacity of the Gaussian N-relay diamond network in $O(N \log N)$ running time and discovers a high-capacity k-relay subnetwork in $O(kN)$ running time.**

This result also provides a new approximation to the capacity of the Gaussian N-relay diamond network which is hybrid in nature: it has both multiplicative and additive gaps. In the intermediate SNR regime, this hybrid approximation is tighter than existing purely additive or purely multiplicative approximations to the capacity of this network.

I. Introduction

Consider a source connected to a destination through a network of wireless relays arranged in an arbitrary topology. There are several ways to use this network. For example, we can route the information from the source to the destination over a single path, using point-to-point connections. Or, following an information theoretic approach, we can seek to optimally utilize all the available relays to achieve the network capacity, the largest end-to-end communication rate this network can support. Clearly the first approach has lower complexity and uses fewer resources of the network, while the second can potentially achieve much higher throughput. In this paper, we aim to understand the fundamental trade-off between using fewer relays and achieving larger rates, and perhaps the possibility of having both at the same time. We ask the following question: can we achieve (a good part of) the capacity of a wireless network by using only a (small) subset of (perhaps a large number of) available relay nodes?

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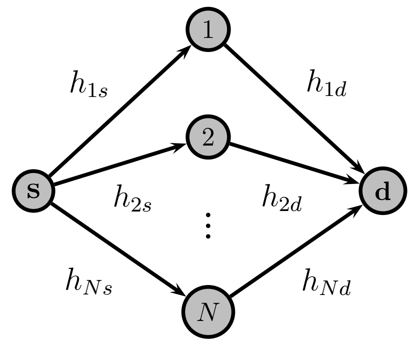

Fig. 1. The Gaussian N-relay diamond network. The source is connected to the relays through a broadcast channel, while the relays are connected to the destination through a multiple-access channel.

Traditionally, network information theory aims to characterize the best end-to-end communication rate we can achieve in a network, without providing any understanding of the importance of each relay for achieving this rate [1], [3], [2], [4], [6]. However, in order to design simple and efficient communication architectures for wireless networks, apart from knowing the capacity of a large network, it may be even more useful to know what is the largest rate we can achieve by using only a given number of the relays. We may want to know how this rate increases if we allow for more relays; how it compares to the capacity of the network; and how to efficiently discover the subset of relays providing the largest capacity.

As a first step in this direction, in this paper, we consider a source that communicates to a destination over the Gaussian N-relay diamond network depicted in Fig. 1. This is a two-stage network, where the source node is connected to N relays through a broadcast channel and the relays are connected to the destination through a multiple-access channel. We ask, what fraction of the capacity we can achieve by using only k out of the N relays (for example, if we route the information between the source node and the destination over a single relay).

The fraction of the capacity we can get with k relays naturally depends on the channel gains. Indeed, consider for example the case where $N = 2$, the diamond network, and the example in Fig. 2. For the identical channel gains in Fig. 2(a) we can show that the communication rate achieved using only one of the relays is only 1 bit/s/Hz away from the cut-set upper bound on the capacity of the network; while for the anti-symmetrical channel gains as in Fig. 2(b) using only one of the relays achieves (within 1 bit/s/Hz) only *half* of the cutset

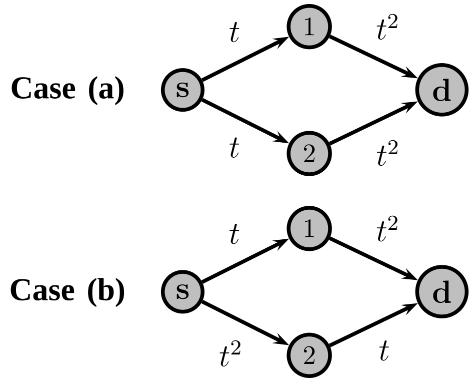

Fig. 2. Two instantiations of a diamond relay network.

upper bound on the capacity of the network.

To avoid channel-specific results, we can try to provide worst-case guarantees that hold universally for all possible channel gains. For example, is it possible that in 2-relay networks, we can always find a single relay to use and still achieve *half* of the capacity of the diamond network within 1 bit/s/Hz (as was the case for the two examples in Fig. 2). We prove in this paper that this is indeed always the case. In fact, we show that even if we have an arbitrary number N of relays, we can remove all but one of them and still achieve approximately half of the capacity of the whole network.

A. Overview of Main Results

Our main result is to show that in every Gaussian N-relay diamond network, there exist a k-relay sub-network whose capacity C_k satisfies

$$C_k \geq \frac{k}{k+1}\,\overline{C} - 1.3k - \frac{3k}{k+1}\log N \tag{1}$$

where $\overline{C}$ is the cut-set upper bound on the capacity of the N-relay diamond network. Intuitively, this holds because if all k-relay subnetworks have small capacity, the capacity of the whole network cannot be too large. As k increases, the difference between the capacity of the best k-relay subnetwork and that of the whole network naturally decreases. The surprising outcome here is that the fraction of the capacity we can get with k relays is independent of the number of available relay nodes N. Moreover, it increases quite quickly with k: in the high-capacity regime, we can get at least half-the capacity of every N-relay diamond network by simply routing information over the best relay, using 2 relays we achieve a fraction of $2/3$, etc.

We also show that the lower bound in (1) is tight in the multiplicative fraction, i.e., for given N and k, it is possible to find N-relay diamond networks where the capacity of *every* k-relay sub-diamond network is at most

$$C_k \leq \frac{k}{k+1}C + 1.3k + 3\log k, \tag{2}$$

where C is the capacity of the whole network. For the case $k = 1$ and $N = 2$, one such example is case (b) in Fig. 2.

We prove the result (1) in two steps. We first show that in every Gaussian N-Relay diamond network, there exists a subset of k-relay nodes such that the information-theoretic cut-set upper bound on the capacity of this k-relay sub-network is larger than $\frac{k}{k+1}(\overline{C} - 3\log N)$; i.e., this step only involves the cut-set upper bounds on the capacities of the corresponding networks. We then use the compress-and-forward type of strategies in [4], [5], [6], over this k-relay sub-network. These strategies are known to achieve the cut-set upper bound on the capacity of any arbitrary Gaussian relay network within a gap that is linear in the number of relay nodes utilized. In particular, the result of [6] implies that we can achieve the cut-set upper bound on the capacity of the k-relay network within $1.3k$ bits/s/Hz. Combining these two steps yields (1).

An alternative relaying strategy that is often considered for the N-relay diamond network in the literature is amplify-and-forward [7], [8], [11]. For example, [11] shows that amplify-and-forward at the relays can achieve the cutset upper bound on the capacity of the N-relay diamond network within 3.6 bits/s/Hz when all channel gains in the first and the second stages are equal. To show that this approximate optimality of amplify-and-forward is only limited to the the case of equal channel gains, we show that the rate achieved by this strategy is approximately equal to the capacity of the best relay alone in any arbitrary N-relay diamond network. More precisely, we show that

$$C_{AF} \leq C_1 + 2\log N$$

where C_{AF} is the best rate achievable with amplify-and-forward at the N relays, and C_1 is the rate achieved by using only the best relay (say, with a decode-and-forward strategy) while keeping the rest of the relays silent. This result says that amplify-and-forward with the N relays can at most provide a beamforming gain, bounded by $2\log N$, over the best relay. Since our result in (2) shows that there are configurations of N-relay diamond networks where the best relay alone can at most provide approximately half the capacity of the whole network, the two results together imply that amplify-and-forward can be limited to approximately half the capacity of the network in certain configurations. This implies that amplify-and-forward fails to provide constant gap approximations for the capacity of the N-relay diamond network, such as those provide by the compress-and-forward type of strategies in [4], [5], [6], [9] or partial-decode-and-forward in [10].

Finally, a natural question given our existence result in (1) is whether such high-capacity k-relay subnetworks can be discovered efficiently. Our existence proof naturally suggests an algorithm for discovering such networks in $O(kN)$ running time given the cutset upper bound $\bar{C}$ and the configuration of the N-relay diamond network. However, a direct computation of $\bar{C}$ itself requires evaluating the cut capacity over exponentially many cuts. [12] shows that the problem of computing a constant gap approximation to $\bar{C}$ can be casted as a minimization of a submodular function and solved in $O(N^5\alpha + N^6)$ running time using state-of-the-art algorithms for submodular function minimization, where α is the time it takes to compute the value of a single cut which is typically polynomial in N. Our work reveals that information flow in wireless networks has much more structure than mere submodularity. We show that the combinatorial structure that allows us to obtain the simplification result in (1) can be also used to devise an algorithm to compute a constant gap approximation to the cutset upper bound on the capacity of the N-relay diamond network in $O(N \log N)$ time. The properties of wireless information flow beyond submodularity are further exploited in [13] where Non-Shannon properties of Gaussian random variables are used to obtain simplification results for the N-relay diamond network with multiple antennas.

II. Related Work and Positioning

Two lines of work have previously looked at a form of network simplification for wireless networks. First, relay selection techniques in [14], [15], [16], design practical algorithms that allow to select the best single relay in an N-relay diamond network, and show that such algorithms provide cooperative diversity. These works look only at maintaining diversity and not capacity. Second, work in [11], [17], [18], [19] looks at selecting a subset of the best relays when restricted to utilize an amplify and forward strategy. Our work differs in that we do not restrict our attention to specific strategies (or a single relay) but instead provide universal capacity results for arbitrary strategies.

Our result can also be regarded as a new approximation to the capacity C of the Gaussian N-Relay diamond network. We show that

$$\frac{k}{k+1}\overline{C} - 1.3k - \frac{3k}{k+1}\log N \ \leq C \leq \overline{C} \quad \forall k,\, 1 \leq k \leq N-1, \tag{3}$$

where $\overline{C}$ denotes the cut-set upper bound. The best of the earlier additive approximation results in [4], [5], [6] yield

$$\overline{C} - 1.3N \leq C \leq \overline{C}. \tag{4}$$

for the N-relay diamond network, while the best multiplicative approximation to the capacity of the Gaussian N-relay diamond network is given by [11]

$$\frac{1}{K(\log N)^4}\overline{C} \leq C \leq \bar{C}, \tag{5}$$

for a constant $K > 0$ independent of everything else.

