Lattices and Quadratic Forms (Summer 2024) - Solutions to Problem Set 3

1. Let us start by recognizing

$$jv = \left(j - \frac{j}{n+1}, -\frac{j}{n+1}, \ldots, -\frac{j}{n+1}\right)$$

so that we can characterize $jv + A_n$ as

$$jv + A_n = \left\{ \left(k_1 - \frac{j}{n+1}, k_2 - \frac{j}{n+1}, \ldots, k_{n+1} - \frac{j}{n+1}\right) : \ k \in \mathbb{Z}^{n+1} \text{ and } k_1 + k_2 + \ldots + k_{n+1} = j \right\}.$$

The norm of an arbitrary vector in $jv + A_n$ as shown above is then given by

$$\sum_{r=1}^{n+1} k_r^2 - \frac{2j}{n+1}\sum_{r=1}^{n+1} k_r + \frac{j^2}{n+1} = \sum_{r=1}^{n+1} k_r^2 - \frac{j^2}{n+1},$$

where we used the fact that $\sum_{r=1}^{n+1} k_r = j$ for the second equality. So to find the shortest norm in $jv + A_n$ our goal is reduced to minimizing the norm $\sum_{r=1}^{n+1} k_r^2$ over integers k_r with the constraint $\sum_{r=1}^{n+1} k_r = j$. Here we find an easy lower bound to the norm since we have $k_r^2 \geq k_r$ when $k_r \in \mathbb{Z}$ with equality if and only if $k_r = 0$ or $k_r = 1$. This yields

$$\sum_{r=1}^{n+1} k_r^2 \geq \sum_{r=1}^{n+1} k_r = j.$$

However then any k with j of its entries equal to 1 and the remaining $n+1-j$ equal to 0 saturates this bound and finishes our proof that the minimum norm in $jv + A_n$ is

$$j - \frac{j^2}{n+1} = \frac{j(n+1-j)}{n+1}.$$

Note that the corresponding vectors in $jv + A_n$ are the vectors that we discussed in class:

$$\left(\underbrace{\frac{n+1-j}{n+1}, \ldots, \frac{n+1-j}{n+1}}_{j \text{ times}}, \underbrace{-\frac{j}{n+1}, \ldots, -\frac{j}{n+1}}_{n+1-j \text{ times}}\right)$$

and their permutations. In fact, our proof shows a bit more than the problem statement: It shows that only these vectors saturate the minimum norm bound, so in $jv + A_n$ there are $\binom{n+1}{j}$ vectors with minimal norm.

Remark. Here is an alternative argument: First let us show that if there are any negative entries in k, then we can find another vector $k' \in \mathbb{Z}^{n+1}$ that also satisfies the constraint and that has a smaller norm. Suppose that $k_{i_1} < 0$. Then pick index i_2 with $k_{i_2} > 0$, which exists because of the constraint

$$\sum_{r \neq i_1} k_r = j - k_{i_1} > 0.$$

Then we can replace k as (with the omitted entries unchanged)

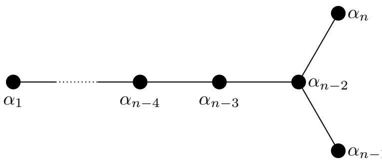
$$k = (\ldots, k_{i_1}, \ldots, k_{i_2}, \ldots) \mapsto k' = (\ldots, 0, \ldots, k_{i_2} + k_{i_1}, \ldots).$$

This k' also satisfies the constraint and it has a smaller norm since

$$(k_{i_1} + k_{i_2})^2 - k_{i_1}^2 - k_{i_2}^2 = 2k_{i_1}k_{i_2} < 0.$$

Since the norm $\sum_{r=1}^{n+1} k_r^2$ is a nonnegative integer, this process can not go on indefinitely and should end up with a vector with no negative entries. Then with the search space restricted to $k \in \mathbb{Z}_{\geq 0}^{n+1}$, we can proceed similarly and show that if we start with a vector that has an entry ≥ 2, then we can always reduce its norm while remaining in $\mathbb{Z}_{\geq 0}^{n+1}$.

2. Let $(\alpha_1, \ldots, \alpha_n)$ be a fundamental system of roots for D_n (so its Gram matrix is described by the Dynkin diagram).



