Lattices and Quadratic Forms (Summer 2024) - Solutions to Problem Set 5

1. Given representatives $m \in L/cL$, we can uniquely write any element $n \in L+\mu$ as $n = r+m+\mu$ where $r \in cL$. Therefore,

$$\Theta_{L,\mu}(\tau/c) = \sum_{n\in L+\mu} e^{\pi i \frac{\tau}{c} n^2} = \sum_{m\in L/cL}\sum_{r\in cL} e^{\pi i \frac{\tau}{c}(r+m+\mu)^2} = \sum_{m\in L/cL}\sum_{r\in \sqrt{c}L} e^{\pi i \frac{\tau}{c}(\sqrt{c}r+m+\mu)^2}$$

and hence

$$\Theta_{L,\mu}(\tau/c) = \sum_{m\in L/cL}\sum_{r\in \sqrt{c}L+\frac{m+\mu}{\sqrt{c}}} e^{\pi i\tau r^2} = \sum_{m\in L/cL}\Theta_{\sqrt{c}L,\frac{1}{\sqrt{c}}(m+\mu)}(\tau),$$

proving the asserted identity. Here we also note that $\sqrt{c}L$ is a positive-definite, integral lattice and $\frac{1}{\sqrt{c}}(m+\mu) \in (\sqrt{c}L)^\sharp$. Further note that the summands $\Theta_{\sqrt{c}L,\frac{1}{\sqrt{c}}(m+\mu)}(\tau)$ are independent of the representatives chosen from L/cL elements and this is why we can view the sum as one over L/cL.

2. We start with the $(\tau,z)\mapsto(\tau+1,z)$ transformation. For θ_{01}, we have

$$\theta_{01}(\tau+1,z) = \sum_{m\in\mathbb{Z}}(-1)^m e^{\pi i m^2} e^{\pi i\tau m^2+2\pi i z m}.$$

Since $(-1)^m e^{\pi i m^2} = (-1)^{m(m+1)} = 1$, we find

$$\boxed{\theta_{01}(\tau+1,z) = \theta_{00}(\tau,z).}$$

For θ_{10} and θ_{11}, the transformation produces the $e^{\pi i(m+\frac{1}{2})^2}$ factor within the summands. Since

$$e^{\pi i\left(m+\frac{1}{2}\right)^2} = e^{\pi i/4}(-1)^{m^2+m} = e^{\pi i/4} \quad \text{for } m\in\mathbb{Z},$$

we find

$$\boxed{\theta_{10}(\tau+1,z) = e^{\pi i/4}\theta_{10}(\tau,z)} \quad \text{and} \quad \boxed{\theta_{11}(\tau+1,z) = e^{\pi i/4}\theta_{11}(\tau,z).}$$

For the $(\tau,z)\mapsto\left(-\frac{1}{\tau},\frac{z}{\tau}\right)$ transformation, we start with the general expression

$$\theta_{ab}(\tau,z) = \sum_{m\in\mathbb{Z}+\frac{a}{2}} e^{\pi i\tau m^2+2\pi i m\left(z+\frac{b}{2}\right)} = e^{-\pi i(z+b/2)^2/\tau}\sum_{m\in\mathbb{Z}+\frac{a}{2}} e^{\pi i\tau\left(m+\frac{z+b/2}{\tau}\right)^2}$$

so that

$$\theta_{ab}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = e^{\pi i(z+b\tau/2)^2/\tau}\sum_{m\in\mathbb{Z}+\frac{a}{2}} e^{-\frac{\pi i}{\tau}\left(m-z-\frac{b\tau}{2}\right)^2} = e^{\pi i(z+b\tau/2)^2/\tau}\sum_{m\in\mathbb{Z}} f_\tau\left(m-z-\frac{b\tau-a}{2}\right),$$

where

$$f_\tau(x) := e^{-\frac{\pi i}{\tau}x^2}.$$

This Gaussian function is a Schwartz function and we use Poisson summation (on the self-dual lattice $\mathbb{Z}$) while noting that the Fourier transform[1] of $f_\tau(x-r)$ is $\sqrt{-i\tau}\, e^{\pi i\tau k^2-2\pi i k r}$ to find[2]

$$\theta_{ab}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = e^{\pi i(z+b\tau/2)^2/\tau}\sqrt{-i\tau}\sum_{k\in\mathbb{Z}} e^{\pi i\tau k^2-2\pi i k\left(z+\frac{b\tau-a}{2}\right)}.$$

[1] The Fourier transform $\widehat{f}(k)$ of $f(x)$ is

$$\widehat{f}(k) = \int_{\mathbb{R}} \mathrm{d}x\, f(x)\, e^{-2\pi i k x}.$$

[2] This is including the case where r is complex as can be confirmed by performing the relevant Gaussian integrals.