The lower bound we provide in (3) is tighter than both (4) and (5) in the intermediate SNR regime and when N is large. The auxiliary parameter k in (3) allows to optimize this lower bound as a function of $\bar{C}$ and N. When N is large, choosing a small k reduces the additive gap from $O(N)$ in (4) to $O(\log N)$. This improvement in the additive gap can be more important than the $\frac{1}{k+1}\bar{C}$ loss due to the multiplicative gap when $\bar{C}$ (and therefore C) is not too large, overall yielding a tighter lower bound than (4). When $\bar{C}$ is large and N is small increasing k to N reduces (3) to (4). Similarly, when N is large and $\bar{C}$ is not too small, (3) can be clearly tighter than (5). This approach suggests a new approximation philosophy to the capacity of wireless networks where multiplicative and additive gaps to the cutset upper bound are allowed simultaneously and are traded through an auxiliary parameter (in our case k). Earlier works in the literature have either aimed to characterize the capacity within an additive gap by allowing no multiplicative gap [4], [5], or vice-a-versa [11]. These purely additive or multiplicative capacity approximations are relevant in the high or the low SNR regimes respectively, while a hybrid approximation can be also useful at intermediate SNR's.

The fact that (3) can be tighter than (4) also implies that employing an unnecessarily large number of relays with the compress-and-forward type of strategies in [4], [5], [6] can indeed deteriorate rather than improve the communication rate. Recall that the result in (3) is obtained by applying these strategies with a carefully chosen subset of k relays, while (4) is obtained by using the same strategy with all the N relays. Motivated by this observation, recent work [10], [21] has demonstrated the need to optimize the quantization levels in these strategies which allows to achieve the information-theoretic cutset upper bound on the capacity of the N-relay diamond network within $O(\log N)$ bits/s/Hz. More precisely, these works show that (4), valid for any wireless network with N relays, can be refined to

$$\overline{C} - \log(N+1) - \log N - 1 \leq C \leq \overline{C}$$

for the N-relay diamond network. This new result can be readily used to tighten our simplification result in (1) to

$$C_k \geq \frac{k}{k+1}\overline{C} - \log(k+1) - \log k - 1 - \frac{3k}{k+1}\log N,$$

by simply using the optimized quantization levels for the k-relay subnetwork.

III. Model

We consider the Gaussian N-relay diamond network depicted in Fig. 1 where the source node s wants to communicate to the destination node d with the help of N relay nodes. Let $X_s[t]$ and $X_i[t]$ denote the signals transmitted by the source node s and the relay node $i \in \{1, \ldots, N\}$ respectively at time instant $t \in \mathbb{N}$. Let $Y_d[t]$ and $Y_i[t]$ denote the signals received by the destination node d and the relay node $i \in \{1, \ldots, N\}$ respectively at time instant t. The transmitted signal $X_i[t]$ by relay i is a causal function of the its corresponding received signal $Y_i[t]$. The received signals relate to the transmitted signals as

$$Y_i[t] = h_{is}X_s[t] + Z_i[t],$$
$$Y_d[t] = \sum_{i=1}^{N} h_{id}X_i[t] + Z[t],$$

where h_{is} denotes the complex channel coefficient between the source node and the relay node i and h_{id} denotes the complex channel coefficient between the relay node i and the destination node. $Z_i[t]$, $i = 1, \ldots, N$ and $Z[t]$ are independent and identically distributed circularly symmetric Gaussian random variables of variance σ^2. All nodes are subject to an average power constraint P and we define $\mathrm{SNR} = P/\sigma^2$. Note that the equal power constraint assumption is without loss of generality as the channel coefficients are arbitrary. We assume that the channel coefficients are known at all the nodes. We are interested in the maximum reliable communication rate C in bits/channel use that can be achieved between the source and the destination node in this network.

IV. Main Results

The main result of this paper is summarized in the following theorems.

Theorem 1: Consider an arbitrary Gaussian N-relay diamond network. Let C_k be the largest rate at which we can communicate from the source node to the destination using only k out of the N relays while the remaining $N-k$ relays are kept silent. Then

$$C_k \geq \frac{k}{k+1}\overline{C} - 1.3k - \frac{3k}{k+1}\log N, \tag{6}$$

where $\overline{C}$ denotes the cut-set upper bound on the capacity of the N-relay network. Moreover, for any given N and k, there exist configurations of the Gaussian N-relay diamond network such that

$$C_k \leq \frac{k}{k+1}C + 1.3k + 3\log k, \tag{7}$$

where C is the capacity of the N-relay network.

Remark 1: For the case $k = 1$, we have the following tighter bound,

$$C_1 \geq \frac{1}{2}\overline{C} - \frac{3}{2}\log N.$$

The theorem states that in every Gaussian N-relay diamond network, there exists a subset of k relays which alone provide approximately a fraction $k/(k+1)$ of the capacity of the whole network. On the other hand, there are also configurations, where each k-relay sub-network alone can at most provide this fraction of the capacity. The approximations are within the beamforming gain, which we upper bound by $3\log N$ for the N-relay diamond network uniformly over all possible channel configurations.[1] The beamforming gain is relatively small when the capacity is large, and indeed is much smaller than this upper bound when channel gains are significantly different. On the other hand, the term $1.3k$ in the gap is not fundamental and reflects the gap between the rate achieved by the state-of-the art relaying strategies [4], [5], [6] and the cutset upper bound on the capacity of the diamond network with k relays.[2]

A key ingredient in the above results is the fact that compress-and-forward type of strategies in [4], [5], [6] can achieve the cut-set upper bound on the capacity of any arbitrary diamond relay network within a gap that is linear in the number of relay nodes utilized, and independent of the channel configurations and the operating SNR. We next show that an amplify-and-forward strategy fails to provide such a universal performance guarantee over the channel configurations, and its performance is approximately bounded by the capacity of the best relay alone.

[1] As can be seen from the proof of Theorem 1, the $3\log N$ bound can be improved to $\max\left(3\log N - \log\frac{27}{4}, 2\log N\right)$. Accordingly, as the subsequent proof makes it obvious, the $\frac{3k}{k+1}\log N$ term in (6) can be tightened to $\frac{k}{k+1}\max\left(3\log N - \log\frac{27}{4}, 2\log N\right)$ and the $3\log k$ term in (7) can be also tightened to $\max\left(3\log k - \log\frac{27}{4}, 2\log k\right)$.

[2] For example, using improved relaying strategies from recent results in [10], [21], it can be readily sharpened from $1.3k$ to $2\log k$.

Theorem 2: In any Gaussian N-relay diamond network, the rate C_{AF} achieved by amplify-and-forward at the N relays is bounded by

$$C_{AF} \leq C_1 + 2\log N,$$

where C_1 is the capacity provided by routing over the best relay.

Finally, we address the algorithmic complexity of discovering a high-capacity k-relay subnetwork in Theorem 1.

Theorem 3: A constant gap approximation to the capacity of the Gaussian N-relay diamond network can be computed in $O(N\log N)$ running time. The k-relay subnetwork satisfying (6) can be discovered in $O(kN)$ running time, given the configuration of the network and the approximation to the cutset upper bound.

Theorem 1 is proven in VI, Theorem 2 is proven in Section VIII, and Theorem 3 is proven in Section VII. The following section derives a simple approximation to the cutset upper bound on the capacity of the N-relay diamond network, which forms the basis for all these results.

V. Approximating the Cut-Set Upper Bound

In this section we derive upper and lower bounds on the cut-set upper bound, that essentially reduce calculating its value to a purely combinatorial problem.

In the following, let $[N] \doteq \{1, 2, \cdots, N\}$. By the cut-set upper bound [22, Theorem 14.10.1], the capacity C of the network is upper bounded by,

$$C \leq \overline{C} \doteq \max_{X_s, X_1, \ldots, X_N} \min_{\Lambda \subseteq [N]} I(X_s, X_\Lambda; Y_d, Y_{[N]\setminus\Lambda} \mid X_{[N]\setminus\Lambda}) \tag{8}$$

where the maximization is over the joint probability distribution of the random variables X_s and $X_1, \ldots, X_N$ satisfying the power constraint P. For a set $S \subseteq [N]$, X_S denotes the corresponding collection of random variables, i.e $X_S \doteq \{X_i\}_{i\in S}$.

A. An Upper Bound on the Cut-Set Upper Bound

The cut-set upper bound in (8) can be upper bounded by exchanging the order of maximization and minimization in (8). For each cut Λ, the resulting maximization of the mutual information can be upper bounded by the capacities of the SIMO (single input multiple output) channel between s and nodes in $[N]\setminus\Lambda$ and the MISO (multiple input single output) channel between nodes in Λ and d. We have,

$$\begin{aligned}\overline{C} &\leq \min_{\Lambda\subseteq[N]} \sup_{X_s, X_\Lambda, X_{[N]\setminus\Lambda}} I(X_s, X_\Lambda; Y, Y_{[N]\setminus\Lambda} \mid X_{[N]\setminus\Lambda}) \\ &\leq \min_{\Lambda\subseteq[N]} \sup_{X_s} I(X_s; Y_{[N]\setminus\Lambda}) + \sup_{X_\Lambda} I(X_\Lambda; \sum_{i\in\Lambda} h_{id}X_i + Z), \\ &\leq \min_{\Lambda\subseteq[N]} C_{SIMO}(s; [N]\setminus\Lambda) + C_{MISO}(\Lambda; d).\end{aligned}$$

The capacities of the corresponding SIMO and MISO channels are well-known [23]. Plugging these expressions yields

$$\overline{C} \leq \min_{\Lambda\subseteq[N]} \log\Big(1 + \text{SNR}\sum_{i\in[N]\setminus\Lambda}|h_{is}|^2\Big) + \log\Big(1 + \text{SNR}\big(\sum_{i\in\Lambda}|h_{id}|\big)^2\Big). \tag{9}$$

We will further develop a simple upper bound on this expression by bounding each term in the above summations by the maximum of the terms that are summed. This gives us the upper bound,

$$\overline{C} \leq \min_{\Lambda \subseteq [N]} \left(\max_{i \in \Lambda} \; R_{id} + \max_{i \in [N] \setminus \Lambda} \; R_{is} \right) + 3 \log N, \qquad (10)$$

where $R_{id} = \log\left(1 + \mathrm{SNR}\,|h_{id}|^2\right)$ and $R_{is} = \log\left(1 + \mathrm{SNR}\,|h_{is}|^2\right)$ are the capacities of the corresponding point-to-point channels.[3] A detailed derivation of the upper bound in this section can be found in Appendix A.

