Due in class on May 2, 2024

Lattices and Quadratic Forms (Summer 2024) - Problem Set 3

1. Consider the realization of A_n as a sublattice of $\mathbb{Z}^{n+1}$ given by

$$A_n := \{k \in \mathbb{Z}^{n+1} : k_1 + k_2 + \ldots + k_{n+1} = 0\}.$$

In class we have seen that the discriminant group is given by $A_n^\sharp / A_n = \{jv + A_n : j = 0, 1, \ldots, n\}$, where

$$v := \left(\frac{n}{n+1}, -\frac{1}{n+1}, \ldots, -\frac{1}{n+1} \right).$$

Prove that the shortest possible norm in the coset $jv + A_n$ is given by $\frac{j(n+1-j)}{n+1}$.

2. Find the size of the automorphism group of the D_n lattice for each $n \geq 4$.
3. Let us consider the E_8 lattice.
   a) How many norm 6 and norm 8 vectors are there in E_8?
   b) Consider the action of the Weyl group $W(E_8)$ on the norm 6 vectors. How many orbits are there under this action? *Hint:* First find the orbits under the action of $W(D_8)$.
4. Let L be an n-dimensional irreducible root lattice (an ADE lattice), let $S(L) = \{\alpha_1, \ldots, \alpha_n\}$ be a set of simple roots and $R^+(L)$ be the corresponding positive roots.
   a) For any positive root $\alpha \in R^+(L) \setminus S(L)$, prove that there is a simple root α_j such that $\alpha - \alpha_j \in R^+(L)$.
   b) Given a positive root $\alpha = \sum_{j=1}^n k_j \, \alpha_j$, its height (relative to the simple roots $\alpha_1, \ldots, \alpha_n$) is defined as the positive integer $\mathrm{ht}(\alpha) := \sum_{j=1}^n k_j$. Positive roots have a positive height and the height is equal to one if and only if the given root is a simple one.
   Use part (a) to describe an algorithm that takes the Coxeter-Dynkin diagram (or equivalently the Gram matrix) corresponding to $\alpha_1, \ldots, \alpha_n$ as input and then generates a list of all the roots in L successively over each height.
   c) Implement your algorithm to find all the roots in the E_6 lattice together with their height. What is the root with the largest height (the highest root) and its height?

Note: You may use computational tools to help with your solutions, but describe your steps. Each problem is worth 10 points.