Recall that automorphisms are in bijective correspondence with n-tuples of D_n vectors with the same Gram matrix as that of $(\alpha_1, \ldots, \alpha_n)$. So to find the size of the automorphism group of D_n, we count the number of such n-tuples of vectors in the realization

$$D_n := \{k \in \mathbb{Z}^n : \, k_1 + k_2 + \ldots + k_n \text{ even}\}.$$

- We start with counting the number of possibilities for α_{n-2}, which is a root (like the other α_j's). Recall that roots in D_n are of the form $\pm\varepsilon_i \pm \varepsilon_j$, where $\varepsilon_i \in \mathbb{Z}^n$ denotes the vector that has all zeros as coordinates except for a '+1' at the i^{th} entry. Also recall that the number of roots, and hence the number of α_{n-2} candidates, is

$$\boxed{4\binom{n}{2} = 2n(n-1).}$$

  - The automorphism group of D_n includes permutations and sign changes of coordinates.[1] So up to the application of an automorphism we can assume that e.g. $\alpha_{n-2} = \varepsilon_1 + \varepsilon_2$.
- Our next goal is to find the number of possibilities for the set $\{\alpha_{n-3}, \alpha_{n-1}, \alpha_n\}$ given α_{n-2} above. The vectors $\alpha_{n-3}, \alpha_{n-1}, \alpha_n$ are pairwise orthogonal roots that have inner product -1 with α_{n-2}. Any root that has inner product -1 with α_{n-2} is of the form $-\varepsilon_1 \pm \varepsilon_r$ or $-\varepsilon_2 \pm \varepsilon_r$ for some $r \in \{3, 4, \ldots, n\}$. So the set $\{\alpha_{n-3}, \alpha_{n-1}, \alpha_n\}$ is of the form

$$\{-\varepsilon_1 + \varepsilon_j, -\varepsilon_1 - \varepsilon_j, -\varepsilon_2 \pm \varepsilon_k\} \quad \text{or} \quad \{-\varepsilon_2 + \varepsilon_j, -\varepsilon_2 - \varepsilon_j, -\varepsilon_1 \pm \varepsilon_k\}$$

  for $j, k \in \{3, 4, \ldots, n\}$ with $j \neq k$. So in total there are

$$\boxed{2 \cdot (n-2) \cdot 2(n-3)}$$

  ways to choose the set $\{\alpha_{n-3}, \alpha_{n-1}, \alpha_n\}$ given α_{n-2}. Here the factor of 2 in front refers to the two possibilities mentioned above, $(n-2)$ counts the number of ways we can choose j from $\{3, 4, \ldots, n\}$, and $2(n-3)$ counts the number of ways we can choose $k \neq j$ together with the choice of sign in front of ε_k.
  - Up to permutations and sign changes that preserve $\alpha_{n-2} = \varepsilon_1 + \varepsilon_2$, we can set

$$\{\alpha_{n-3}, \alpha_{n-1}, \alpha_n\} = \{-\varepsilon_1 + \varepsilon_3, -\varepsilon_1 - \varepsilon_3, -\varepsilon_2 - \varepsilon_4\}.$$

- Now we found the possibilities for the unordered set $\{\alpha_{n-3}, \alpha_{n-1}, \alpha_n\}$, which we set to $\{-\varepsilon_1 + \varepsilon_3, -\varepsilon_1 - \varepsilon_3, -\varepsilon_2 - \varepsilon_4\}$ up to automorphisms, we need to assign individual vectors $\alpha_{n-3}, \alpha_{n-1}, \alpha_n$ to individual elements of this set. There are 3! ways to perform this assignment and this completes our considerations for $n = 4$.
  - If $n > 4$, however, four of these assignments can not be continued to build the n-tuple $(\alpha_1, \ldots, \alpha_n)$. In particular, if α_{n-3} is $-\varepsilon_1 \pm \varepsilon_3$, then it is not possible to find a root α_{n-4} consistent with this choice since any root that has inner product -1 with $-\varepsilon_1 \pm \varepsilon_3$ necessarily has nonzero inner product with $-\varepsilon_1 \mp \varepsilon_3$ (but α_{n-4} should be orthogonal to $-\varepsilon_1 \mp \varepsilon_3$ since we assign it to either α_{n-1} or α_n).

[1]However recall that the Weyl group only includes evenly many sign changes.