B. A Lower Bound on the Cut-Set Upper Bound

The cut-set upper bound $\overline{C}$ above can be lower bounded by choosing $X_s, \{X_i\}_{i \in [N]}$ to be independent circularly-symmetric Gaussian random variables with variance P, in which case

$$I(X_s, X_\Lambda; Y, Y_{[N]\setminus\Lambda} \,|\, X_{[N]\setminus\Lambda}) = \log\left(1 + \mathrm{SNR} \sum_{i \in \Lambda} |h_{id}|^2\right) + \log\left(1 + \mathrm{SNR} \sum_{i \in [N]\setminus\Lambda} |h_{is}|^2\right).$$

Retaining only the maximum terms in the summations, we obtain

$$\overline{C} \geq \min_{\Lambda \subseteq [N]} \left(\max_{i \in \Lambda} R_{id} + \max_{i \in [N]\setminus\Lambda} R_{is} \right). \qquad (11)$$

Note that this lower bound for $\overline{C}$ differs from the upper bound in (10) only by the gap term $3 \log N$. This implies that within a gap of $3 \log N$ bits/s/Hz, the cut-set upper bound on the capacity of the N-relay diamond network behaves like the lower bound in (11). Since recent results [4], [5], [6], [10], [21] show that the actual capacity of the network is within a constant gap to the cutset upper bound, this also provides an approximation to the capacity of the N-relay diamond network, i.e.,

$$C \approx \min_{\Lambda \subseteq [N]} \left(\max_{i \in \Lambda} R_{id} + \max_{i \in [N]\setminus\Lambda} R_{is} \right). \qquad (12)$$

This reveals a peculiar combinatorial structure for the capacity of the diamond network in terms of the point-to-point capacities of the individual channels. Our main result is based on exploiting this combinatorial structure.

C. The Cut-Set Upper Bound for a k-Relay Sub-network

Consider a subset $\Gamma \subseteq [N]$ of the relay nodes such that $|\Gamma| = k$. Let C_Γ be the capacity of the k-relay diamond sub-network where the source node s wants to communicate to the destination node d by using only these k relay nodes. The rest $N - k$ relays are not used. The cut-set upper bound on the capacity of this k-relay network yields

$$C_\Gamma \leq \overline{C}_\Gamma \doteq \sup_{X, X_\Gamma} \min_{\Lambda \subseteq \Gamma} I(X, X_\Lambda; Y, Y_{\Gamma\setminus\Lambda} \,|\, X_{\Gamma\setminus\Lambda}). \qquad (13)$$

Note that (10) and (11) can be applied to Γ to obtain correspondingly upper and lower bounds on $\overline{C}_\Gamma$.

[3]Note that the N-relay diamond network can be equivalently characterized in terms of these point-to-point channel capacities.

Among all $\Gamma \subseteq [N]$ with $|\Gamma| = k$, consider the one that has the largest cut-set upper bound $\overline{C}_\Gamma$. Let $\overline{C}_k$ denote the cut-set upper bound on the capacity of this this sub-network. Formally, we define

$$\overline{C}_k = \max_{\substack{\Gamma \subseteq [N] \\ |\Gamma| = k}} \overline{C}_\Gamma. \qquad (14)$$

Combining (11) and (14), we have

$$\overline{C}_k \geq \max_{\substack{\Gamma \subseteq [N] \\ |\Gamma| = k}} \min_{\Lambda \subseteq \Gamma} \left(\max_{i \in \Lambda} R_{id} + \max_{i \in \Gamma\setminus\Lambda} R_{is} \right). \qquad (15)$$

Let C_k be the capacity of the best k-relay sub-network. In the sequel, we will be interested in lower bounding C_k in terms of $\bar{C}$, the cutset upper bound on the capacity of the network. For this, we will first relate C_k to $\overline{C}_k$ and then make use of the above lower bound for $\overline{C}_k$.

VI. k Relays Approximately Achieve $\frac{k}{k+1}$ Fraction of the Capacity

In this section, we prove Theorem 1. However, before going into the formal proof, let us illustrate the main idea for $k = 1$. Assume the capacity C of the N-relay diamond network were given exactly by (12), while the capacity obtained by using relay $i \in [N]$ alone is given by

$$C_i = \min(R_{is}, R_{id}).$$

Note that this is the capacity approximation in (12) evaluated for a single relay ($N = 1$), but in this particular case it indeed corresponds to the exact capacity of a single relay (2-hop) network. Can we argue that there exists a relay $i \in [N]$ such that $C_i \geq C/2$? This is easy. If this were not the case, it would imply that

$$\forall i \in [N], \qquad \text{either} \qquad R_{is} < \frac{C}{2} \quad \text{or} \quad R_{id} < \frac{C}{2}.$$

This would allow us to construct a cut of the network Λ which crosses only links with capacities strictly smaller than $C/2$, both on the source side and the destination side, i.e., $R_{id} < C/2 \; \forall i \in \Lambda$ and $R_{is} < C/2 \; \forall i \in [N] \setminus \Lambda$. Hence the value of this cut is strictly smaller than C and this contradicts with our initial assumption that the capacity of the N-relay diamond network is C. Therefore, there exists at least one relay $i \in [N]$ such that $C_i \geq C/2$. To prove the converse statement in Theorem 1, we need to create examples where each relay alone only provides half the capacity of whole network: consider a configuration where $R_{is} = C/2$ and $R_{id} = C$ for some of the relays and $R_{is} = C$ and $R_{id} = C/2$ for the rest. The capacity of the whole network is C by (12), while each relay alone can only provide capacity $C/2$.

The formal proof of Theorem 1 is based on the following two technical lemmas.

Lemma 1: Let R_{id} and R_{is} be arbitrary positive real numbers for $i = 1, 2, \cdots, N$. For $k \in [N]$, let

$$r_k \doteq \frac{\max\limits_{\substack{\Gamma \subseteq [N] \\ |\Gamma| = k}} \min\limits_{\Lambda \subseteq \Gamma} \left(\max\limits_{i \in \Lambda} R_{id} + \max\limits_{i \in \Gamma\setminus\Lambda} R_{is} \right)}{\min\limits_{\Lambda \subseteq [N]} \left(\max\limits_{i \in \Lambda} R_{id} + \max\limits_{i \in [N]\setminus\Lambda} R_{is} \right)}. \qquad (16)$$

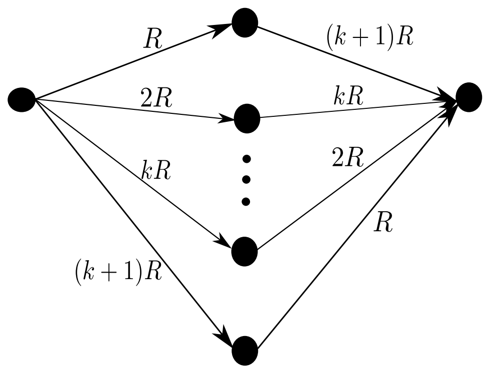

Fig. 3. A $(k+1)$-relay diamond network where every subset of k relays achieve approximately $\frac{k}{k+1}$ of the capacity. The labels indicate the capacity of the corresponding links.

Then,

$$r_k \geq \frac{k}{k+1}.$$

Lemma 2: Let $R_{is} = i\,R$ and $R_{id} = (k+2-i)\,R$ for $i \in [k+1]$ where R is an arbitrary positive number. Let r_k be defined as in (16) with $N = k+1$. Then,

$$r_k = \frac{k}{k+1}.$$

The configuration in Lemma 2 is depicted in Fig. 3.

Proof of Theorem 1: From (10) and (15), we have

$$\frac{\overline{C}_k}{\overline{C} - 3\log N} \geq r_k.$$

Combining this with the result of Lemma 1, we obtain

$$\overline{C}_k \geq \frac{k}{k+1}\overline{C} - \frac{3k}{k+1}\log N. \tag{17}$$

This proves that in every N relay diamond network, there exists a subset of k relays, such that the cut-set upper bound on the capacity of the corresponding k relay subnetwork is lower bounded by approximately a fraction $\frac{k}{k+1}$ of the cut-set upper bound on the capacity of the whole network. Let $\Gamma^* \subset [N]$ be the maximizing term in (14), i.e., $\overline{C}_{\Gamma^*} = \overline{C}_k$, and let C_{Γ^*} be the actual capacity of this network. From [6], $C_{\Gamma^*} \geq \overline{C}_{\Gamma^*} - 1.3k$, for any k-relay network, which is achieved by a noisy network coding strategy generalizing the quantize-map-and-forward strategy of [4]. Let C_k be the capacity of the best k-relay subnetwork. Since $C_k \geq C_{\Gamma^*}$ by definition, we have

$$C_k \geq \overline{C}_k - 1.3k.$$

Together with (17) this yields the result (6) in Theorem 1.

