Lattices and Quadratic Forms (Summer 2024) - Solutions to Problem Set 6

1. a) Recalling the realization of D_n as the lattice

$$D_n := \{k \in \mathbb{Z}^n : k_1 + k_2 + \ldots + k_n \text{ even}\}$$

we have

$$\Theta_{D_n}(\tau) = \sum_{\substack{k \in \mathbb{Z}^n \\ k_1+\ldots+k_n \in 2\mathbb{Z}}} e^{\pi i \tau (k_1^2 + \ldots + k_n^2)}.$$

We can implement the condition $k_1 + \ldots + k_n \in 2\mathbb{Z}$ by extending the sum to all elements of $\mathbb{Z}^n$ and then inserting a factor of $\frac{1}{2}\left(1 + (-1)^{k_1+\ldots+k_n}\right)$. This leads to

$$\Theta_{D_n}(\tau) = \frac{1}{2}\left(\sum_{k \in \mathbb{Z}} e^{\pi i \tau k^2}\right)^n + \frac{1}{2}\left(\sum_{k \in \mathbb{Z}} (-1)^k e^{\pi i \tau k^2}\right)^n,$$

which can be expressed in terms of Jacobi theta functions

$$\theta_{ab}(\tau) := \sum_{m \in \mathbb{Z} + \frac{a}{2}} e^{\pi i \tau m^2 + \pi i b m}$$

as

$$\boxed{\Theta_{D_n}(\tau) = \frac{1}{2}\left(\theta_{00}(\tau)^n + \theta_{01}(\tau)^n\right).}$$

For $v = (1, 0, \ldots, 0)$, the theta function for the corresponding coset $v + D_n$ is similarly given by

$$\Theta_{D_n,v}(\tau) = \sum_{\substack{k \in \mathbb{Z}^n \\ k_1+\ldots+k_n \in 2\mathbb{Z}}} e^{\pi i \tau ((k_1+1)^2 + k_2^2 + k_3^2 + \ldots + k_n^2)}.$$

Shifting $k \mapsto k - 1$, we can rewrite this as

$$\sum_{\substack{k \in \mathbb{Z}^n \\ k_1+\ldots+k_n \in 2\mathbb{Z}+1}} e^{\pi i \tau (k_1^2 + k_2^2 + k_3^2 + \ldots + k_n^2)} = \frac{1}{2} \sum_{k \in \mathbb{Z}^n} \left(1 - (-1)^{k_1+\ldots+k_n}\right) e^{\pi i \tau (k_1^2 + \ldots + k_n^2)}$$

and then express it in terms of Jacobi theta functions as

$$\boxed{\Theta_{D_n,v}(\tau) = \frac{1}{2}\left(\theta_{00}(\tau)^n - \theta_{01}(\tau)^n\right).}$$

Next we consider the coset for $s = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ and find

$$\begin{aligned}\Theta_{D_n,s}(\tau) = &\sum_{\substack{k \in \mathbb{Z}^n \\ k_1+\ldots+k_n \in 2\mathbb{Z}}} e^{\pi i \tau ((k_1+\frac{1}{2})^2 + (k_2+\frac{1}{2})^2 + \ldots + (k_n+\frac{1}{2})^2)} \\ &= \frac{1}{2} \sum_{k \in \mathbb{Z}^n} \left(1 + (-1)^{k_1+\ldots+k_n}\right) e^{\pi i \tau ((k_1+\frac{1}{2})^2 + (k_2+\frac{1}{2})^2 + \ldots + (k_n+\frac{1}{2})^2)}.\end{aligned}$$

The sum over each k_j can now be performed separately as above. The contribution of the second term is zero since

$$-i\theta_{11}(\tau) = \sum_{k \in \mathbb{Z}} (-1)^k e^{\pi i \tau (k+\frac{1}{2})^2} = 0$$

as can be seen by taking $k \mapsto -k - 1$. From the first term we then find

$$\boxed{\Theta_{D_n,s}(\tau) = \frac{1}{2}\theta_{10}(\tau)^n.}$$

Finally, for $c = \left(-\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$ we have

$$
\begin{aligned}
\Theta_{D_n,c}(\tau) &= \sum_{\substack{k\in\mathbb{Z}^n\\ k_1+\ldots+k_n\in 2\mathbb{Z}}} e^{\pi i\tau((k_1-\frac{1}{2})^2+(k_2+\frac{1}{2})^2+\ldots+(k_n+\frac{1}{2})^2)}\\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}^n}\left(1+(-1)^{k_1+\ldots+k_n}\right) e^{\pi i\tau((k_1-\frac{1}{2})^2+(k_2+\frac{1}{2})^2+\ldots+(k_n+\frac{1}{2})^2)}.
\end{aligned}
$$