So for $n > 4$ only two of the assignments are viable (setting $\alpha_{n-3} = -\varepsilon_2 - \varepsilon_4$ and assigning $-\varepsilon_1 \pm \varepsilon_3$ to α_{n-1} and α_n) and these two assignments are equivalent because we can change the sign of ε_3 if necessary and set

$$\alpha_{n-1} = -\varepsilon_1 + \varepsilon_3 \quad \text{and} \quad \alpha_n = -\varepsilon_1 - \varepsilon_3.$$

In summary, there are

$$\boxed{\begin{cases} 3! & \text{if } n = 4 \\ 2 & \text{if } n > 4 \end{cases}}$$

ways to choose $\alpha_{n-3}, \alpha_{n-1}, \alpha_n$ from the set $\{-\varepsilon_1 + \varepsilon_3, -\varepsilon_1 - \varepsilon_3, -\varepsilon_2 - \varepsilon_4\}$ described above (in a way that can be continued with an α_{n-4}).

- At this point we continue with α_{n-4} assuming that $n > 4$ and that

$$\alpha_{n-3} = -\varepsilon_2 - \varepsilon_4, \quad \alpha_{n-2} = \varepsilon_1 + \varepsilon_2, \quad \alpha_{n-1} = -\varepsilon_1 + \varepsilon_3, \quad \alpha_n = -\varepsilon_1 - \varepsilon_3.$$

The orthogonality of the root α_{n-4} to $\alpha_{n-2}, \alpha_{n-1}, \alpha_n$ means that the first three entries of α_{n-4} should be zero. The requirement that $\alpha_{n-4} \cdot \alpha_{n-3} = -1$ then requires α_{n-3} to be of the form $\varepsilon_4 \pm \varepsilon_j$ for some $j \in \{5, \ldots, n\}$. There are

$$\boxed{2(n-4)}$$

ways to choose such vectors. Moreover, up to sign changes and permutations of $\{5, \ldots, n\}$ (which preserve $\alpha_{n-3}, \ldots, \alpha_n$) we can set

$$\alpha_{n-4} = \varepsilon_4 - \varepsilon_5.$$

- From now on the argument is the same as that for α_{n-4}. Orthogonality with $\alpha_{n-3}, \ldots, \alpha_n$ and having inner product -1 with α_{n-4} requires the root α_{n-5} to be of the form $\varepsilon_5 \pm \varepsilon_j$ for some $j \in \{6, \ldots, n\}$. So there are

$$2(n-5)$$

ways to choose such vectors and up to sign changes and permutations of $\{6, \ldots, n\}$ (which preserve $\alpha_{n-4}, \ldots, \alpha_n$) we can set $\alpha_{n-5} = \varepsilon_5 - \varepsilon_6$. Continuing this way, we find for each successive $5 \le j \le n-1$ that there are $2(n-j)$ ways to choose α_{n-j} and up to automorphisms fixing the earlier vectors we can set it to $\alpha_{n-j} = \varepsilon_j - \varepsilon_{j+1}$.

Now we have completed our counting of ordered n-tuples of vectors $(\alpha_1, \ldots, \alpha_n)$ with inner products described by the Dynkin diagram, we can combine the factors we have found above to find that there are

$$[2n(n-1)] \cdot [2 \cdot (n-2) \cdot 2(n-3)] \cdot \begin{Bmatrix} 3! & \text{if } n = 4 \\ 2 & \text{if } n > 4 \end{Bmatrix} \cdot \prod_{j=4}^{n-1} [2(n-j)] \, .$$

of them. Therefore, we finally have

$$|\mathrm{Aut}(D_n)| = 2^n \, n! \cdot \begin{Bmatrix} 3 & \text{if } n = 4 \\ 1 & \text{if } n > 4 \end{Bmatrix}.$$

3. As in class we use the realization of E_8 as the lattice D_8^+, i.e.

$$E_8 = D_8 \cup (s + D_8),$$

where

$$D_8 := \{k \in \mathbb{Z}^8 : k_1 + k_2 + \ldots + k_8 \text{ even}\} \quad \text{and} \quad s = \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$$

so that

$$E_8 = \left\{ k \in \mathbb{Z}^8 \text{ or } k \in \left(\mathbb{Z} + \frac{1}{2}\right)^8 : k_1 + k_2 + \ldots + k_8 \text{ even} \right\}.$$

a) We start by counting norm 6 vectors:

   i. Vectors of the form $(\pm1,\pm1,\pm1,\pm1,\pm1,\pm1,0,0)$ with any sign choice and all of their permutations: There are $\binom{8}{6}$ ways to choose the nonzero entries and 2^6 ways to choose the signs. So in total there are $2^6\binom{8}{6}=1792$ such vectors.