Next, we prove the existence of a $(k+1)$-relay diamond network where the capacity of each k-relay sub-network satisfies (7), i.e., for now we assume $N = k+1$. To prove this, we require an upper bound on C_k and a lower bound on C. The lower bound on C can be obtained by combining (11) with the fact that $C \geq \overline{C} - 1.3(k+1)$ from [6] (since $N = k+1$), which yields

$$C \geq \min_{\Lambda\subseteq[N]} \left(\max_{i\in\Lambda} R_{id} + \max_{i\in[N]\setminus\Lambda} R_{is} \right) - 1.3(k+1). \tag{18}$$

On the other hand, applying (10) for any $\Gamma \subseteq [k+1]$ s.t $|\Gamma| = k$, we obtain

$$\overline{C}_\Gamma \leq \min_{\Lambda\subseteq\Gamma} \left(\max_{i\in\Lambda} R_{id} + \max_{i\in\Gamma\setminus\Lambda} R_{is} \right) + 3\log k.$$

Therefore,

$$\overline{C}_k \leq \max_{\substack{\Gamma\subseteq[k+1] \\ |\Gamma|=k}} \min_{\Lambda\subseteq\Gamma} \left(\max_{i\in\Lambda} R_{id} + \max_{i\in\Gamma\setminus\Lambda} R_{is} \right) + 3\log k. \tag{19}$$

Combining (18) and (19), we obtain

$$\frac{\overline{C}_k - 3\log k}{C + 1.3(k+1)} \leq r_k.$$

Lemma 2 demonstrates a configuration where $r_k = \frac{k}{k+1}$. For such configurations, the above inequality yields

$$\overline{C}_k \leq \frac{k}{k+1} C + 1.3k + 3\log k.$$

Since $C_k \leq \overline{C}_k$, this proves that there exist $k+1$-relay diamond networks such that the capacity of each k-relay subnetwork satisfies the bound (7) in Theorem 1. However, Theorem 1 claims the existence of N-relay diamond networks where each k-relay subnetwork satisfies (7). To extend the proof to any $N > k$, simply consider augmenting the $k+1$ relay diamond network of Fig. 3 by adding relay nodes with zero capacities. Whatever holds for the $k+1$-relay network also holds for this trivially augmented N-relay network. This completes the proof of Theorem 1. $\square$

We will next prove Lemma 1 for the case $k=1$ and $k=2$. The proof of Lemma 1 for $k > 2$ and the proof of Lemma 2 are provided in Appendix B.

Proof of Lemma 1: We introduce the following notation. Let

$$\omega(\Gamma) \doteq \min_{\Lambda\subseteq\Gamma} \left(\max_{i\in\Lambda} R_{id} + \max_{i\in\Gamma\setminus\Lambda} R_{is} \right) \tag{20}$$

$$\omega \doteq \min_{\Lambda\subseteq[N]} \left(\max_{i\in\Lambda} R_{id} + \max_{i\in[N]\setminus\Lambda} R_{is} \right), \tag{21}$$

and $\omega_k \doteq \max_{\substack{\Gamma\subseteq[N] \\ |\Gamma|=k}} \omega(\Gamma)$. Note that r_k in Lemma 1 is defined as $r_k = \frac{w_k}{\omega}$.

The first thing we note is that $r_k \leq 1$. This follows from the fact that every subset of Γ is necessarily a subset of $[N]$,i.e., if $\Lambda \subseteq \Gamma$ then $\Lambda \subseteq [N]$ and $\Gamma \setminus \Lambda \subseteq [N] \setminus \Lambda$. Therefore, the value of each cut $\Lambda \subseteq \Gamma$ in Γ is smaller than or equal to the value of the same cut in $[N]$. The same reasoning also implies that for $k_1 \geq k_2$ we have $r_{k_1} \geq r_{k_2}$. Both properties are to be naturally satisfied by a capacity function: by using more relays we can only increase the capacity.

- For $k = 1$, the lemma claims that $w_1 \geq \frac{1}{2}\omega$. Since

$$w_1 = \max_{i\in[N]} \min\left(R_{id}, R_{is}\right),$$

this is equivalent to saying that $\exists\, i \in [N]$ s.t. $R_{id} \geq \frac{1}{2}\omega$ and $R_{is} \geq \frac{1}{2}\omega$. We will prove this by contradiction. Assume

$$\forall i \in [N],\ R_{id} < \frac{1}{2}\omega \quad \text{or} \quad R_{is} < \frac{1}{2}\omega. \tag{22}$$

Let $\Lambda_0 = \left\{i \in [N] : R_{id} < \frac{1}{2}\omega\right\}$. The assumption in (22) implies that $R_{is} < \frac{1}{2}\omega,\ \forall i \in \overline{\Lambda}_0$. Note that ω in (21) can be upper bounded by considering only the cut Λ_0 among all possible cuts $\Lambda \subseteq [N]$. We obtain

$$\omega \leq \max_{i\in\Lambda_0} R_{id} + \max_{i\in[N]\setminus\Lambda_0} R_{is} < \omega$$

since each of the two terms are strictly smaller than $\frac{1}{2}\omega$. This contradiction proves the lemma for $k = 1$.

- For $k = 2$, the lemma claims that $w_2 \geq \frac{2}{3}\omega$. We can prove this by establishing a number of properties for a network with ω.

  Property: $\exists\, p \in [N]$ s.t. $R_{ps} \geq \frac{2}{3}\omega$ and $R_{pd} \geq \frac{1}{3}\omega$.

  We prove this by contradiction. Assume

$$\forall i \in [N]\,,\ R_{is} < \frac{2}{3}\omega \quad \text{or} \quad R_{id} < \frac{1}{3}\omega.$$

  Consider the cut $\Lambda_0 = \left\{i \in [N] : R_{id} < \frac{1}{3}\omega\right\}$. Then $R_{is} < \frac{2}{3}\omega,\ \forall i \in \overline{\Lambda}_0$. Considering only the cut Λ_0 we obtain

$$\omega \leq \max_{i\in\Lambda_0} R_{id} + \max_{i\in[N]\setminus\Lambda_0} R_{is} < \omega,$$

  which is a contradiction.

  We next proceed by investigating two separate cases:

  - *Case 1:* $R_{pd} \geq \frac{2}{3}\omega$. Then, the proof of the lemma is complete since we have $w_2 \geq w_1 \geq \frac{2}{3}\omega$.
  - *Case 2:* $R_{pd} < \frac{2}{3}\omega$. Then we establish the following property:

    Property: $\exists\, m \in [N],\ m \neq p$ s.t. $R_{ms} \geq \frac{1}{3}\omega$ and $R_{md} \geq \frac{2}{3}\omega$.

    Again, we can prove this property by contradiction. Assume the contrary and consider $\Lambda_1 = \left\{i \in [N] : R_{id} < \frac{2}{3}\omega\right\}$. Note that $p \in \Lambda_1$ since we are in Case 2 and $R_{is} < \frac{1}{3}\omega,\ \forall i \in \overline{\Lambda}_1$. The value of the cut Λ_1 is strictly smaller than ω, which is a contradiction.

    Finally, consider the 2-relay sub-network composed of m and p. It can be easily verified that $\omega(\{m,p\}) \geq \frac{2}{3}\omega$, completing the proof of the lemma for $k = 2$.

The proof of the lemma for the general case follows similar lines. The main idea is to show that given any arbitrary real numbers R_{id} and R_{is} for $i = 1, 2, \cdots, N$, we can gradually discover a k-relay subnetwork Γ^* such that $C_{\Gamma^*} \geq \frac{k}{k+1}\omega$. Details for the proof in general case is given in Appendix B and the general flow of the proof is as follows:

Assume that there is a configuration such that all subsets $\Gamma \subseteq [N]$ with $|\Gamma| \leq k$ satisfy $\omega(\Gamma) < \frac{k}{k+1}\omega$.

- There exists a node $i \in [N]$ such that

$$R_{is} \geq \frac{k}{k+1}\omega \quad \text{and} \quad R_{id} \geq \frac{1}{k+1}\omega$$

  because otherwise $\omega([N]) < \omega$ which is a contradiction. We relabel this node as node N.
- Either $R_{Nd} \geq \frac{k}{k+1}\omega$ and we are done since $\omega(\{N\}) \geq \frac{k}{k+1}\omega$ or we can find a natural number a such that $1 \leq a \leq k-1$ and

$$\frac{k-a+1}{k+1}\omega > R_{Nd} \geq \frac{k-a}{k+1}\omega.$$

- At this point we define $a_0 = 0$ and starting with $l = 1$ we recursively prove the following steps for an increasing integer l:
  (a) There exists a node $y \in [l, N-1]$ such that

$$R_{ys} \geq \frac{a_{l-1}+1}{k+1}\omega \text{ and } R_{yd} \geq \frac{k-a_{l-1}}{k+1}\omega,$$

  because otherwise $\omega([N]) < \omega$ which is a contradiction. We relabel this node as node l.

  (b) If $R_{ls} \geq \frac{a}{k+1}\omega$ for this node, $\omega([l] \cup \{N\}) \geq \frac{k}{k+1}\omega$ and the proof of the lemma is complete. Otherwise, we can find a natural number a_l with $a_{l-1} < a_l < a$ such that

$$\frac{a_l+1}{k+1}\omega > R_{ls} \geq \frac{a_l}{k+1}$$

  and reiterate steps (a) and (b) for $l+1$.

  If we can iterate the above steps l times, we generate a sequence of natural numbers with

$$a_0 = 0 < a_1 < \ldots < a_{l-1} < a.$$

  Since for $l = k$, the sequence of inequalities

$$a_0 = 0 < a_1 < \ldots < a_{k-1} < a \leq k-1,$$

  is contradictory, this implies that the above steps can be iterated at most $k-1$ times. In other words, for some $1 \leq l \leq k-1$, the condition $R_{ls} \geq \frac{a}{k+1}\omega$ in the beginning of step (b) should be true and therefore $\omega([l] \cup \{N\}) \geq \frac{k}{k+1}\omega$.

VII. ALGORITHMIC COMPLEXITY

Given an arbitrary N-relay diamond network, characterized by the point-to-point capacities of the individual links $R_{is}, R_{id},\ i \in [N]$, can we efficiently discover a k-relay subnetwork whose capacity satisfies (6)? In this section, we prove Theorem 3.