Letting $k_1 \mapsto k_1+1$, this is equal to

$$
\frac{1}{2}\sum_{k\in\mathbb{Z}^n}\left(1-(-1)^{k_1+\ldots+k_n}\right) e^{\pi i\tau((k_1+\frac{1}{2})^2+(k_2+\frac{1}{2})^2+\ldots+(k_n+\frac{1}{2})^2)}
$$

and the second term again gives a vanishing contribution as discussed above to yield

$$
\boxed{\Theta_{D_n,c}(\tau) = \frac{1}{2}\theta_{10}(\tau)^n.}
$$

Remark. The equality of $\Theta_{D_n,s}(\tau)$ and $\Theta_{D_n,c}(\tau)$ is of course expected since there is an automorphism of D_n that maps s to c, namely $(k_1, k_2, \ldots, k_n) \mapsto (-k_1, k_2, \ldots, k_n)$.

b) Since $E_8 = D_8 \cup (D_8 + s)$ we have

$$
\Theta_{E_8}(\tau) = \Theta_{D_8}(\tau) + \Theta_{D_8,s}(\tau),
$$

which according to part (a) yields

$$
\boxed{\Theta_{E_8}(\tau) = \frac{1}{2}\left(\theta_{00}(\tau)^8 + \theta_{01}(\tau)^8 + \theta_{10}(\tau)^8\right).}
$$

Plugging the q-expansions of $\theta_{00}(\tau)$, $\theta_{01}(\tau)$, and $\theta_{10}(\tau)$ in up to order q^{20} we get

$$
\begin{aligned}
\Theta_{E_8}(\tau) = \frac{1}{2}&\left(1+2q^{\frac{1}{2}}+2q^2+2q^{\frac{9}{2}}+2q^8+2q^{\frac{25}{2}}+2q^{18}+\ldots\right)^8\\
&+\frac{1}{2}\left(1-2q^{\frac{1}{2}}+2q^2-2q^{\frac{9}{2}}+2q^8-2q^{\frac{25}{2}}+2q^{18}+\ldots\right)^8\\
&+\frac{1}{2}\left(2q^{\frac{1}{8}}+2q^{\frac{9}{8}}+2q^{\frac{25}{8}}+2q^{\frac{49}{8}}+2q^{\frac{81}{8}}+2q^{\frac{121}{8}}+\ldots\right)^8
\end{aligned}
$$

which can be summed to

$$
\begin{aligned}
\Theta_{E_8}(\tau) = 1 &+ 240q + 2160q^2 + 6720q^3 + 17520q^4 + 30240q^5 + 60480q^6 + 82560q^7\\
&+ 140400q^8 + 181680q^9 + 272160q^{10} + 319680q^{11} + 490560q^{12}\\
&+ 527520q^{13} + 743040q^{14} + 846720q^{15} + 1123440q^{16}\\
&+ 1179360q^{17} + 1635120q^{18} + 1646400q^{19} + 2207520q^{20} + O\left(q^{21}\right).
\end{aligned}
$$

In particular, E_8 has 2207520 vectors of norm 40.

c) Since there is an automorphism of D_4 that maps v to s, we have the identity

$$
\Theta_{D_4,v}(\tau) = \Theta_{D_4,s}(\tau).
$$

In terms of Jacobi theta functions this gives the “Riemann theta relation”

$$
\boxed{\theta_{00}(\tau)^4 = \theta_{01}(\tau)^4 + \theta_{10}(\tau)^4.}
$$

d) By part (b) we have $\Theta_{E_8\perp E_8}(\tau) = \Theta_{E_8}(\tau)^2$ equal to

$$\Theta_{E_8\perp E_8}(\tau) = \frac{1}{4}\left(\theta_{00}(\tau)^8 + \theta_{01}(\tau)^8 + \theta_{10}(\tau)^8\right)^2.$$