Taking $k \mapsto -k$, we rewrite this as

$$\theta_{ab}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = e^{\pi i (z+b\tau/2)^2/\tau}\sqrt{-i\tau}\sum_{k\in\mathbb{Z}} e^{\pi i\tau k^2+2\pi i k\left(z+\frac{b\tau-a}{2}\right)}$$

and then note

$$e^{\frac{\pi i}{\tau}(z+b\tau/2)^2}e^{\pi i\tau k^2+2\pi i k\left(z+\frac{b\tau-a}{2}\right)} = e^{\frac{\pi i}{\tau}z^2}e^{\pi i\tau\left(k+\frac{b}{2}\right)^2}e^{2\pi i\left(k+\frac{b}{2}\right)\left(z+\frac{a}{2}\right)}e^{-2\pi i k a-\frac{\pi i}{2}ab}.$$

Since $a \in \{0,1\}$, we have $e^{-2\pi i k a} = 1$ for $k \in \mathbb{Z}$ and this leads to

$$\theta_{ab}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = e^{-\frac{\pi i}{2}ab}e^{\pi i z^2/\tau}\sqrt{-i\tau}\sum_{k\in\mathbb{Z}+\frac{b}{2}} e^{\pi i\tau k^2+2\pi i k\left(z+\frac{a}{2}\right)}$$

and hence

$$\boxed{\theta_{ab}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = e^{-\frac{\pi i}{2}ab}e^{\pi i z^2/\tau}\sqrt{-i\tau}\,\theta_{ba}(\tau,z).}$$

In particular,

$$\boxed{\theta_{01}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = e^{\pi i z^2/\tau}\sqrt{-i\tau}\,\theta_{10}(\tau,z),} \qquad \boxed{\theta_{10}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = e^{\pi i z^2/\tau}\sqrt{-i\tau}\,\theta_{01}(\tau,z),}$$

and

$$\boxed{\theta_{11}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = -i\,e^{\pi i z^2/\tau}\sqrt{-i\tau}\,\theta_{11}(\tau,z).}$$

3. Under $\tau \mapsto \tau+1$ we have

$$\Psi_{\boldsymbol{\mu}}(\tau+1) = \sum_{\mathbf{n}\in L+\boldsymbol{\mu}+\frac{\boldsymbol{\ell}}{2}} e^{\pi i\mathbf{n}^2}\, e^{\pi i\tau\mathbf{n}^2+\pi i\mathbf{n}\cdot\boldsymbol{\ell}}.$$

Now note that for $\mathbf{n} = \mathbf{m}+\boldsymbol{\mu}+\frac{\boldsymbol{\ell}}{2}$ with $\mathbf{m}\in L$ we have

$$e^{\pi i\left(\mathbf{m}+\boldsymbol{\mu}+\frac{\boldsymbol{\ell}}{2}\right)^2} = e^{\pi i\left(\boldsymbol{\mu}+\frac{\boldsymbol{\ell}}{2}\right)^2+\pi i(\mathbf{m}^2+\mathbf{m}\cdot\boldsymbol{\ell})+2\pi i\mathbf{m}\cdot\boldsymbol{\mu}}.$$

Since $\boldsymbol{\mu}\in L^\sharp$ we have $\mathbf{m}\cdot\boldsymbol{\mu}\in\mathbb{Z}$ and since $\boldsymbol{\ell}$ is a characteristic vector in the integral lattice L we have $\mathbf{m}^2+\mathbf{m}\cdot\boldsymbol{\ell}\in 2\mathbb{Z}$. Therefore, $e^{\pi i\left(\mathbf{m}+\boldsymbol{\mu}+\frac{\boldsymbol{\ell}}{2}\right)^2} = e^{\pi i\left(\boldsymbol{\mu}+\frac{\boldsymbol{\ell}}{2}\right)^2}$ factors out as an overall factor and we get

$$\boxed{\Psi_{\boldsymbol{\mu}}(\tau+1) = e^{\pi i\left(\boldsymbol{\mu}+\frac{\boldsymbol{\ell}}{2}\right)^2}\Psi_{\boldsymbol{\mu}}(\tau).}$$