Note that from the proof of Theorem 1, the k-relay subnetwork $\Gamma^* \subseteq [N]$ whose capacity C_{Γ^*} satisfies (6) is the one for which $w(\Gamma^*)/\omega \geq \frac{k}{k+1}$, where $w(\Gamma^*)$ and ω are defined in (21). The proof of Lemma 1 suggests a natural algorithm to discover this network.

- For $k = 1$, the lemma proves that

$$\exists i \in [N]\,,\ R_{id} \geq \frac{1}{2}\omega \quad \text{and} \quad R_{is} \geq \frac{1}{2}\omega.$$

  This node i can be discovered by making $2N$ comparisons in the worst case.
- For $k = 2$, the lemma first proves that

$$\exists p \in [N]\,,\ R_{pd} \geq \frac{2}{3}\omega \quad \text{and} \quad R_{ps} \geq \frac{1}{3}\omega.$$

  Then either $R_{ps} \geq \frac{2}{3}\omega$ or

$$\exists m \in [N]\,, m \neq p \quad \text{and} \quad R_{md} \geq \frac{1}{3}\omega \quad \text{and} \quad R_{ms} \geq \frac{2}{3}\omega.$$

  We can follow this flow to discover relays p and m for which we have $\omega(\{m,p\}) \geq \frac{2}{3}\omega$. p can be discovered in at most $2N$ comparisons. An extra comparison determines whether $R_{ps} \geq \frac{2}{3}\omega$ or $\frac{1}{3}\omega \leq R_{ps} < \frac{2}{3}\omega$. In the first case, the algorithm terminates. Otherwise, we need at most $2(N-1)$ additional comparisons to discover m. This yields $4N-1$ comparisons in the worst case.

- For general $k < N$ the flow in the proof of the lemma suggests a natural algorithm to find a subset $\Gamma^* \subseteq [N]$ with $\omega(\Gamma^*) \geq \frac{k}{k+1}\omega$.
  (a) Find a node $i \in [N]$ such that
  $$R_{is} \geq \frac{k}{k+1}\omega \quad \text{and} \quad R_{id} \geq \frac{1}{k+1}\omega,$$
  and relabel it as node N.
  (b) If $R_{Nd} \geq \frac{k}{k+1}\omega$, terminate the algorithm and declare $\Gamma^* = \{N\}$. Otherwise, set $a_0 = 0$ and determine a such that $1 \leq a \leq k-1$ and
  $$R_{Ns} \geq \frac{k}{k+1}\omega \text{ and } \frac{k-a+1}{k+1}\omega > R_{Nd} \geq \frac{k-a}{k+1}\omega.$$
  (c) For $1 \leq r \leq k-1$,
  (c-1) Find a node $i \in [r, N-1]$ such that
  $$R_{is} \geq \frac{a_{r-1}+1}{k+1}\omega \text{ and } R_{id} \geq \frac{k-a_{r-1}}{k+1}\omega,$$
  and label it node r.
  (c-2) If $R_{rs} \geq \frac{a}{k+1}\omega$, terminate the algorithm and declare $\Gamma^* = [r] \cup \{N\}$.
  (c-3) Otherwise (if $R_{rs} < \frac{a}{k+1}\omega$), determine a_r such that $a_{r-1} < a_r < a$, and
  $$\frac{a_r+1}{k+1}\omega > R_{rs} \geq \frac{a_r}{k+1}\omega \text{ and } R_{rd} \geq \frac{k-a_{r-1}}{k+1}\omega$$
  and then set $r \leftarrow r+1$.

  The total number of comparisons to be made by the algorithm can be upper bounded as follows:
  - Step (a): at most $2N$ comparisons
  - Step (b): at most $k-1$ comparisons
  - Step (c-1): at most $2(N-r)$ comparisons
  - Step (c-3): at most $k-1-r$ comparisons

  Assuming that step (c) makes the maximum number of iterations $k-1$, the total number of comparisons to be made by the algorithm is upper bounded by
  $$2N + (k-1) + \sum_{r=1}^{k-1}[2(N-r) + (k-1-r)] = 2Nk - \frac{(k-1)k}{2}.$$

However, the above discussion assumes that ω is given. Given the set of real numbers R_{is}, R_{id}, $i = 1, \ldots, N$, a straightforward approach to computing ω in (21) requires the evaluation of 2^N cuts, while computing the value of each cut requires N comparisons. Instead, the following algorithm allows to compute ω in $N \log N$ running time.

First, sort (rearrange) the nodes in the order of increasing R_{is}, i.e., $R_{1s} \leq \cdots \leq R_{Ns}$. For this sorted configuration, observe that the cut with the minimum value in (21), i.e., the cut Λ^* for which
$$\omega = \max_{i \in \Lambda^*} R_{id} + \max_{i \in [N]\setminus\Lambda^*} R_{is},$$

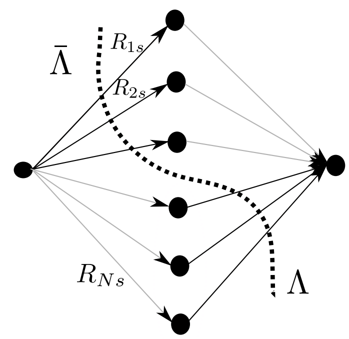

Fig. 4. The minimum cut on a configuration such that $R_{1s} \leq \cdots \leq R_{Ns}$.

is necessarily of the form in Figure 4. More precisely, $\Lambda^* = \Lambda_m \triangleq [m+1, N]$ and $[N] \setminus \Lambda^* = [m]$ for some $1 \leq m \leq N$. This is easy to see: consider any cut $\Lambda \subseteq [N]$ not necessarily of the form in Figure 4. Let m be the node in $[N] \setminus \Lambda$ with the largest index, i.e., $m = \max\{i \in [N] \setminus \Lambda\}$ and let $\Lambda_m = \{m+1, \ldots, N\}$. We have
$$\max_{i \in \Lambda_m} R_{id} + \max_{i \in [N]\setminus\Lambda_m} R_{is} \leq \max_{i \in \Lambda} R_{id} + \max_{i \in [N]\setminus\Lambda} R_{is}.$$
The second terms are equal because R_{is} are sorted in increasing order and the first term can be only smaller for Λ_m since it is a subset of Λ by construction. This reduces the number of candidate cuts for the min cut from 2^N to N.

In other words, the mincut can be calculated by making N comparisons of two numbers: the maximum value R_{is}, $i \in [N] \setminus \Lambda_m$, with the maximum R_{id}, with $i \in \Lambda_m$, for $\Lambda_m = [m+1, N]$, $m = 0, \ldots, N$. Assume that the R_{is} values are sorted as previously described - this can be done using $N \log N$ comparisons, for example with the heap sort algorithm. Thus for the set $[N] \setminus \Lambda_m$, the value we would use is R_{ms}. But we can also keep a sorted heap of the R_{id} values, that again can be created using $N \log N$ operations. Then for Λ_1 we would use the max value, for Λ_2 the max value after removing R_{1d}, etc. That is, we can take advantage of the fact that each subset of R_{id}'s would also be ordered, to extract the max value of the subset. Thus in total of $N + 2\log N$ comparisons, we can compute ω.

This implies that with at most $(2k+1)N + 2N \log N$ comparisons we can compute ω, a constant gap approximation to the capacity of the N-relay diamond network, and identify a k-relay subnetwork that approximately achieves a fraction $k/(k+1)$ of ω. This completes the proof of Theorem 3.

VIII. Amplify-and-Forward with N Relays vs. Routing over the Best Relay

In this section, we derive an upper bound on the rate achieved by amplify-forward over the Gaussian N-relay diamond network in terms of the capacity of the best relay. With amplify-forward, the transmitted signals from the relay nodes are nothing but the scaled versions of the received signals from the source, $X_i[t] = \beta_i Y_i[t]$. Here, the coefficients β_i should be chosen such that the transmitted signals X_i satisfy the average power constraint P, i.e., $\mathbb{E}[||X_i||^2] \leq P$. Note that we allow each relay to optimize its transmit power, i.e. we allow the average transmit power used by each node to be strictly smaller than P, which as pointed out in [20] leads to a higher achievable rate as compared to simply amplifying received signals at full-power. This amplify-and-forward operation at the relays induces a point-to-point link between the source node and destination given by,

$$Y_d[t] = \left(\sum_{i=1}^{N} h_{id} h_{is} \beta_i\right) X_s[t] + \left(Z[t] + \sum_{i=1}^{N} h_{id} \beta_i Z_i[t]\right).$$

Using the familiar capacity expression for a point-to-point AWGN channel, we get

$$C_{AF} = \log\left(1 + \frac{\left|\sum_{i=1}^{N} h_{id} h_{is} \beta_i\right|^2 \text{SNR}}{1 + \sum_{i=1}^{N} |h_{id}|^2 |\beta_i|^2}\right). \quad (23)$$

The β_i's in the above expression can be optimized to get the largest communication rate subject to the power constraint at the relays. Since $\mathbb{E}[|X_i|^2] \leq P$, we can write

$$|\beta_i|^2 = \frac{\text{SNR}}{1 + |h_{is}|^2 \text{SNR}} |\alpha_i|^2,$$

where $|\alpha_i| \leq 1$ for each i. Next, we first upper bound the rate in (23) and then express it in terms of the new variables α_i. Applying the Cauchy-Schwarz inequality on the numerator of the fractional term inside the logarithm, we get

$$\begin{aligned} C_{AF} &\leq \log\left(1 + \frac{N \sum_{i=1}^{N} |h_{id}|^2 |h_{is}|^2 |\beta_i|^2 \,\text{SNR}}{1 + \sum_{i=1}^{N} |h_{id}|^2 |\beta_i|^2}\right) \\ &\leq \log\left(1 + \frac{N^2 \max_{i\in[N]} |h_{id}|^2 |h_{is}|^2 |\beta_i|^2 \,\text{SNR}}{\max\left(1, \max_{i\in[N]} |h_{id}|^2 |\beta_i|^2\right)}\right). \end{aligned}$$