Also as in part (b) we can compute the theta function for $D_{16}^+ = D_{16}\cup(D_{16}+s)$ as $\Theta_{D_{16}}(\tau)+\Theta_{D_{16},s}(\tau)$ to get

$$\Theta_{D_{16}^+}(\tau) = \frac{1}{2}\left(\theta_{00}(\tau)^{16} + \theta_{01}(\tau)^{16} + \theta_{10}(\tau)^{16}\right).$$

Using the Riemann theta relation from part (c), we can simplify these expressions by eliminating $\theta_{10}(\tau)^4 = \theta_{00}(\tau)^4 - \theta_{01}(\tau)^4$. This leads to

$$\Theta_{E_8\perp E_8}(\tau) = \frac{1}{4}\left(\theta_{00}(\tau)^8 + \theta_{01}(\tau)^8 + \left(\theta_{00}(\tau)^4 - \theta_{01}(\tau)^4\right)^2\right)^2 = \left(\theta_{00}(\tau)^8 - \theta_{00}(\tau)^4\theta_{01}(\tau)^4 + \theta_{01}(\tau)^8\right)^2$$

and

$$\begin{aligned}\Theta_{D_{16}^+}(\tau) &= \frac{1}{2}\left(\theta_{00}(\tau)^{16} + \theta_{01}(\tau)^{16} + \left(\theta_{00}(\tau)^4 - \theta_{01}(\tau)^4\right)^4\right)\\ &= \theta_{00}(\tau)^{16} - 2\theta_{00}(\tau)^{12}\theta_{01}(\tau)^4 + 3\theta_{00}(\tau)^8\theta_{01}(\tau)^8 - 2\theta_{00}(\tau)^4\theta_{01}(\tau)^{12} + \theta_{01}(\tau)^{16},\end{aligned}$$

which are equal.

2. The functions θ_{00}^4, θ_{01}^4, θ_{10}^4 are holomorphic functions on the upper half-plane. For their modular transformations, we first recall from class that $\Gamma(2)$ is generated by $-I_2$, $T^2 = \left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)$, $U^2 = \left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)$. So we simply check next that θ_{00}^4, θ_{01}^4, θ_{10}^4 transform like weight 2 modular forms under T^2 and U^2 ($-I_2$ acts trivially). For this, we also recall from the previous problem set the behavior of θ_{00}, θ_{01}, θ_{10} under translation and inversion:

$$\begin{aligned}\theta_{00}(\tau+1) &= \theta_{01}(\tau), & \theta_{00}(-1/\tau) &= \sqrt{-i\tau}\,\theta_{00}(\tau),\\ \theta_{01}(\tau+1) &= \theta_{00}(\tau), & \theta_{01}(-1/\tau) &= \sqrt{-i\tau}\,\theta_{10}(\tau),\\ \theta_{10}(\tau+1) &= e^{\pi i/4}\theta_{10}(\tau), & \theta_{10}(-1/\tau) &= \sqrt{-i\tau}\,\theta_{01}(\tau).\end{aligned}$$

This then implies

$$\begin{aligned}\theta_{00}^4(\tau+1) &= \theta_{01}^4(\tau), & \theta_{00}^4(-1/\tau) &= -\tau^2\,\theta_{00}^4(\tau),\\ \theta_{01}^4(\tau+1) &= \theta_{00}^4(\tau), & \theta_{01}^4(-1/\tau) &= -\tau^2\,\theta_{10}^4(\tau),\\ \theta_{10}^4(\tau+1) &= -\theta_{10}^4(\tau), & \theta_{10}^4(-1/\tau) &= -\tau^2\,\theta_{01}^4(\tau).\end{aligned}$$

- Under T^2 we have

$$\theta_{00}^4(\tau+2) = \theta_{00}^4(\tau), \quad \theta_{01}^4(\tau+2) = \theta_{01}^4(\tau), \quad \text{and} \quad \theta_{10}^4(\tau+2) = \theta_{10}^4(\tau),$$

which does conform to the pattern $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 f(\tau)$ for $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) := T^2$.