   ii. Vectors of the form $(\pm2,\pm1,\pm1,0,0,0,0,0)$ with any sign choice and their permutations: We first choose the position of the ±2 entry in $\binom{8}{1}$ ways. Then we choose the position of the ±1 entries in $\binom{7}{2}$ ways. Finally we have 2^3 choices for the signs. In total this yields $2^3\binom{8}{1}\binom{7}{2}=1344$ vectors.

   iii. Vectors of the form $\left(\varepsilon_1\,\frac{3}{2},\varepsilon_2\,\frac{3}{2},\varepsilon_3\,\frac{1}{2},\varepsilon_4\,\frac{1}{2},\varepsilon_5\,\frac{1}{2},\varepsilon_6\,\frac{1}{2},\varepsilon_7\,\frac{1}{2},\varepsilon_8\,\frac{1}{2}\right)$, where the signs $\varepsilon_j\in\{\pm1\}$ are required to satisfy $\prod_{j=1}^{8}\varepsilon_j=+1$ (since $2\times\frac{3}{2}+6\times\frac{1}{2}$ is even and flipping the sign of either $\frac{1}{2}$ or $\frac{3}{2}$ terms changes this sum by an odd number) and we can permute the positions of the $\frac{3}{2}$ and $\frac{1}{2}$ entries. There are $\binom{8}{2}$ ways to choose the position of the $\frac{3}{2}$ entries and then there are 2^7 sign choices (we can choose $\varepsilon_1,\dots,\varepsilon_7$ arbitrarily and that uniquely determines ε_8). So overall we find $2^7\binom{8}{2}=3584$ such vectors.

   In total this means that E_8 has

$$1792+1344+3584=6720$$

   norm 6 vectors. We next apply the same argument to norm 8 vectors.

   i. Vectors of the form $(\pm1,\pm1,\pm1,\pm1,\pm1,\pm1,\pm1,\pm1)$ with any sign choice: There are $2^8=256$ such vectors.

   ii. Vectors of the form $(\pm2,\pm1,\pm1,\pm1,\pm1,0,0,0)$ with any sign choice and all their permutations: We first choose the position of the ±2 entry in $\binom{8}{1}$ ways. Then we choose the position of the ±1 entries in $\binom{7}{4}$ ways. Finally we have 2^5 choices for the signs. In total this yields $2^5\binom{8}{1}\binom{7}{4}=8960$ vectors.

   iii. Vectors of the form $(\pm2,\pm2,0,0,0,0,0,0)$ with any sign choice and all their permutations: There are $\binom{8}{2}$ ways to choose the position of the ±2 entries and 2^2 sign choices. So there are $2^2\binom{8}{2}=112$ such vectors.

   iv. Vectors of the form $\left(\varepsilon_1\,\frac{3}{2},\varepsilon_2\,\frac{3}{2},\varepsilon_3\,\frac{3}{2},\varepsilon_4\,\frac{1}{2},\varepsilon_5\,\frac{1}{2},\varepsilon_6\,\frac{1}{2},\varepsilon_7\,\frac{1}{2},\varepsilon_8\,\frac{1}{2}\right)$, where the signs $\varepsilon_j\in\{\pm1\}$ are required to satisfy $\prod_{j=1}^{8}\varepsilon_j=-1$ (since $3\times\frac{3}{2}+5\times\frac{1}{2}$ is odd) and we can permute the positions of the $\frac{3}{2}$ and $\frac{1}{2}$ entries. There are $\binom{8}{3}$ ways to choose the position of the $\frac{3}{2}$ entries and there are 2^7 sign choices. So overall we find $2^7\binom{8}{3}=7168$ such vectors.

   v. Vectors of the form $\left(\varepsilon_1\,\frac{5}{2},\varepsilon_2\,\frac{1}{2},\varepsilon_3\,\frac{3}{2},\varepsilon_4\,\frac{1}{2},\varepsilon_5\,\frac{1}{2},\varepsilon_6\,\frac{1}{2},\varepsilon_7\,\frac{1}{2},\varepsilon_8\,\frac{1}{2}\right)$, where the signs $\varepsilon_j\in\{\pm1\}$ are required to satisfy $\prod_{j=1}^{8}\varepsilon_j=+1$ (since $\frac{5}{2}+7\times\frac{1}{2}$ is even) and we can permute the positions of the $\frac{5}{2}$ and $\frac{1}{2}$ entries. There are $\binom{8}{1}$ ways to choose the position of the $\frac{5}{2}$ entry and there are 2^7 sign choices. So overall we find $2^7\binom{8}{1}=1024$ such vectors.