The second inequality is obtained by upper bounding each term of the sum in the numerator by the maximum term and taking only the maximum element for the sum in the denominator. In terms of α_i, this last upper bound can be expressed as

$$C_{AF} \leq \log\left(1 + \frac{N^2 \max_{i\in[N]} \frac{|h_{id}|^2 |h_{is}|^2 |\alpha_i|^2 \text{SNR}^2}{1+|h_{is}|^2 \text{SNR}}}{\max\left(1, \max_{i\in[N]} \frac{|h_{id}|^2 |\alpha_i|^2 \text{SNR}}{1+|h_{is}|^2 \text{SNR}}\right)}\right).$$

In Lemma 3 below, we show that for any arbitrary positive real numbers u_{id}, u_{is} and $0 \leq b_i \leq 1$, $i = 1, 2, \cdots, N$, we have

$$\max\left(1, \max_{i\in[N]} \frac{u_{id} b_i}{1 + u_{is}}\right) \max_{i\in[N]} \left(\min(u_{id}, u_{is})\right) \geq \max_{i\in[N]} \frac{b_i u_{id} u_{is}}{1 + u_{is}} \quad (24)$$

Plugging $u_{id} = |h_{id}|^2 \text{SNR}$, $u_{is} = |h_{is}|^2 \text{SNR}$ and $b_i = |\alpha_i|^2$ in this relation, we get

$$\begin{aligned} C_{AF} &\leq \log\left(1 + N^2 \max_{i\in[N]} \min(|h_{id}|^2 \text{SNR}, |h_{is}|^2 \text{SNR})\right) \\ &\leq \max_{i\in[N]} \min(R_{is}, R_{id}) + 2\log N \\ &= C_1 + 2\log N. \end{aligned}$$

This proves Theorem 2. Lastly, we prove the inequality in (24).

Lemma 3: Let u_{id}, u_{is} be arbitrary positive real numbers and b_i be a real number in the interval $[0, 1]$ for $i = 1, 2, \cdots, N$. Then,

$$\max\left(1, \max_{i\in[N]} \frac{u_{id} b_i}{1 + u_{is}}\right) \max_{i\in[N]} \left(\min(u_{id}, u_{is})\right) \geq \max_{i\in[N]} \frac{b_i u_{id} u_{is}}{1 + u_{is}}. \quad (25)$$

Proof of Lemma 3: The expression on the left-hand side of (25) can be rewritten as

$$\begin{aligned} \gamma = \max_{i\in[N]} \max\{ &\min(u_{id}, u_{is}), \frac{u_{id} b_i}{1 + u_{is}} \min(u_{id}, u_{is}), \\ &\min(u_{id}, u_{is}) \max_{j\in[N], j\neq i} \frac{u_{jd} b_j}{1 + u_{js}}\}. \end{aligned}$$

If $u_{is} < u_{id}$, $\frac{u_{id} b_i}{1+u_{is}} \min(u_{id}, u_{is}) = \frac{u_{id} u_{is} b_i}{1+u_{is}}$ is among the terms to be maximized in γ. If $u_{is} \geq u_{id}$, $\min(u_{id}, u_{is}) = u_{id}$ is among the terms to be maximized in γ and it satisfies $u_{id} > \frac{u_{id} u_{is} b_i}{1+u_{is}}$. Therefore, we can immediately conclude that

$$\gamma \geq \max_{i\in[N]} \frac{b_i u_{id} u_{is}}{1 + u_{is}}.$$

IX. Conclusions and Discussion

In this paper, we showed that in an N-relay diamond network we can use k of the N relays and approximately maintain a fraction $\frac{k}{k+1}$ of the total capacity. In particular, we can use a single relay and approximately achieve half the capacity of any diamond network. Our proof was based on reducing the network simplification to a combinatorial problem.

We believe that simplification is ubiquitous in wireless networks, fundamentally due to the broadcast nature of the wireless medium. Note that if the network in Fig 1 were a wired network with N independent paths between the source and the destination, the capacity loss incurred by removing a relay would be proportional to the individual capacity of this path. In wireless, the received signals by different nodes are not independent, but being functions of the same transmitted signals, are necessarily correlated. As a result, there is lot of redundancy in the information carried by these signals which creates opportunities for simplification. However, quantifying this redundancy and evaluating the usefulness of each relay can be highly nontrivial. Note that the conclusion of this paper that half the capacity of the N-relay diamond network is always carried by a single relay is a priori highly non-obvious. It does not imply that the rest $N - 1$ relays do not carry much information, but rather the amount of fresh information they can bring to the destination is only as much as that of this single relay.

Following our work, numerical simulations in [12] give evidence of the simplification phenomena in more general networks: in a network of 40 nodes with channel gains drawn from an i.i.d. Gaussian distribution, only 4 nodes are needed on the average to approach capacity. However, establishing theoretical simplification results for more general networks is non-trivial due to the difficulty in capturing the correlation structure of the signals received at different relays and quantifying its resultant implications for simplifying the network. Nevertheless, we believe better understanding this simplification phenomenon is paramount to efficient relay and power utilization in wireless networks as well as in building high-capacity cooperative relaying strategies with minimal network and code complexity. Extending our simplification results to more general networks remains as a future work. As a first step in this direction, we provide simplification results for the Gaussian diamond network with multiple antennas in [13].

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APPENDIX A
AN UPPER BOUND ON THE CUT-SET UPPER BOUND
(detailed derivation of Section V-A)

The cut-set upper bound in (8) can be further upper bounded by

$$\overline{C} \leq \min_{\Lambda\subseteq[N]} \sup_{X_s,X_\Lambda,X_{[N]\setminus\Lambda}} I(X_s,X_\Lambda;Y_d,Y_{[N]\setminus\Lambda}\,|\,X_{[N]\setminus\Lambda}) \tag{26}$$

$$\leq \min_{\Lambda\subseteq[N]} \sup_{X_s,X_\Lambda} I(X_s,X_\Lambda;\sum_{i\in\Lambda}h_{id}X_i+Z,Y_{[N]\setminus\Lambda}) \tag{27}$$

$$\leq \min_{\Lambda\subseteq[N]} \sup_{X} I(X_s;Y_{[N]\setminus\Lambda}) + \sup_{X_\Lambda} I(X_\Lambda;\sum_{i\in\Lambda}h_{id}X_i+Z), \tag{28}$$

where (26) follows by changing the order of maximization and minimization in (8); (27) follows because

$$\begin{aligned}
&I(X_s,X_\Lambda;Y_d,Y_{[N]\setminus\Lambda}\,|\,X_{[N]\setminus\Lambda})\\
&=I(X_s,X_\Lambda;Y_d-\sum_{i\in[N]\setminus\Lambda}h_{id}X_i,Y_{[N]\setminus\Lambda}\,|\,X_{[N]\setminus\Lambda})\\
&=h(Y_d-\sum_{i\in[N]\setminus\Lambda}h_{id}X_i,Y_{[N]\setminus\Lambda}\,|\,X_{[N]\setminus\Lambda})\\
&\qquad-h(Y_d-\sum_{i\in[N]\setminus\Lambda}h_{id}X_i,Y_{[N]\setminus\Lambda}\,|\,X_s,X_\Lambda,X_{[N]\setminus\Lambda})\\
&=h(Y_d-\sum_{i\in[N]\setminus\Lambda}h_{id}X_i,Y_{[N]\setminus\Lambda}\,|\,X_{[N]\setminus\Lambda})-h(Z,Z_{[N]\setminus\Lambda})\\
&\leq h(Y_d-\sum_{i\in[N]\setminus\Lambda}h_{id}X_i,Y_{[N]\setminus\Lambda})-h(Z,Z_{[N]\setminus\Lambda})\\
&=I(X_s,X_\Lambda;\sum_{i\in\Lambda}h_{id}X_i+Z,Y_{[N]\setminus\Lambda}).
\end{aligned}$$

Note that this last expression maximized over all random variables X_s,X_Λ is the capacity of the point to point channel between $\{s,\Lambda\}$ and $\{[N]\setminus\Lambda,d\}$. The capacity of this channel can be further upper bounded by the sum of the capacities of the SIMO channel between s and $\{[N]\setminus\Lambda\}$ and the MISO channel between $\{\Lambda\}$ and d which is the result stated in (28). Formally, this follows because

$$\begin{aligned}
&I(X_s,X_\Lambda;\sum_{i\in\Lambda}h_{id}X_i+Z,Y_{[N]\setminus\Lambda})\\
&\leq h(\sum_{i\in\Lambda}h_{id}X_i+Z)+h(Y_{[N]\setminus\Lambda})-h(Z)-h(Z_{[N]\setminus\Lambda})\\
&=I(X_s;Y_{[N]\setminus\Lambda})+I(X_\Lambda;\sum_{i\in\Lambda}h_{id}X_i+Z).
\end{aligned}$$

The solutions to the maximization of these mutual informations over the input distributions are well-know and yield the capacities of the corresponding SIMO and MISO channels [23]. Therefore, (28) can be further upper bounded as

$$\overline{C} \leq \min_{\Lambda\subseteq[N]} \Bigg(\log\Big(1+\mathrm{SNR}\sum_{i\in[N]\setminus\Lambda}|h_{is}|^2\Big) + \log\Big(1+\mathrm{SNR}\big(\sum_{i\in\Lambda}|h_{id}|\big)^2\Big)\Bigg) \qquad (29)$$

where $\mathrm{SNR}\doteq\frac{P}{N_0W}$. We will further develop a trivial upper bound on this expression. For simplicity of notation, let us introduce $t_{is}\doteq\sqrt{\mathrm{SNR}}|h_{is}|$ and $t_{id}\doteq\sqrt{\mathrm{SNR}}|h_{id}|$. Separating the cases $\Lambda=\emptyset$ and $\Lambda=[N]$, which correspond to the pure SIMO and pure MISO cuts respectively in (29), we have,

$$\overline{C} \leq \min\Bigg\{ \log\Big(1+\sum_{i\in[N]}t_{is}^2\Big), \log\Big(1+\big(\sum_{i\in[N]}t_{id}\big)^2\Big), \min_{\substack{\Lambda\subseteq[N]\\|\Lambda|\neq 0,N}} \Big(\log\Big(1+\sum_{i\in[N]\setminus\Lambda}t_{is}^2\Big)+\log\Big(1+\big(\sum_{i\in\Lambda}t_{id}\big)^2\Big)\Big)\Bigg\}.$$