- Under U^2 we have (for ab equal to 00, 01, or 10)

$$\begin{aligned}\theta_{ab}^4\left(\frac{\tau}{2\tau+1}\right) &= -\left(-\frac{2\tau+1}{\tau}\right)^2\theta_{ba}^4\left(-\frac{2\tau+1}{\tau}\right)\\ &= -\left(-\frac{2\tau+1}{\tau}\right)^2\theta_{ba}^4\left(-\frac{1}{\tau}\right) = \left(-\frac{2\tau+1}{\tau}\right)^2\tau^2\,\theta_{ab}^4(\tau),\end{aligned}$$

where we apply inversion, translation by two, and inversion again in succession to obtain these equalities. Consequently, we have

$$\boxed{\theta_{ab}^4\left(\frac{\tau}{2\tau+1}\right) = (2\tau+1)^2\,\theta_{ab}^4(\tau),}$$

which conforms to the pattern $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 f(\tau)$ for $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) := U^2$.

So in summary we have confirmed that

$$\boxed{\theta_{ab}^4\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2\,\theta_{ab}^4(\tau)} \quad \text{where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2).$$

Our final task is to confirm that $\theta_{ab}^4\big|_2\gamma$ is bounded as $\tau \to i\infty$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Now recall from class that $\Gamma(2)$ is a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z})/\Gamma(2)$ is of order 6 with representatives from each coset given by

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad STS = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix},$$
$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \qquad TS = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

So we simply need to check the boundedness condition for these six $\mathrm{SL}_2(\mathbb{Z})$ elements.[1]

- For $\gamma = I_2$, the boundedness as $\tau \to i\infty$ follows from the q-expansions

$$\begin{aligned}
\theta_{00}^4(\tau) &= 1 + 8q^{1/2} + 24q + 32q^{3/2} + 24q^2 + 48q^{5/2} + 96q^3 + \dots, \\
\theta_{01}^4(\tau) &= 1 - 8q^{1/2} + 24q - 32q^{3/2} + 24q^2 - 48q^{5/2} + 96q^3 + \dots, \\
\theta_{10}^4(\tau) &= 16q^{1/2} + 64q^{3/2} + 96q^{5/2} + 128q^{7/2} + 208q^{9/2} + \dots,
\end{aligned}$$

  which are all bounded as q tends to zero.
- For $\gamma = T$, this follows from the fact that

$$\theta_{ab}^4\big|_2 T = (-1)^a\,\theta_{ab'}^4 \text{ with } b' = 1 - a - b$$

  and that the right hand side is bounded as $\tau \to i\infty$ as we have already found out above with q-expansions.
- For $\gamma = S$, this follows from the fact that

$$\theta_{ab}^4\big|_2 S = -\theta_{ba}^4.$$

- Since the three functions θ_{00}^4, θ_{01}^4, θ_{10}^4 transform among themselves under S and T transformations, the results above also confirm boundedness of $\theta_{ab}^4\big|_2\gamma$ as $\tau \to i\infty$ for the remaining three $\mathrm{SL}_2(\mathbb{Z})$ elements above (which are generated by S and T).

3. First we make sure that $P_{k,m}$ is well-defined by showing that for each of the summands in

$$P_{k,m}(\tau) := \frac{1}{2}\sum_{\substack{(c,d)\in\mathbb{Z}^2\setminus\{0\} \\ \gcd(c,d)=1}} \frac{e^{2\pi i m \frac{a\tau+b}{c\tau+d}}}{(c\tau+d)^k}$$

there do exist integers $a, b \in \mathbb{Z}$ as described in the problem and the summand is independent of the choice: Given any $(c,d) \in \mathbb{Z}^2 \setminus \{0\}$ with $\gcd(c,d) = 1$, there do exist $a, b \in \mathbb{Z}$ with $ad - bc = 1$ and hence $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Furthermore, for any other such integers $a', b' \in \mathbb{Z}$ we have $a' = a + nc$ and $b' = b + nd$ for some $n \in \mathbb{Z}$ so that

$$e^{2\pi i m \frac{a'\tau+b'}{c\tau+d}} = e^{2\pi i m\left(n + \frac{a\tau+b}{c\tau+d}\right)} = e^{2\pi i m \frac{a\tau+b}{c\tau+d}}$$

since $m \in \mathbb{Z}$, which proves that the summand is independent of the choice of a, b. Furthermore, we have

$$\left|e^{2\pi i m \frac{a\tau+b}{c\tau+d}}\right| = e^{-2\pi m \,\mathrm{Im}\left(\frac{a\tau+b}{c\tau+d}\right)} \le 1,$$