   In total this yields

$$256+8960+112+7168+1024=17520$$

   norm 8 vectors in E_8.

b) Recall that the Weyl group of D_8 is generated by permutations and evenly many sign changes.[2]

   - Using elements of $W(D_8)$ we can bring the 1792 vectors of the form $(\pm1,\pm1,\pm1,\pm1,\pm1,\pm1,0,0)$ discussed above into the form $(1,1,1,1,1,1,0,0)$.[3]
   - The 1344 vectors of the form $(\pm2,\pm1,\pm1,0,0,0,0,0)$ can be transformed to $(2,1,1,0,0,0,0,0)$.
   - Finally, 3584 vectors of the form $\left(\varepsilon_1\,\frac{3}{2},\varepsilon_2\,\frac{3}{2},\varepsilon_3\,\frac{1}{2},\varepsilon_4\,\frac{1}{2},\varepsilon_5\,\frac{1}{2},\varepsilon_6\,\frac{1}{2},\varepsilon_7\,\frac{1}{2},\varepsilon_8\,\frac{1}{2}\right)$ can be transformed to[4] $\left(\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ with $W(D_8)$ action.

[2] Note that in our realization the roots in the D_8 sublattice are of the form $\pm\varepsilon_i\pm\varepsilon_j$ with $1\le i\ne j\le 8$. The Weyl reflection associated with $\pm(\varepsilon_i-\varepsilon_j)$ exchanges the i^{th} and j^{th} coordinates, whereas the Weyl reflection associated with $\pm(\varepsilon_i+\varepsilon_j)$ exchanges the i^{th} and j^{th} coordinates while also changing their sign.

[3] Note that there are zero entries so we can perform any sign change we want on the nonzero entries.

[4] Here we successively apply sign changes on the first entry and one another to remove the signs on entries other than the first. Note that the product of the signs is $+1$ so the remaining sign at the first entry has to be $+1$ as well.

So under the action of $W(D_8)$ there are three orbits. However we would like to study the orbits under $W(E_8)$, so we next check what happens when we apply a Weyl reflection associated with one of the roots of E_8 that is not in D_8. So let us take the root $s = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

- Applying the corresponding Weyl reflection on $v := (1, 1, 1, 1, 1, 1, 0, 0)$ we find

$$\sigma_s(v) = v - (s \cdot v)s = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{3}{2}\right).$$

  So the $W(D_8)$ orbits of $(1, 1, 1, 1, 1, 1, 0, 0)$ and $\left(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ are connected by $W(E_8)$.
- With $w := (2, 1, 1, 0, 0, 0, 0, 0)$ we have

$$\sigma_s(w) = w - (s \cdot w)s = (1, 0, 0, -1, -1, -1, -1, -1).$$

  So the $W(D_8)$ orbits associated with $(1, 1, 1, 1, 1, 1, 0, 0)$ and $(2, 1, 1, 0, 0, 0, 0, 0)$ are also connected by $W(E_8)$.

Therefore, the norm 6 vectors of E_8 form a single orbit under the action of the Weyl group $W(E_8)$.

Remark. At norm 8 however there will be more than one orbit. This can be shown like in our argument above. First we find the orbits under $W(D_8)$ and then it is enough to check how σ_s acts on these $W(D_8)$ orbits since $W(E_8)$ is generated by $W(D_8)$ and σ_s (because all of the new roots of E_8 can be obtained from s with evenly many sign changes, i.e. with the action a $W(D_8)$ element).