Note that the variables t_{is} and t_{id} are real and positive. The sums over the variables t_{id} and t_{is} can be increased by setting each summand to the maximum of the variables that are summed. For example, using also the fact that log is strictly increasing we can write,

$$\log\Big(1+\big(\sum_{i\in\Lambda}t_{id}\big)^2\Big) \leq \log\big(|\Lambda|^2+|\Lambda|^2\max_{i\in\Lambda}t_{id}^2\big) \text{ if } |\Lambda|>0.$$

Using similar arguments we get the following inequality,

$$\overline{C} \leq \min\Bigg\{ \log\Big(1+\max_{i\in[N]}t_{is}^2\Big)+\log N, \log\Big(1+\max_{i\in[N]}t_{id}^2\Big)+2\log N, \min_{\substack{\Lambda\subseteq[N]\\|\Lambda|\neq 0,N}}\Big(\log\Big(1+\max_{i\in[N]\setminus\Lambda}t_{is}^2\Big) + \log\Big(1+\max_{i\in\Lambda}t_{id}^2\Big)+\log\big(|\Lambda|^2|\,[N]\setminus\Lambda|\big)\Big)\Bigg\}.$$

Let us first focus on the $\log\big(|\Lambda|^2|\,[N]\setminus\Lambda|\big)$ term. We have $|\Lambda|+|\,[N]\setminus\Lambda|=N$ and hence

$$\log\big(|\Lambda|^2|\,[N]\setminus\Lambda|\big)=\log\big(N|\Lambda|^2-|\Lambda|^3\big).$$

This term is maximized when $|\Lambda|=\frac{2N}{3}$. Hence,

$$\log\big(|\Lambda|^2|\,[N]\setminus\Lambda|\big)\leq 3\log N-\log\frac{27}{4}.$$

Noting that

$$\log\big(1+\max_{i\in\Lambda}t_{id}^2\big)=\max_{i\in\Lambda}\log\big(1+t_{id}^2\big),$$

we obtain the following upper bound,

$$\overline{C} \leq \min_{\Lambda\subseteq[N]}\ \max_{i\in\Lambda}\log\big(1+t_{id}^2\big)+\max_{i\in[N]\setminus\Lambda}\log\big(1+t_{is}^2\big) + \max\Big(3\log N-\log\frac{27}{4}, 2\log N\Big). \qquad (30)$$

Appendix B
A combinatorial Problem
(proofs of Lemmas 1 and 2)

In addition to $\omega(\Gamma)$, ω, ω_k defined in Section VI, in the due analysis we also use the notation $[a,a+b]=\{a,a+1,\cdots,a+b\}$ for $a\geq 1$ and $b\geq 0$.

Proof of Lemma 1: Given any set of real numbers R_{is}, R_{id}, $i\in[N]$ giving ω in (21), we will prove the lemma by establishing a number of properties for the these numbers in terms of ω. These properties naturally suggest an algorithm to discover a subset $\Gamma\in[N]$ such that $|\Gamma|\leq k$ and $\omega(\Gamma)\geq\frac{k}{k+1}\omega$.

Given any set of real numbers R_{is}, R_{id}, $i\in[N]$, we have the following property

- Property (1): $\exists p\in[N]$ such that $R_{ps}\geq\frac{k}{k+1}\omega$ and $R_{pd}\geq\frac{1}{k+1}\omega$. If not, we would have the following contradictory argument: Assume for all $i\in[N]$, we either have $R_{is}<\frac{k}{k+1}\omega$ or $R_{id}<\frac{1}{k+1}\omega$. Let $S=\{i: R_{is}\geq\frac{k}{k+1}\omega\}$. By the assumption, this means that $\forall i\in S$, $R_{yd}<\frac{1}{k+1}\omega$. Therefore considering the subset $S\subseteq[N]$, we can upper bound ω as,

$$\omega\leq\max_{i\in S}R_{id}+\max_{i\in\overline{S}}R_{is} < \frac{1}{k+1}\omega+\frac{k}{k+1}\omega=\omega.$$

  which is a contradiction.
- Case 1: $R_{pd}\geq\frac{k}{k+1}\omega$. In this case, the lemma is proved since $\omega(\{p\})=\min(R_{ps},R_{pd})\geq\frac{k}{k+1}\omega$, and therefore $\omega_k\geq\omega_1\geq\frac{k}{k+1}\omega$.

Note the proof is complete for $k=1$ at this point, since $R_{pd}\geq\frac{k}{k+1}\omega$ is necessarily the case. We assume that $k>1$ in the remaining discussion.

- Case 2: $R_{pd}<\frac{k}{k+1}\omega$. Then we have the following property.
  Property (2): $\exists m\in[N]$, $m\neq p$ such that $R_{ms}\geq\frac{1}{k+1}\omega$ and $R_{md}\geq\frac{k}{k+1}\omega$. Otherwise, we would have the following contradiction: Assume for all $i\in[N]$, $i\neq p$, we either have $R_{is}<\frac{1}{k+1}\omega$ or $R_{id}<\frac{k}{k+1}\omega$. Let $S=\{i\in[N]: R_{is}\geq\frac{1}{k+1}\omega\}$. By Property (1) above, $p\in S$. Moreover, $\forall i\in S$, $R_{id}<\frac{k}{k+1}\omega$. For p this follows since we are in Case 2 and for other $i\in S$ it follows by the assumption. Therefore we can upper bound ω by

$$\omega\leq\max_{i\in S}R_{id}+\max_{i\in\bar{S}}R_{is} < \frac{k}{k+1}\omega+\frac{1}{k+1}\omega=\omega$$

  which is a contradiction.

Without loss of generality we can rearrange $i\in[N]$ and assume that $p=N$, i.e., $R_{Ns}\geq\frac{k}{k+1}\omega$ and $\frac{k}{k+1}\omega>R_{Nd}\geq\frac{1}{k+1}\omega$. Equivalently,

$$R_{Ns}\geq\frac{k}{k+1}\omega \quad\text{and}\quad \frac{k-a+1}{k+1}\omega>R_{Nd}\geq\frac{k-a}{k+1}\omega,$$

for an integer a such that $1 \leq a \leq k-1$. Similarly, we can also assume that $m = 1$, i.e., $R_{1s} \geq \frac{1}{k+1}\omega$ and $R_{1d} \geq \frac{k}{k+1}\omega$. We proceed by investigating two possible case for R_{1s}.

- Case 1: $R_{1s} \geq \frac{a}{k+1}\omega$. In this case, the lemma is proved since we would have
$$\omega(\{1, N\}) > \frac{k}{k+1}\omega,$$
which means $w_k \geq w_2 \geq \frac{k}{k+1}\omega$.

Note that the proof is complete for $k = 2$ at this point, since $1 \leq a \leq k-1$ yields $a = 1$ and $R_{1s} \geq \frac{a}{k+1}\omega$ is necessarily the case. We assume that $k > 2$ in the remaining discussion.

- Case 2: $\frac{a}{k+1}\omega > R_{1s} \geq \frac{1}{k+1}\omega$. Equivalently,
$$\frac{a_1+1}{k+1}\omega > R_{1s} \geq \frac{a_1}{k+1}\omega \quad \text{and} \quad R_{1d} \geq \frac{k-a_0}{k+1}\omega,$$
for integers a_1 and a_0 such that $1 \leq a_1 < a$ and $a_0 = 0$. We investigate this case, by proving the following proposition.

Proposition 1: Given positive real numbers R_{is}, R_{id}, $i \in [N]$, assume that we can arrange them in the following form.

- $R_{Ns} \geq \frac{k}{k+1}\omega$ and $\frac{k-a+1}{k+1}\omega > R_{Nd} \geq \frac{k-a}{k+1}\omega$ for some $a \in \mathbb{N}$ such that $1 \leq a \leq k-1$.
- For any r such that $1 \leq r \leq l$, $\frac{a_r+1}{k+1}\omega > R_{rs} \geq \frac{a_r}{k+1}\omega$ and $R_{rd} \geq \frac{k-a_{r-1}}{k+1}\omega$ for some $l \in \mathbb{N}$, $1 \leq l \leq k-2$, and $a_0, a_1, \ldots, a_l \in \mathbb{N}$ such that $a_0 = 0 < a_1 < \cdots < a_{l-1} < a_l < a$.

Then, there exists a $y \in [l+1, N-1]$ such that $R_{ys} \geq \frac{a_l+1}{k+1}\omega$ and $R_{yd} \geq \frac{k-a_l}{k+1}\omega$.

Before proving the proposition, we first use it to complete the proof of Lemma 1. Note that we have currently proven that for any positive real numbers R_{is}, R_{id}, $i \in [N]$, either $r_k \geq \frac{k}{k+1}$, or the assumptions of the proposition are satisfied for $l = 1$.

