Lattices and Quadratic Forms (Summer 2024) - Solutions to Problem Set 4

1. Since we are trying to build the E_8 lattice, which is an even, self-dual lattice, as a lattice that contains the A_8 lattice, we will be looking at lattices L such that $A_8 \leq L \leq A_8^\sharp$. Such lattices are in one-to-one correspondence with the subgroups of the discriminant group $A_8^\sharp/A_8$ (by the correspondence theorem). We know that

$$A_8^\sharp/A_8 \simeq \mathbb{Z}/9\mathbb{Z}$$

with the cyclic group generated by $[v] := v + A_8$ where

$$v := \left(\frac{8}{9}, -\frac{1}{9}, \ldots, -\frac{1}{9}\right).$$

The group $A_8^\sharp/A_8$ has three subgroups: the trivial subgroup, the full group $A_8^\sharp/A_8$, and the order three cyclic subgroup generated by $[3v]$ or $[6v]$. The first two subgroups correspond to the lattices A_8 and $A_8^\sharp$, which are not self-dual. So we restrict our attention to

$$A_8 \leq L := A_8 \cup (\beta + A_8) \cup (2\beta + A_8) \leq A_8^\sharp$$

where

$$\beta := \left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) \equiv 3v \pmod{A_8}.$$

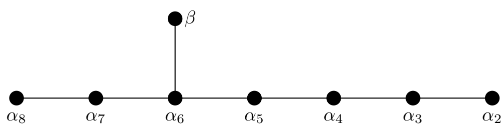
The lattice L is generated by the vectors β and $\alpha_1, \ldots, \alpha_8$, where α_j denotes the simple roots of A_8 given by

$$\alpha_1 = (1, -1, 0, \ldots, 0), \quad \alpha_2 = (0, 1, -1, 0, \ldots, 0), \quad \ldots \quad, \alpha_8 = (0, \ldots, 0, 1, -1).$$

Also note that

$$\alpha_1 = -2\alpha_2 - 3\alpha_3 - 4\alpha_4 - 5\alpha_5 - 6\alpha_6 - 4\alpha_7 - 2\alpha_8 - 3\beta$$

So the vectors $\alpha_2, \alpha_3, \ldots, \alpha_8$ and β suffice to generate L. Moreover, these are linearly independent vectors since they form a basis for the eight dimensional real vector space spanned by $\alpha_1, \ldots, \alpha_8$. So $(\alpha_2, \alpha_3, \ldots, \alpha_8, \beta)$ forms an ordered basis for L consisting of roots (note that $\beta^2 = 2$). Since $\alpha_6 \cdot \beta = -1$ with all the other $\alpha_j \cdot \beta$'s zero, we find that the Gram matrix for this basis of L is the same as the one described by the E_8 Dynkin diagram if we identify



This proves that L is isometric to the E_8 lattice.

Remark. Alternatively, we can use the following argument:

- Let $\ell \in L$ be an arbitrary element. We can write it (uniquely) as $\ell = v + j\beta$ for some $v \in A_8$ and $j \in \{0, 1, 2\}$. Then we have $\ell^2 = v^2 + 2jv \cdot \beta + j^2\beta^2 \in 2\mathbb{Z}$ because $v \cdot \beta \in \mathbb{Z}$ and $\beta^2 = 2$. So L is an even lattice (and in particular an integral lattice).
- Using the relation $\det(L') = |L/L'|^2 \det(L)$ (for L' a sublattice of L of the same rank) with $L' = A_8$ so that $\det(L') = 9$ and $|L/L'| = 3$ we find $\det(L) = 1$. So the discriminant group is trivial and L is a self-dual lattice.

Finally, note that the integral lattice L is generated by the roots of the A_8 lattice and β, which is also a root since $\beta^2 = 2$. So L is a root lattice (that is possibly reducible). However, the only self-dual root lattice of dimension 8 is the E_8 lattice since by the classification theorem, root lattices are orthogonal direct sums of irreducible root lattices (with ADE classification) and only E_8 factors can appear in the decomposition if the resulting root lattice is self-dual (so the only self-dual root lattices are $E_8, E_8 \perp E_8, \ldots$).