[1] Any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ is of the form $\gamma = \gamma_2\gamma'$ with $\gamma_2 \in \Gamma(2)$ and γ' is one of the six $\mathrm{SL}_2(\mathbb{Z})$ elements above. Since $\theta_{ab}^4\big|_2\gamma_2 = \theta_{ab}^4$ as we found above, it is enough to check the boundedness condition for γ'.

where the final inequality follows from the assumption $m > 0$ and that $\operatorname{Im}\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{\operatorname{Im}(\tau)}{|c\tau+d|^2} > 0$. So we have

$$\left|\frac{e^{2\pi i m \frac{a\tau+b}{c\tau+d}}}{(c\tau+d)^k}\right| \leq \frac{1}{|c\tau+d|^k}$$

and the arguments we have given for Eisenstein series immediately apply to show that for $k > 2$ the series here is absolutely and uniformly convergent over the sets

$$S_{A,B} := \{\tau \in \mathbb{H} : |\tau_1| \leq A \text{ and } \tau_2 \geq B\}$$

and hence locally uniformly convergent over $\mathbb{H}$. Consequently, $P_{k,m}(\tau)$ is holomorphic on $\mathbb{H}$ (the summands are holomorphic).

Next we check that this holomorphic function transforms like a weight k modular form by computing its translation and inversion properties. For this we will be more careful about our summands and write $a_{c,d}$ and $b_{c,d}$ instead of a and b to emphasize their dependence on c and d:

$$P_{k,m}(\tau) = \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2\setminus\{0\} \\ \gcd(c,d)=1}} \frac{e^{2\pi i m \frac{a_{c,d}\tau+b_{c,d}}{c\tau+d}}}{(c\tau+d)^k}$$

- For translation we have

  $$P_{k,m}(\tau+1) = \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2\setminus\{0\} \\ \gcd(c,d)=1}} \frac{e^{2\pi i m \frac{a_{c,d}\tau+a_{c,d}+b_{c,d}}{c\tau+c+d}}}{(c\tau+c+d)^k}$$

  The change of variables $(c,d) \mapsto (c',d') := (c, c+d)$ preserves the set of (c,d) we are summing over and furthermore

  $$\begin{pmatrix} a_{c,d} & a_{c,d}+b_{c,d} \\ c & c+d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

  so that

  $$(a_{c,d}, a_{c,d}+b_{c,d}) = (a_{c',d'}, b_{c',d'}) + n(c',d')$$

  for some $n \in \mathbb{Z}$. Accordingly, we have

  $$\frac{e^{2\pi i m \frac{a_{c,d}\tau+a_{c,d}+b_{c,d}}{c\tau+c+d}}}{(c\tau+c+d)^k} = \frac{e^{2\pi i m \frac{a_{c',d'}\tau+b_{c',d'}}{c'\tau+d'}}}{(c'\tau+d')^k}$$

  and hence

  $$\boxed{P_{k,m}(\tau+1) = P_{k,m}(\tau).}$$

- For inversion we have

  $$P_{k,m}(-1/\tau) = \tau^k \, \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2\setminus\{0\} \\ \gcd(c,d)=1}} \frac{e^{2\pi i m \frac{b_{c,d}\tau-a_{c,d}}{d\tau-c}}}{(d\tau-c)^k}$$