To study the behavior under the inversion $\tau\mapsto -\frac{1}{\tau}$ we first rewrite

$$\Psi_{\boldsymbol{\mu}}(\tau) = \sum_{\mathbf{n}\in L+\boldsymbol{\mu}+\frac{\boldsymbol{\ell}}{2}} e^{\pi i\tau\mathbf{n}^2+\pi i\mathbf{n}\cdot\boldsymbol{\ell}} = e^{-\pi i\frac{\boldsymbol{\ell}^2}{4\tau}}\sum_{\mathbf{n}\in L+\boldsymbol{\mu}+\frac{\boldsymbol{\ell}}{2}} e^{\pi i\tau\left(\mathbf{n}+\frac{\boldsymbol{\ell}}{2\tau}\right)^2} = e^{-\pi i\frac{\boldsymbol{\ell}^2}{4\tau}}\sum_{\mathbf{n}\in L} e^{\pi i\tau\left(\mathbf{n}+\boldsymbol{\mu}+\frac{\boldsymbol{\ell}}{2}+\frac{\boldsymbol{\ell}}{2\tau}\right)^2}$$

so that

$$\Psi_{\boldsymbol{\mu}}\left(-\frac{1}{\tau}\right) = e^{\pi i\tau\frac{\boldsymbol{\ell}^2}{4}}\sum_{\mathbf{n}\in L} f_\tau\left(\mathbf{n}+\boldsymbol{\mu}+\frac{\boldsymbol{\ell}}{2}-\frac{\boldsymbol{\ell}\tau}{2}\right)$$

where

$$f_\tau(\mathbf{x}) := e^{-\frac{\pi i}{\tau}\mathbf{x}^2}.$$

This Gaussian function is a Schwartz function and we use Poisson summation while noting that the Fourier transform of $f_\tau(\mathbf{x}+\mathbf{r})$ is $(-i\tau)^{n/2}\,e^{\pi i\tau\mathbf{k}^2+2\pi i\mathbf{k}\cdot\mathbf{r}}$ (also when $\mathbf{r}$ is complex) to find

$$\Psi_{\boldsymbol{\mu}}\left(-\frac{1}{\tau}\right) = e^{\pi i\tau\frac{\boldsymbol{\ell}^2}{4}}\frac{(-i\tau)^{n/2}}{\sqrt{\det L}}\sum_{\mathbf{k}\in L^\sharp} e^{\pi i\tau\mathbf{k}^2+2\pi i\mathbf{k}\cdot\left(\boldsymbol{\mu}+\frac{\boldsymbol{\ell}}{2}-\frac{\boldsymbol{\ell}\tau}{2}\right)}.$$

Letting $\mathbf{k} \mapsto -\mathbf{k}$ and then writing the sum $\sum_{\mathbf{k}\in L^\sharp}$ as $\sum_{\boldsymbol{\nu}\in L^\sharp/L} \sum_{\mathbf{k}\in L+\boldsymbol{\nu}}$ we find

$$\Psi_{\boldsymbol{\mu}}\left(-\frac{1}{\tau}\right) = e^{\pi i \tau \frac{\boldsymbol{\ell}^2}{4}} \frac{(-i\tau)^{n/2}}{\sqrt{\det L}} \sum_{\boldsymbol{\nu}\in L^\sharp/L} \sum_{\mathbf{k}\in L+\boldsymbol{\nu}} e^{\pi i \tau \mathbf{k}^2 - 2\pi i \mathbf{k}\cdot\left(\boldsymbol{\mu}+\frac{\boldsymbol{\ell}}{2}-\frac{\boldsymbol{\ell}\tau}{2}\right)}.$$

Now note that

$$e^{\pi i \tau \frac{\boldsymbol{\ell}^2}{4}} e^{\pi i \tau \mathbf{k}^2 - 2\pi i \mathbf{k}\cdot\left(\boldsymbol{\mu}+\frac{\boldsymbol{\ell}}{2}-\frac{\boldsymbol{\ell}\tau}{2}\right)} = e^{\pi i \tau \left(\mathbf{k}+\frac{\boldsymbol{\ell}}{2}\right)^2} e^{-\pi i \mathbf{k}\cdot\boldsymbol{\ell}} e^{-2\pi i \mathbf{k}\cdot\boldsymbol{\mu}},$$

where

$$e^{-2\pi i \mathbf{k}\cdot\boldsymbol{\mu}} = e^{-2\pi i \boldsymbol{\nu}\cdot\boldsymbol{\mu}} \quad \text{for } \mathbf{k} \in L+\boldsymbol{\nu}.$$

Moreover, for any $\mathbf{k} \in L+\boldsymbol{\nu}$, which we can write as $\mathbf{k} = \mathbf{m}+\boldsymbol{\nu}$ with $\mathbf{m} \in L$, we have $\mathbf{k}\cdot\boldsymbol{\ell} = (\mathbf{m}+\boldsymbol{\nu})\cdot\boldsymbol{\ell} \in \mathbb{Z}$ since $\mathbf{m}\cdot\boldsymbol{\ell} \in \mathbb{Z}$ for $\mathbf{m}, \boldsymbol{\ell} \in L$ with L an integral lattice and since $\boldsymbol{\nu}\cdot\boldsymbol{\ell} \in \mathbb{Z}$ because $\boldsymbol{\nu}$ is in the dual lattice. Therefore,

$$e^{-\pi i \mathbf{k}\cdot\boldsymbol{\ell}} = e^{+\pi i \mathbf{k}\cdot\boldsymbol{\ell}} = e^{\pi i \left(\mathbf{k}+\frac{\boldsymbol{\ell}}{2}\right)\cdot\boldsymbol{\ell} - \frac{\pi i}{2}\boldsymbol{\ell}^2}. \tag{0.1}$$