2. Given an arbitrary element γ of the lattice L, the transformation $\sigma := \sigma_{\alpha_i}\sigma_{\alpha_j} \in W(L)$ acts as

$$\sigma(\gamma) = \sigma_{\alpha_i}(\sigma_{\alpha_j}(\gamma)) = \sigma_{\alpha_i}(\gamma - (\gamma \cdot \alpha_j)\alpha_j)$$

so that

$$\sigma(\gamma) = \gamma - (\gamma \cdot \alpha_j)\alpha_j - (\gamma \cdot \alpha_i)\alpha_i + (\gamma \cdot \alpha_j)(\alpha_j \cdot \alpha_i)\alpha_i$$

- Let us first handle the trivial case $i = j$ where $\alpha_i \cdot \alpha_j = 2$. From our computation (or from the fact that σ_{α_i} are reflections that have order two), we find that $\sigma(\gamma) = \gamma$ and hence in this case it has order one.

Assuming $1 \leq i \neq j \leq n$ from now on (for which σ acts nontrivially), the only possibilities for $\alpha_i \cdot \alpha_j$ are 0 and -1 since these are simple roots.

- If $\alpha_i \cdot \alpha_j = 0$, then our expression for $\sigma_{\alpha_i}(\sigma_{\alpha_j}(\gamma))$ becomes symmetric in i and j. So we have $\sigma_{\alpha_i}\sigma_{\alpha_j} = \sigma_{\alpha_j}\sigma_{\alpha_i}$. This then immediately shows

$$\sigma^2 = \sigma_{\alpha_i}\sigma_{\alpha_j}\sigma_{\alpha_i}\sigma_{\alpha_j} = \sigma_{\alpha_i}\sigma_{\alpha_i}\sigma_{\alpha_j}\sigma_{\alpha_j} = 1.$$

  So $\sigma = \sigma_{\alpha_i}\sigma_{\alpha_j}$ has order two in this case.
- Finally, we consider the case $\alpha_i \cdot \alpha_j = -1$, where we have

$$\sigma(\gamma) = \gamma - (\gamma \cdot \alpha_j)(\alpha_i + \alpha_j) - (\gamma \cdot \alpha_i)\alpha_i.$$

  The easiest way to understand what σ is doing is to restrict to the plane spanned by α_i and α_j (since σ acts identically on the orthogonal complement) and to pick an orthonormal basis for that plane where $\alpha_j = (\sqrt{2}, 0)$ and $\alpha_i = \left(-\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right)$ with respect to that basis. Then $\gamma = (\gamma_1, \gamma_2)$ (where γ_j are the components with respect to this basis) is transformed as

$$\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix} \mapsto \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix} - \gamma_1\sqrt{2}\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \sqrt{\frac{3}{2}}\end{pmatrix} - \left(-\frac{\gamma_1}{\sqrt{2}} + \gamma_2\sqrt{\frac{3}{2}}\right)\begin{pmatrix}-\frac{1}{\sqrt{2}}\\ \sqrt{\frac{3}{2}}\end{pmatrix} = \begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}.$$

  The transformation matrix is simply a rotation by 120^0. So σ in this case has order three.
  - Alternatively, we can explicitly compute using $\sigma(\gamma) \cdot \alpha_j = \gamma \cdot \alpha_i$ and $\sigma(\gamma) \cdot \alpha_i = -\gamma \cdot \alpha_i - \gamma \cdot \alpha_j$ as

$$\begin{aligned}\sigma^2(\gamma) = \gamma - (\gamma \cdot \alpha_j)(\alpha_i + \alpha_j) - (\gamma \cdot \alpha_i)\alpha_i - (\gamma \cdot \alpha_i)(\alpha_i + \alpha_j) + (\gamma \cdot \alpha_i + \gamma \cdot \alpha_j)\alpha_i\\ = \gamma - (\gamma \cdot \alpha_j)\alpha_j - (\gamma \cdot \alpha_i)(\alpha_i + \alpha_j).\end{aligned}$$

    Correspondingly, we can then use $\sigma^2(\gamma) \cdot \alpha_j = -\gamma \cdot \alpha_i - \gamma \cdot \alpha_j$ and $\sigma^2(\gamma) \cdot \alpha_i = \gamma \cdot \alpha_j$ to compute

$$\sigma^3(\gamma) = \gamma - (\gamma \cdot \alpha_j)\alpha_j - (\gamma \cdot \alpha_i)(\alpha_i + \alpha_j) + (\gamma \cdot \alpha_i + \gamma \cdot \alpha_j)(\alpha_i + \alpha_j) - (\gamma \cdot \alpha_j)\alpha_i = \gamma.$$

