Due in class on June 10, 2024

Lattices and Quadratic Forms (Summer 2024) - Problem Set 6

1. Recall that for any $n \geq 4$, we can realize the D_n lattice as a sublattice of $\mathbb{Z}^n$ defined by

$$D_n := \{k \in \mathbb{Z}^n : k_1 + k_2 + \ldots + k_n \text{ even}\}$$

and the representatives for the nontrivial elements of $D_n^\#/D_n$ are given by the vectors

$$s = \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right), \quad c = \left(-\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right), \quad v = (1, 0, \ldots, 0).$$

   a) Express the theta functions $\Theta_{D_n}(\tau)$, $\Theta_{D_n,s}(\tau)$, $\Theta_{D_n,c}(\tau)$, $\Theta_{D_n,v}(\tau)$ (here $\Theta_{D_n,w}$ refers to the theta function for the coset $w + D_n$), in terms of the Jacobi theta functions $\theta_{00}(\tau)$, $\theta_{01}(\tau)$, $\theta_{10}(\tau)$ that we discussed in Problem Set 5 (with $\theta_{ab}(\tau) := \theta_{ab}(\tau, 0)$).
   b) Recall that the E_8 lattice can be realized as the lattice D_8^+ where $D_n^+ := D_n \cup (s + D_n)$. Use this to express $\Theta_{E_8}(\tau)$ in terms of Jacobi theta functions and use this expression to compute this theta function up to order q^{20}. How many norm 40 vectors are there in the E_8 lattice?
   c) Consider the $n = 4$ case and recall from Problem Set 2 that the cosets $s + D_4$ and $v + D_4$ can be mapped to each other with an automorphism of the D_4 lattice. What is the identity on Jacobi theta functions implied by this automorphism?
   d) Prove that the theta functions $\Theta_{E_8 \perp E_8}(\tau)$ and $\Theta_{D_{16}^+}(\tau)$ are equal to each other.

2. Show that the functions $\theta_{00}(\tau)^4$, $\theta_{01}(\tau)^4$, and $\theta_{10}(\tau)^4$ are modular forms on the congruence subgroup $\Gamma(2)$ of weight 2.

3. Let $k \geq 4$ be an even integer and let m be a positive integer. Define the Poincaré series

$$P_{k,m}(\tau) := \frac{1}{2} \sum_{\substack{(c,d) \in \mathbb{Z}^2 \setminus \{0\} \\ \gcd(c,d)=1}} \frac{e^{2\pi i m \frac{a\tau+b}{c\tau+d}}}{(c\tau + d)^k}$$

where a, b are any two integers such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Show that this description is well-defined. Prove that this series is absolutely and locally uniformly convergent on $\mathbb{H}$ by reducing it to the case for Eisenstein series. Finally show that $P_{k,m}(\tau)$ is a weight k cusp form.

4. Find all the subgroups of $\mathrm{SL}_2(\mathbb{Z})$ that contain $\Gamma(2)$.

Note: You may use computational tools to help with your solutions, but describe your steps. Each problem is worth 10 points.