4.

a) Recall that the inner product between two roots α, β can only take value in $\{-2, -1, 0, +1, +2\}$ with equality to ± 2 if and only $\alpha = \pm\beta$. Now let $\alpha \in R^+(L)/S(L)$ and $\beta \in S(L)$. Since $-\beta$ is not a positive root, our assumption tells us $\alpha \neq \pm\beta$ and hence $\alpha \cdot \beta \in \{-1, 0, +1\}$ as discussed above.
Since α is a positive root, it has an expansion to simple roots as $\alpha = \sum_{j=1}^{n} k_j \alpha_j$ with k_j nonnegative integers. If we had $\alpha \cdot \alpha_j \in \{-1, 0\}$ for all j, this would lead to a contradiction since

$$\alpha \cdot \alpha = \sum_{j=1}^{n} k_j (\alpha_j \cdot \alpha) \leq 0,$$

but we need to have $\alpha \cdot \alpha = 2$. So there is at least one j for which $\alpha \cdot \alpha_j = +1$. That immediately implies that $\beta := \alpha - \alpha_j$ is a root for that j.
Moreover, in the expansion $\alpha = \sum_{j=1}^{n} k_j \alpha_j$ one of the entries k_i for some $i \neq j$ has to be positive since α is not a simple root and the only scalar multiple of α_j that is a positive root is α_j itself (which is simple). Correspondingly, this $k_i > 0$ coefficient also appears in the expansion of β to simple roots. So β should also be a positive root.

b) For any positive integer k, let us define $R_k^+(L)$ as the set of positive roots with height k (with respect to a given set of simple roots). Then we know that $R_1^+(L)$ is simply the set of simple roots and by part (a) for any vector $\beta \in R_k^+(L)$ with $k > 1$, there is a vector $\gamma \in R_{k-1}^+(L)$ and a simple root $\alpha \in S(L)$ such that $\beta = \alpha + \gamma$. Here we note that $\alpha + \gamma$ is a root (which is then necessarily in $R_k^+(L)$) if and only if $\alpha \cdot \gamma = -1$. This then leads to the following algorithm for finding all the positive roots (which together with their negatives give all the roots) starting with simple roots:

- **Step 1:** Set $R_1^+(L) = S(L)$ and $k = 1$.
- **Step 2:** Initialize $R_{k+1}^+(L)$ to empty set.
- **Step 3:** For all pairs of vectors $\alpha \in S(L)$ and $\gamma \in R_k^+(L)$, add $\alpha + \gamma$ to the set $R_{k+1}^+(L)$ if $\alpha \cdot \gamma = -1$.
- **Step 4:** If $R_{k+1}^+(L)$ is nonempty, increase k by 1 and return to Step 2. If not, terminate.

c) Implementing the algorithm for E_6 where the Dynkin diagram describes the inner products between simple roots $\{\alpha_1, \alpha_2, \dots, \alpha_6\}$ through the Gram matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix},$$

we find the following positive roots:

- **Height 1:** $\{\alpha_1,\, \alpha_2,\, \alpha_3,\, \alpha_4,\, \alpha_5,\, \alpha_6\}$
- **Height 2:** $\{\alpha_1 + \alpha_2,\, \alpha_2 + \alpha_3,\, \alpha_3 + \alpha_4,\, \alpha_3 + \alpha_6,\, \alpha_4 + \alpha_5\}$
- **Height 3:** $\{\alpha_1 + \alpha_2 + \alpha_3,\, \alpha_2 + \alpha_3 + \alpha_4,\, \alpha_2 + \alpha_3 + \alpha_6,\, \alpha_3 + \alpha_4 + \alpha_5,\, \alpha_3 + \alpha_4 + \alpha_6\}$
- **Height 4:** $\{\alpha_1+\alpha_2+\alpha_3+\alpha_4,\, \alpha_1+\alpha_2+\alpha_3+\alpha_6,\, \alpha_2+\alpha_3+\alpha_4+\alpha_5,\, \alpha_2+\alpha_3+\alpha_4+\alpha_6,\, \alpha_3+\alpha_4+\alpha_5+\alpha_6\}$
- **Height 5:** $\{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5,\, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_6,\, \alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6,\, \alpha_2+2\alpha_3+\alpha_4+\alpha_6\}$
- **Height 6:** $\{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6,\, \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_6,\, \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6\}$
- **Height 7:** $\{\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6,\, \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_6,\, \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6\}$
- **Height 8:** $\{\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6,\, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6\}$
- **Height 9:** $\{\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6\}$
- **Height 10:** $\{\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6\}$
- **Height 11:** $\{\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6\}$

In particular, the highest root is $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6$ and it has height 11.