Consequently, we find

$$\Psi_{\boldsymbol{\mu}}\left(-\frac{1}{\tau}\right) = e^{-\frac{\pi i}{2}\boldsymbol{\ell}^2} \frac{(-i\tau)^{n/2}}{\sqrt{\det L}} \sum_{\boldsymbol{\nu}\in L^\sharp/L} e^{-2\pi i \boldsymbol{\mu}\cdot\boldsymbol{\nu}} \sum_{\mathbf{k}\in L+\boldsymbol{\nu}+\frac{\boldsymbol{\ell}}{2}} e^{\pi i \tau \mathbf{k}^2 + \pi i \mathbf{k}\cdot\boldsymbol{\ell}}$$

and hence

$$\boxed{\Psi_{\boldsymbol{\mu}}\left(-\frac{1}{\tau}\right) = e^{-\frac{\pi i}{2}\boldsymbol{\ell}^2} \frac{(-i\tau)^{n/2}}{\sqrt{\det L}} \sum_{\boldsymbol{\nu}\in L^\sharp/L} e^{-2\pi i \boldsymbol{\mu}\cdot\boldsymbol{\nu}} \Psi_{\boldsymbol{\nu}}(\tau).}$$

4. Let us consider the one dimensional integral lattice $L \simeq \mathbb{Z}$ with the symmetric bilinear form $B(n,m) = 3nm$ for $n, m \in \mathbb{Z}$. We have $\det L = 3$ with representatives of the discriminant group $L^\sharp/L$ given by $\mu = 0, +\frac{1}{3}, -\frac{1}{3}$. Also noting that $\ell = 1$ is a characteristic vector for this lattice (since $3n^2 + 3n \in 2\mathbb{Z}$), the corresponding theta functions $\Psi_\mu(\tau)$ (of the form we considered in Problem 3) are

$$\begin{aligned}
\Psi_0(\tau) &= \sum_{n\in\mathbb{Z}} e^{3\pi i \tau (n+\frac{1}{2})^2 + 3\pi i (n+\frac{1}{2})} = -i \sum_{n\in\mathbb{Z}} (-1)^n\, e^{3\pi i \tau (n+\frac{1}{2})^2} \\
\Psi_{1/3}(\tau) &= \sum_{n\in\mathbb{Z}} e^{3\pi i \tau (n+\frac{5}{6})^2 + 3\pi i (n+\frac{5}{6})} = i \sum_{n\in\mathbb{Z}} (-1)^n e^{3\pi i \tau (n+\frac{5}{6})^2} \\
\Psi_{-1/3}(\tau) &= \sum_{n\in\mathbb{Z}} e^{3\pi i \tau (n+\frac{1}{6})^2 + 3\pi i (n+\frac{1}{6})} = i \sum_{n\in\mathbb{Z}} (-1)^n e^{3\pi i \tau (n+\frac{1}{6})^2}.
\end{aligned}$$

Importantly, we note that

$$\Psi_{-1/3}(\tau) = i\,\eta(\tau).$$

Moreover, by letting $n \mapsto -n-1$ we find

$$\Psi_0(\tau) = 0 \quad \text{and} \quad \Psi_{1/3}(\tau) = -i\,\eta(\tau).$$

Therefore, the corresponding transformation in Problem 3 implies

$$\Psi_{-1/3}(-1/\tau) = e^{-\frac{3\pi i}{2}} \frac{(-i\tau)^{1/2}}{\sqrt{3}} \left(\Psi_0(\tau) + e^{-6\pi i \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)} \Psi_{-1/3}(\tau) + e^{-6\pi i \left(-\frac{1}{3}\right)\left(\frac{1}{3}\right)} \Psi_{1/3}(\tau)\right).$$

This yields

$$i\,\eta(-1/\tau) = i \frac{(-i\tau)^{1/2}}{\sqrt{3}} \left(i e^{-\frac{2\pi i}{3}} \eta(\tau) - i e^{\frac{2\pi i}{3}} \eta(\tau)\right).$$

Since $i e^{-\frac{2\pi i}{3}} - i e^{\frac{2\pi i}{3}} = \sqrt{3}$, we finally find

$$\boxed{\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)}$$

as claimed.