3. We start with the $B_{n\geq 2}$ root system in $\mathbb{R}^n$ given by

$$\{\pm\varepsilon_i \pm \varepsilon_j,\, \pm\varepsilon_j : 1 \leq i < j \leq n\}.$$

Note that the vectors $\pm\varepsilon_j$ span the Euclidean lattice $\mathbb{Z}^n$ and the vectors $\pm\varepsilon_i \pm \varepsilon_j$ are contained in this lattice. So we have (identifying the notation for the lattice and the root system)

$$\boxed{B_n = \mathbb{Z}^n.}$$

Next we consider the $C_{n\geq 3}$ root system in $\mathbb{R}^n$ given by

$$\{\pm\varepsilon_i \pm \varepsilon_j,\, \pm 2\varepsilon_j : 1 \leq i < j \leq n\}.$$

For $n \geq 4$, the vectors $\pm\varepsilon_i \pm \varepsilon_j$ are the roots of the D_n lattice and the vectors $\pm 2\varepsilon_j$ are already spanned by vectors of the form $\pm\varepsilon_i \pm \varepsilon_j$. So for $n \geq 4$ we have

$$\boxed{C_n = D_n.}$$

If we extend the definition of the D_n lattice to $n = 3$ as well, then we would again see that the $C_3 = D_3$. One thing to note however is that the D_3 lattice is not a new lattice but it is equivalent to the A_3 lattice. One quick way to confirm this is to note that $C_3 = D_3$ is spanned by the vectors $\alpha_1 = \varepsilon_2 + \varepsilon_3$, $\alpha_2 = \varepsilon_1 - \varepsilon_2$, $\alpha_3 = \varepsilon_2 - \varepsilon_3$ which have the same Gram matrix as that described by the A_3 Dynkin diagram. So we also note

$$\boxed{C_3 = A_3.}$$

Next we consider the G_2 root system in $\mathbb{R}^3$ given by

$$\{\pm(\varepsilon_2 - \varepsilon_3), \pm(\varepsilon_1 - \varepsilon_3), \pm(\varepsilon_1 - \varepsilon_2), \pm(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3), \pm(2\varepsilon_2 - \varepsilon_1 - \varepsilon_3), \pm(2\varepsilon_3 - \varepsilon_1 - \varepsilon_2)\}.$$

Now note that the vectors $\pm(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3), \pm(2\varepsilon_2 - \varepsilon_1 - \varepsilon_3), \pm(2\varepsilon_3 - \varepsilon_1 - \varepsilon_2)$ are in the linear span of $\pm(\varepsilon_2 - \varepsilon_3), \pm(\varepsilon_1 - \varepsilon_3), \pm(\varepsilon_1 - \varepsilon_2)$ with integral coefficients. So we can restrict our attention to the latter set of vectors and these are nothing but the roots of A_2 in its usual realization in $\mathbb{Z}^3$. So we have

$$\boxed{G_2 = A_2.}$$

We finally consider the F_4 root system in $\mathbb{R}^4$ given by

$$\{\pm\varepsilon_i \text{ for } 1 \leq i \leq 4,\ \pm\varepsilon_i \pm \varepsilon_j \text{ for } 1 \leq i < j \leq 4,\ \frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\}.$$

Note that the vectors $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ span all the vectors of the form $\pm\varepsilon_i$ and $\pm\varepsilon_i \pm \varepsilon_j$. Moreover, adding $\beta := -\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$ to the list and noting that $\varepsilon_1 = -2\beta - \varepsilon_2 - \varepsilon_3 - \varepsilon_4$ we find that

$$\beta,\ \varepsilon_2,\ \varepsilon_3,\ \varepsilon_4$$

forms a basis for the lattice F_4. All four of these vectors have norm 1 and we have

$$\beta \cdot \varepsilon_j = -\frac{1}{2} \quad \text{for all } j \in \{2, 3, 4\}$$

with the inner products between $\varepsilon_2, \varepsilon_3, \varepsilon_4$ equal to zero. So the corresponding Gram matrix is equal to that described by the D_4 Dynkin diagram up to a rescaling by $\frac{1}{2}$ (with β corresponding to the trivalent node in the diagram). So we have

$$\boxed{F_4 = D_4\langle 1/2 \rangle.}$$