Due in class on May 13, 2024

Lattices and Quadratic Forms (Summer 2024) - Problem Set 4

1. In class we constructed the E_8 lattice as the lattice D_8^+. Show that one can also construct the E_8 lattice by using the A_8 lattice and its cosets in the dual lattice $A_8^\#$.
2. Let L be a root lattice and let $\{\alpha_1, \ldots, \alpha_n\}$ be a set of simple roots. Compute the order of the elements $\sigma_{\alpha_i}\sigma_{\alpha_j}$ in the Weyl group for each $1 \leq i, j \leq n$.
3. In class we have defined roots as norm 2 vectors of a root lattice. There is a more general definition which defines a root system R (whose elements are called roots) as a finite subset of nonzero vectors in the Euclidean space $\mathbb{R}^n$ obeying the following conditions:
   - Elements of R span $\mathbb{R}^n$.
   - The only multiples of $\alpha \in R$ that are also in R are $\pm\alpha$.
   - If $\alpha, \beta \in R$, then $2\frac{\alpha\cdot\beta}{\alpha\cdot\alpha} \in \mathbb{Z}$.
   - If $\alpha, \beta \in R$, then $\sigma_\alpha(\beta) = \beta - 2\frac{\alpha\cdot\beta}{\alpha\cdot\alpha}\alpha \in R$.

   In particular, the roots inside a root lattice (as we defined in class) do in fact form a root system with respect to this more general definition.

   Many of the notions we have introduced to study root lattices, such as positive/negative roots and simple roots, can be introduced for these more general root systems. Then following similar arguments to the ones we had in class, one can find a complete classification of irreducible root systems (i.e. those root systems that can not be partitioned into two orthogonal subsets) up to bijective mappings that leave the numbers $2\frac{\alpha\cdot\beta}{\alpha\cdot\alpha}$ invariant (so scaling a root system by a positive number gives an equivalent root system). In addition to the root systems

   $$A_{n\geq 1},\ D_{n\geq 4},\ E_6,\ E_7,\ E_8$$

   that we found in class, one finds the following root systems (where we are fixing an appropriate scale)
   - $B_{n\geq 2}$ in $\mathbb{R}^n$ given by
     $$\{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_j : 1 \leq i < j \leq n\}.$$
   - $C_{n\geq 3}$ in $\mathbb{R}^n$ given by
     $$\{\pm\varepsilon_i \pm \varepsilon_j,\ \pm 2\varepsilon_j : 1 \leq i < j \leq n\}.$$
   - G_2 in $\mathbb{R}^3$ given by
     $$\{\pm(\varepsilon_2 - \varepsilon_3), \pm(\varepsilon_1 - \varepsilon_3), \pm(\varepsilon_1 - \varepsilon_2), \pm(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3), \pm(2\varepsilon_2 - \varepsilon_1 - \varepsilon_3), \pm(2\varepsilon_3 - \varepsilon_1 - \varepsilon_2)\}.$$
   - F_4 in $\mathbb{R}^4$ given by
     $$\{\pm\varepsilon_i \text{ for } 1 \leq i \leq 4,\ \pm\varepsilon_i \pm \varepsilon_j \text{ for } 1 \leq i < j \leq 4,\ \frac{1}{2}\left(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4\right)\}.$$

     Identify the lattices generated by these new root systems with lattices we have already constructed.

Note: You may use computational tools to help with your solutions, but describe your steps. Each problem is worth 10 points.