Due in class on May 27, 2024

Lattices and Quadratic Forms (Summer 2024) - Problem Set 5

1. Let L be an integral, positive definite lattice and let us consider the theta functions
$$\Theta_{L,\mu}(\tau) := \sum_{\mathbf{n}\in L+\mu} e^{\pi i\tau\mathbf{n}^2},$$
where $\tau \in \mathbb{H}$ and $\mu \in L^\sharp$. For any given $c \in \mathbb{Z}_{>0}$, prove the identity
$$\Theta_{L,\mu}(\tau/c) = \sum_{m\in L/cL} \Theta_{\sqrt{c}L,\frac{1}{\sqrt{c}}(m+\mu)}(\tau).$$

2. Consider the Jacobi theta functions (for $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$)
$$\theta_{00}(\tau,z) := \sum_{m\in\mathbb{Z}} e^{\pi i\tau m^2+2\pi izm}, \qquad \theta_{01}(\tau,z) := \sum_{m\in\mathbb{Z}} (-1)^m e^{\pi i\tau m^2+2\pi izm},$$
$$\theta_{10}(\tau,z) := \sum_{m\in\mathbb{Z}} e^{\pi i\tau\left(m+\frac{1}{2}\right)^2+2\pi iz\left(m+\frac{1}{2}\right)}, \qquad \theta_{11}(\tau,z) := i\sum_{m\in\mathbb{Z}} (-1)^m e^{\pi i\tau\left(m+\frac{1}{2}\right)^2+2\pi iz\left(m+\frac{1}{2}\right)},$$
which we can alternatively write as
$$\theta_{ab}(\tau,z) := \sum_{m\in\mathbb{Z}+\frac{a}{2}} e^{\pi i\tau m^2+2\pi i\left(z+\frac{b}{2}\right)m} \qquad \text{for } a,b \in \{0,1\}.$$
In class we found the transformations
$$\theta_{00}(\tau+1,z) = \theta_{01}(\tau,z) \quad \text{and} \quad \theta_{00}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = e^{\pi iz^2/\tau}\sqrt{-i\tau}\,\theta_{00}(\tau,z).$$
Find the analogous transformations under $(\tau,z) \mapsto (\tau+1,z)$ and $(\tau,z) \mapsto \left(-\frac{1}{\tau},\frac{z}{\tau}\right)$ for θ_{01}, θ_{10}, and θ_{11}.

3. Let L be an n-dimensional integral (but necessarily even), positive definite lattice and let $\boldsymbol{\ell} \in L$ be a *characteristic (or parity) vector*, i.e. a vector satisfying $\mathbf{n}^2 \equiv \mathbf{n}\cdot\boldsymbol{\ell} \pmod 2$ for all $\mathbf{n} \in L$. Then for any $\boldsymbol{\mu} \in L^\sharp$ define the theta function
$$\Psi_{\boldsymbol{\mu}}(\tau) := \sum_{\mathbf{n}\in L+\boldsymbol{\mu}+\frac{\boldsymbol{\ell}}{2}} e^{\pi i\tau\mathbf{n}^2+\pi i\mathbf{n}\cdot\boldsymbol{\ell}}.$$
Prove that these functions satisfy the transformations
$$\Psi_{\boldsymbol{\mu}}(\tau+1) = e^{\pi i\left(\boldsymbol{\mu}+\frac{\boldsymbol{\ell}}{2}\right)^2}\Psi_{\boldsymbol{\mu}}(\tau) \quad \text{and} \quad \Psi_{\boldsymbol{\mu}}\left(-\frac{1}{\tau}\right) = e^{-\frac{\pi i}{2}\boldsymbol{\ell}^2}\frac{(-i\tau)^{n/2}}{\sqrt{\det L}}\sum_{\boldsymbol{\nu}\in L^\sharp/L} e^{-2\pi i\boldsymbol{\mu}\cdot\boldsymbol{\nu}}\,\Psi_{\boldsymbol{\nu}}(\tau).$$

4. The Dedekind eta function is defined through the infinite product (for $\tau \in \mathbb{H}$ and $q := e^{2\pi i\tau}$)
$$\eta(\tau) := e^{\frac{\pi i\tau}{12}}\prod_{n=1}^{\infty}(1-e^{2\pi in\tau}) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n).$$
Thanks to Euler's pentagonal theorem, one can rewrite the Dedekind eta function as a theta series:
$$\eta(\tau) = q^{\frac{1}{24}}\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{1}{2}n(3n+1)} = \sum_{n\in\mathbb{Z}} e^{3\pi i\left(n+\frac{1}{6}\right)^2\tau+\pi in}.$$
Use this representation to prove the transformation
$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau).$$

Note: You may use computational tools to help with your solutions, but describe your steps. Each problem is worth 10 points.