Due in class on April 25, 2024

Lattices and Quadratic Forms (Summer 2024) - Problem Set 2

1. Let L be an integral lattice with arbitrary signature (not necessarily nondegenerate).
   a) If $v \in L$ is a vector with norm ± 1, show that L can be written as $\mathbb{Z}v \perp L'$ for a sublattice L' of L.
   b) As a corollary, if L is an integral, positive-definite lattice that is generated by its norm 1 and norm 2 vectors, show that L can be written as $\mathbb{Z}^k \perp L'$ for some k and a sublattice L' that is a root lattice.
2. Determine whether the following two lattices are isometric.
   a) Lattices L_1 and L_2 with respective Gram matrices
$$G_1 = \begin{pmatrix} 4 & 3 & 0 \\ 3 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad \text{and} \quad G_2 = \begin{pmatrix} -2 & 2 & -2 \\ 2 & 2 & 1 \\ -2 & 1 & -1 \end{pmatrix}.$$
   b) $\mathbb{Z}\langle 2\rangle \perp \mathbb{Z}\langle -1\rangle$ and $\mathbb{Z}\langle -2\rangle \perp \mathbb{Z}$.
3. For any $n \geq 4$, let us define the D_n lattice as a sublattice of $\mathbb{Z}^n$ defined by
$$D_n := \{k \in \mathbb{Z}^n : k_1 + k_2 + \ldots + k_n \text{ even}\}.$$
   (a) Find all the roots in this lattice. How many of them are there? Prove that D_n is generated by these vectors and hence it is a root lattice.
   (b) Show that the following set of n vectors
$$\alpha_1 = (1, -1, 0, \ldots, 0), \quad \alpha_2 = (0, 1, -1, 0, \ldots, 0), \ \ldots\ , \alpha_{n-1} = (0, \ldots, 0, 1, -1), \quad \alpha_n = (0, \ldots, 0, 1, 1)$$
   form a fundamental system of roots. Verify the corresponding Coxeter-Dynkin diagram.
   (c) What is the discriminant of D_n? *Hint:* Use that it as a sublattice of $\mathbb{Z}^n$.
   (d) Show that a complete set of representatives for the nontrivial elements of $D_n^\#/D_n$ is given by the vectors
$$v = \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right), \quad w = \left(-\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right), \quad e = (1, 0, \ldots, 0).$$
   What is the discriminant group $D_n^\#/D_n$ and the corresponding discriminant form?
   (e) Find an automorphism of D_n that maps $v + D_n$ to $w + D_n$.
   (f) For $8 \mid n$, show that
$$D_n^+ := D_n \cup (v + D_n)$$
   is an even, self-dual lattice. The lattice D_8^+ is indeed one of the constructions of the E_8 lattice.
   (g) For $n > 4$ show that there is no automorphism of D_n that can map $v + D_n$ (or $w + D_n$) to $e + D_n$. Does your argument work for $n = 4$?
   (h) For $n = 4$, show that the transformations $k \mapsto Uk$ and $k \mapsto Vk$ with
$$U := \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad V := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$
   are automorphisms of D_4 of order 3 and 2, respectively. Show that they generate a subgroup of the automorphism group isomorphic to S_3 and that they permute $v + D_4$, $e + D_4$, and $w + D_4$ among each other. These extra symmetries of D_4 are called "triality automorphisms" and are related to the enhanced symmetries of the Dynkin diagram of D_4.

Note: You may use computational tools to help with your solutions, but describe your steps. Problems 1 and 2 are worth 10 points. Problem 3 is worth 20 points.