  As above we can change variables $(c,d) \mapsto (c',d') := (d,-c)$, which preserves the set of (c,d) we are summing over and furthermore

  $$\begin{pmatrix} b_{c,d} & -a_{c,d} \\ d & -c \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

  so that

  $$(b_{c,d}, -a_{c,d}) = (a_{c',d'}, b_{c',d'}) + n(c',d')$$

for some $n \in \mathbb{Z}$. Consequently, we have

$$\frac{e^{2\pi i m \frac{b_{c,d}\tau - a_{c,d}}{d\tau - c}}}{(d\tau - c)^k} = \frac{e^{2\pi i m \frac{a_{c',d'}\tau + b_{c',d'}}{c'\tau + d'}}}{(c'\tau + d')^k}$$

and hence

$$\boxed{P_{k,m}(\tau+1) = \tau^k \, P_{k,m}(\tau).}$$

Now we checked the modular transformations, we finally check that $P_{k,m}(\tau)$ tends to zero as $\tau \to i\infty$ to ensure we have a cusp form. Let us assume $\tau_2 \geq 1$. Since $P_{k,m}(\tau+1) = P_{k,m}(\tau)$, let us also assume $|\tau_1| \leq \frac{1}{2}$. Separating the contributions of the $c=0$ terms and noting that the summands are invariant under $(c,d) \mapsto -(c,d)$ we have

$$P_{k,m}(\tau) = e^{2\pi i m\tau} + \sum_{\substack{c \in \mathbb{Z}^+,\, d \in \mathbb{Z} \\ \gcd(c,d)=1}} \frac{e^{2\pi i m \frac{a\tau+b}{c\tau+d}}}{(c\tau+d)^k}$$

The first term here tends to zero as $\tau \to i\infty$ since $|e^{2\pi i m\tau}| = e^{-2\pi m \tau_2}$ and $m > 0$. Moreover, each of the summands in the second sum tend to zero as $\tau \to i\infty$ because of the denominator. Since the convergence is uniform over $S_{\frac{1}{2},1}$, the sum then also tends to zero.

4. As discussed in class, $\Gamma(2)$ is the kernel of the onto homomorphism $\varphi : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$ that reduces entries modulo 2. In particular, the quotient group $\mathrm{SL}_2(\mathbb{Z})/\Gamma(2)$ is isomorphic to

$$\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\},$$

where the elements $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ have order two and $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ have order three.

By the correspondence theorem, the subgroups of $\mathrm{SL}_2(\mathbb{Z})$ that contain $\Gamma(2)$ correspond to the subgroups of $\mathrm{SL}_2(\mathbb{Z})/\Gamma(2)$. In particular, these subgroups are given by the inverse images of subgroups of $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$ under the homomorphism described above. Aside from the trivial subgroup G_1 and the full subgroup $G_6 := \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$, the other subgroups (which have order 2 or 3) are the cyclic subgroups:

- Three order 2 subgroups generated by each one of the three order two elements:

$$G_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}, \quad G_3 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}, \quad G_4 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}.$$

- The order 3 subgroup

$$G_5 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$

So there are six subgroups of $\mathrm{SL}_2(\mathbb{Z})$ that contain $\Gamma(2)$:

$$\varphi^{-1}(G_j) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \gamma \pmod 2 \text{ for some } \gamma \in G_j \right\} \quad \text{for } j \in \{1, 2, \ldots, 6\}.$$

In more detail, we can describe these subgroups as follows:

- $\mathrm{SL}_2(\mathbb{Z})$ itself corresponding to $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$.
- $\Gamma(2)$ corresponding to the trivial subgroup of $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$.
- Since $\varphi(S) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, the subgroup $\varphi^{-1}(G_2)$ is $\Gamma(2) \cup S\,\Gamma(2)$ and this is the subgroup $\Gamma_\vartheta := \langle S, T^2 \rangle$ discussed in class.

- For G_3, the subgroup $\varphi^{-1}(G_3)$ is

$$\Gamma_1(2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod 2 \right\}$$

  Note that the conditions $a \equiv 1 \pmod 2$ and $d \equiv 1 \pmod 2$ do in fact follow as a consequence of $c \equiv 0 \pmod 2$. So this subgroup is also equal to

$$\Gamma_0(2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod 2 \right\}.$$

  One can define $\Gamma_0(N)$ and $\Gamma_1(N)$ for other values of N as well, but when $N > 2$, they are not equal.
- Similarly, $\varphi^{-1}(G_4)$ is the subgroup

$$\Gamma^0(2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : b \equiv 0 \pmod 2 \right\}.$$

- Finally, the subgroup $\varphi^{-1}(G_5)$ is the index two subgroup of $\mathrm{SL}_2(\mathbb{Z})$ given by

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \text{ or } \pmod 2 \right\}.$$