Assume that the assumptions of the proposition are satisfied for some $1 \leq l \leq k-2$. Then the proposition asserts the existence of $y \in [l+1, N-1]$ such that $R_{ys} \geq \frac{a_l+1}{k+1}\omega$ and $R_{yd} \geq \frac{k-a_l}{k+1}\omega$ for some $a_{l+1} \in \mathbb{N}$ such that $a_l < a_{l+1} < a$. This leads to two possible cases for the newly discovered $y \in [l+1, N-1]$:

- Case 1: $R_{ys} \geq \frac{a}{k+1}\omega$. In this case, the proof of the lemma is completed, because
$$\omega([l] \cup \{y, N\}) \geq \frac{k}{k+1}\omega,$$
and $|\,[l] \cup \{y, N\}| \leq k$. This can be observed as follows: Assume $R_{ys} \geq \frac{a}{k+1}\omega$ and $R_{yd} \geq \frac{k-a_l}{k+1}\omega$ for some $y \in [l+1, N-1]$. Note that if $\omega([l] \cup \{y, N\}) < \frac{k}{k+1}\omega$, there exists at least one set $S \subseteq [l] \cup \{y, N\}$ such that
$$\left(\max_{i \in S} R_{id} + \max_{i \in [l] \cup \{y,N\} \setminus S} R_{is}\right) < \frac{k}{k+1}\omega. \qquad (31)$$
We argue below that such a set S does not exist. Since $R_{Ns} \geq \frac{k}{k+1}\omega$ we should have $N \in S$. Then also $y \in S$, since otherwise we get the contradiction,
$$\begin{aligned}\max_{i \in S} R_{id} + \max_{i \in [l] \cup \{y,N\} \setminus S} R_{is} &\geq R_{Nd} + R_{ys} \\ &\geq \frac{k-a}{k+1}\omega + \frac{a}{k+1}\omega \\ &= \frac{k}{k+1}\omega.\end{aligned}$$
Then by the same reasoning, we also have $l \in S$. Otherwise,
$$\begin{aligned}\max_{i \in S} R_{id} + \max_{i \in [l] \cup \{y,N\} \setminus S} R_{is} &\geq R_{yd} + R_{ls} \\ &\geq \frac{k-a_l}{k+1}\omega + \frac{a_l}{k+1}\omega \\ &= \frac{k}{k+1}\omega.\end{aligned}$$
Similarly for every $r \in [l-1]$, we should also have $r \in S$. This is because if $r+1 \in S$ and $r \in [l] \cup \{y, N\} \setminus S$ we have the following contradiction,
$$\begin{aligned}\max_{i \in S} R_{id} + \max_{i \in [l] \cup \{y,N\} \setminus S} R_{is} &\geq R_{r+1,d} + R_{rs} \\ &\geq \frac{k-a_r}{k+1}\omega + \frac{a_r}{k+1}\omega \\ &= \frac{k}{k+1}\omega.\end{aligned}$$
Therefore $S = [l] \cup \{y, N\}$. However, then we have
$$\max_{i \in S} R_{id} \geq \frac{k-a_0}{k+1}\omega,$$
which contradicts (31) since $a_0 = 0$.
- Case 2: $\frac{a}{k+1}\omega > R_{ys} \geq \frac{a_l+1}{k+1}\omega$. Without loss of generality we can rearrange $y \in [l+1, N-1]$ and assume that $\frac{a}{k+1}\omega > R_{l+1,s} \geq \frac{a_l+1}{k+1}\omega$ and $R_{l+1,d} \geq \frac{k-a_l}{k+1}\omega$. Equivalently,
$$\frac{a_{l+1}+1}{k+1}\omega > R_{l+1,s} \geq \frac{a_{l+1}}{k+1}\omega \quad \text{and} \quad R_{l+1,d} \geq \frac{k-a_l}{k+1}\omega,$$
for some $a_{l+1} \in \mathbb{N}$ such that $a_l < a_{l+1} < a$. Therefore, we have proven that the assumptions of the proposition should indeed be satisfied with $l+1$ in this case.

This implies that starting with $l = 1$, we can apply the proposition recursively as long as $l \leq k-2$. At each step of the recursion, either we prove that $r_k \geq \frac{k}{k+1}\omega$ and the proof of the lemma is complete or l is increased by 1. Assume that $l = k-2$ and applying the proposition still does not prove the lemma (i.e., the k-relays discovered do not satisfy $w(\Gamma) \geq \frac{k}{k+1}\omega$). Then the proposition establishes the existence of a sequence of positive numbers $a_0, a_1, a_2, \cdots, a_{k-1}$ such that
$$a_0 = 0 < a_1 < \cdots < a_{k-2} < a_{k-1} < a \leq k-1,$$
which is a contradiction. This implies that Case 1 should have been true in one of the earlier iterations of the proposition, which proves the lemma.

To summarize the conclusions from Case 1 in the above discussion, we have shown that given any positive real numbers R_{is}, R_{id}, $i \in [N]$ and $1 \leq k < N$, they can be either arranged as
$$R_{Ns} \geq \frac{k}{k+1}\omega \quad \text{and} \quad R_{Nd} \geq \frac{k}{k+1}\omega,$$

or

- $R_{Ns} \geq \frac{k}{k+1}\omega$ and $\frac{k-a+1}{k+1}\omega > R_{Nd} \geq \frac{k-a}{k+1}\omega$ for some $a \in \mathbb{N}$ such that $1 \leq a \leq k-1$,
- and for $1 \leq r \leq l$, $\frac{a_r+1}{k+1}\omega > R_{rs} \geq \frac{a_r}{k+1}\omega$ and $R_{rd} \geq \frac{k-a_{r-1}}{k+1}\omega$ for some $l \in \mathbb{N}$ such that $1 \leq l \leq k-2$, and $a_0, a_1, \ldots, a_l \in \mathbb{N}$ such that $a_0 = 0 < a_1 < \cdots < a_{l-1} < a_l < a$,
- and $R_{l+1,d} \geq \frac{a}{k+1}\omega$ and $R_{yd} \geq \frac{k-a_l}{k+1}\omega$.

For these $l+2 \leq k$ nodes $\Gamma = [l+1] \cup \{N\}$, we have $w(\Gamma) \geq \frac{k}{k+1}\omega$. □

Proof of Proposition 1: If the proposition were not true, then we would have the following contradictory argument: Assume for all $i \in [l+1, N-1]$, we either have $R_{is} < \frac{a_l+1}{k+1}\omega$ or $R_{id} < \frac{k-a_l}{k+1}\omega$. Let $S = \{i \in [l+1, N-1] : R_{is} \geq \frac{a_l+1}{k+1}\omega\}$. This means that $\forall i \in S$, $R_{id} < \frac{k-a_l}{k+1}\omega$ and $\forall i \in [l+1, N-1] \setminus S$, $R_{is} < \frac{a_l+1}{k+1}\omega$. Therefore considering the subset $S \cup \{N\} \subseteq [N]$, we can upper bound ω as,

$$\begin{aligned}\omega &\leq \max_{i \in S \cup \{N\}} R_{id} + \max_{i \in [N] \setminus S \setminus \{N\}} R_{is} \\ &= \max_{i \in S \cup \{N\}} R_{id} + \max_{i \in [l] \cup ([l+1, N-1] \setminus S)} R_{is} \\ &< \max\left(\frac{k-a_l}{k+1}\omega, \frac{k-a+1}{k+1}\omega\right) + \max_{1 \leq r \leq l} \frac{a_r+1}{k+1}\omega \\ &= \frac{k-a_l}{k+1}\omega + \frac{a_l+1}{k+1}\omega = \omega,\end{aligned}$$

which is a contradiction. □

Proof of Lemma 2: We will prove that for the configuration $R_{is} = i\,R$ and $R_{id} = (k+2-i)\,R$ for $1 \leq i \leq k+1$, we have $\omega_k = \frac{k}{k+1}\omega$.

We first show that for this particular configuration $\omega = (k+1)R$. Let Λ be any subset of $[k+1]$ and let $y(\Lambda) = \max_{i \in \bar{\Lambda}} R_{is}$. Then, $\max_{i \in \Lambda} R_{id} \geq (k+2)R - (y(\Lambda) + R)$. Note that the last inequality holds even if $y(\Lambda) = (k+1)R$. Therefore, we have

$$\begin{aligned}\omega &= \min_{\Lambda \subseteq [k+1]} \left(\max_{i \in \Lambda} R_{id} + \max_{i \in [k+1] \setminus \Lambda} R_{is}\right) \\ &\geq \min_{\Lambda \subseteq [k+1]} \left[(k+1 - y(\Lambda)) + y(\Lambda)\right] = (k+1)R.\end{aligned}$$

On the other hand, $\omega \leq (k+1)R$. Therefore, $\omega = (k+1)R$.

We now prove that for any $\Gamma \subset [k+1]$ with $|\Gamma| = k$, we have $\omega(\Gamma) = kR$. Let Λ be any subset of Γ and let $y(\Lambda) = \max_{i \in \Gamma \setminus \Lambda} R_{is}$. Then $\max_{i \in \Lambda} R_{id} \geq (k+2)R - (y(\Lambda) + 2R)$. Note that this inequality holds even if $y(\Lambda) = (k+1)R$. The reason that we have used $y(\Lambda) + 2R$ this time is because of the possibility that $\arg\max_{i \in \Gamma \setminus \Lambda} R_{is} + 1 \notin \Gamma$. Therefore, we have,

$$\begin{aligned}\omega(\Gamma) &= \min_{\Lambda \subseteq \Gamma} \left(\max_{i \in \Lambda} R_{id} + \max_{i \in \Gamma \setminus \Lambda} R_{is}\right) \\ &\geq \min_{\Lambda \subseteq \Gamma} \left[(kR - y(\Lambda)) + y(\Lambda)\right] = kR.\end{aligned}$$

Now, for any $\Gamma \subseteq [k+1]$ with $|\Gamma| = k$ there exists a $j(\Gamma) \in [k+1]$ such that $\Gamma = [k+1] \setminus \{j(\Gamma)\}$. Then, we have

$$\begin{aligned}\omega(\Gamma) &= \min_{\Lambda \subseteq \Gamma} \left(\max_{i \in \Lambda} R_{id} + \max_{i \in \Gamma \setminus \Lambda} R_{is}\right) \\ &\leq \max_{i \in [j(\Gamma)-1]} R_{is} + \max_{i \in [j(\Gamma)+1, k+1]} R_{id} \\ &= (j(\Gamma) - 1)\,R + (k + 2 - (j(\Gamma) + 1))\,R = kR.\end{aligned}$$

Note that this reasoning holds even if $j(\Gamma) = 1$ or $j(\Gamma) = k+1$.

Therefore, we have proved that

$$\omega_k = \max_{\substack{\Gamma \subseteq [k+1] \\ |\Gamma| = k}} \omega(\Gamma) = kR.$$