Due in class on June 20, 2024

Lattices and Quadratic Forms (Summer 2024) - Problem Set 7

1. In class we have found that extending the definition of Eisenstein series E_k to the case $k=2$ as
$$E_2(\tau) := 1 + \frac{1}{\zeta(2)} \sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^2} = 1 - 24 \sum_{n=1}^{\infty} \frac{n\,q^n}{1-q^n}$$
gives a "quasimodular" object transforming as
$$E_2(\tau+1) = E_2(\tau) \quad \text{and} \quad E_2\left(-\frac{1}{\tau}\right) = \tau^2\,E_2(\tau) - \frac{6i\tau}{\pi}.$$
Show that the non-holomorphic function $\widehat{E}_2(\tau) := E_2(\tau) - \frac{3}{\pi\tau_2}$ transforms like a modular form of weight two.
Remark: There is in fact an alternative approach (called Hecke's trick) to the modular properties of E_2, where one starts with the absolutely convergent series
$$\widehat{E}_2(\tau,s) := \sum_{\substack{(m,n)\in\mathbb{Z}^2\setminus\{\mathbf{0}\}\\ \gcd(m,n)=1}} \frac{\tau_2^s}{(m\tau+n)^2\,|m\tau+n|^{2s}} \quad \text{for } s>0$$
which transforms like a weight two modular form and then shows $\lim_{s\to 0^+} \widehat{E}_2(\tau,s)$ is the function $\widehat{E}_2(\tau)$ we defined above.
2. Prove that any 16 dimensional lattice L that is positive definite, even, and self-dual is isometric to either $E_8 \perp E_8$ or to D_{16}^+.
3. Recall that the theta series for a positive definite, odd, and self-dual lattice L satisfies the modular transformations
$$\Theta_L(\tau+2) = \Theta_L(\tau) \quad \text{and} \quad \Theta_L\left(-\frac{1}{\tau}\right) = (-i\tau)^{n/2}\,\Theta_L(\tau)$$
or more generally under $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\vartheta := \langle S, T^2\rangle$
$$\Theta_L\left(\frac{a\tau+b}{c\tau+d}\right) = \epsilon_{c,d}^n (c\tau+d)^{\frac{n}{2}}\,\Theta_L(\tau)$$
with $\epsilon_{c,d}$ the corresponding multiplier system. Also recall that we checked that $\Theta_L|_k\gamma(\tau)$ is bounded as $\tau \to i\infty$ for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Here an important detail was that
$$(-i\tau)^{-n/2}\Theta_L\left(1-\frac{1}{\tau}\right) = \Theta_{S(L)}(\tau)$$
where the shadow of L is the coset $S(L) := L^\sharp_{\text{even}} \setminus L$. These properties then show that $\Theta_L \in M_{\frac{n}{2}}(\Gamma_\vartheta, \epsilon)$, where $M_k(\Gamma_\vartheta,\epsilon)$ is the $\mathbb{C}$ vector space of modular forms of weight $k \in \frac{\mathbb{Z}}{2}$ and multiplier system ϵ under Γ_ϑ.
   - With methods similar to those we covered in class for $\mathrm{SL}_2(\mathbb{Z})$, one can show that the algebra of such modular forms is freely generated by θ_{00} and E_4.

   Use this fact along with the fact that TS transforms Θ_L to $\Theta_{S(L)}$ as described above, to prove that a positive definite, odd, self-dual lattice with no norm 1 or norm 2 vectors should have at least 23 dimensions.
   Remark: There is in fact such a lattice in 23 dimensions called the *odd Leech lattice*.

Note: You may use computational tools to help with your solutions, but describe your steps. Problem 1 is worth 10 points and problems 2 and 3 are worth 15 points.