Lattices and Quadratic Forms (Summer 2024) - Solutions to Problem Set 7

1. To confirm that $\widehat{E}_2(\tau)$ transforms like a modular form of weight two, we check its behavior under translation and inversion:
   - Since τ_2 is invariant under $\tau \mapsto \tau+1$, we have
   $$\widehat{E}_2(\tau+1) = E_2(\tau+1) - \frac{3}{\pi\tau_2} = E_2(\tau) - \frac{3}{\pi\tau_2} = \widehat{E}_2(\tau).$$
   - Under inversion $\tau \mapsto -1/\tau$ we have $\tau_2 \mapsto \mathrm{Im}(-1/\tau) = \frac{\tau_2}{|\tau|^2}$ and hence
   $$\widehat{E}_2(-1/\tau) = E_2(-1/\tau) - \frac{3|\tau|^2}{\pi\tau_2} = \tau^2 E_2(\tau) - \frac{6i\tau}{\pi} - \frac{3|\tau|^2}{\pi\tau_2}.$$
   Note that
   $$\frac{6i\tau}{\pi} + \frac{3|\tau|^2}{\pi\tau_2} = \frac{3\tau^2}{\pi\tau_2}\left(\frac{2i\tau_2}{\tau} + \frac{\overline{\tau}}{\tau}\right) \quad \text{and} \quad \frac{2i\tau_2}{\tau} + \frac{\overline{\tau}}{\tau} = 1,$$
   which confirms
   $$\widehat{E}_2(-1/\tau) = \tau^2 \widehat{E}_2(\tau).$$

2. Let L be a 16 dimensional, positive definite, even, self-dual lattice as posited in the problem. Then its theta function $\Theta_L(\tau)$ is a modular form of weight 8. Since the vector space of modular forms with weight 8 is spanned by $E_4(\tau)^2$ and since the constant term of $\Theta_L(\tau)$ in its q-expansion is 1 just like $E_4(\tau)^2$ (which is the contribution from the origin), we find
   $$\Theta_L(\tau) = E_4(\tau)^2.$$
   In particular, we have
   $$\Theta_L(\tau) = 1 + 480q + \ldots$$
   and L necessarily has 480 roots. Now we consider the sublattice $R \leq L$ generated by these roots, which is a root lattice. According to the classification of root lattices we have[1]
   $$R = X_1 \perp \ldots \perp X_t,$$
   where X_j's are irreducible root lattices, which are either $A_{n\geq 1}$, $D_{n\geq 4}$, E_6, E_7, or E_8. Letting r_j denote the rank of X_j and h_j the Coxeter number of X_j, we have
   $$r_1 + \ldots + r_t \leq 16 \quad \text{and} \quad h_1 r_1 + \ldots + h_t r_t = 480.$$
   In particular, we have
   $$h_1 r_1 + \ldots + h_t r_t \geq 30(r_1 + \ldots + r_t).$$
   The Coxeter numbers of ADE lattices with rank ≤ 16 are (recalling $h_{A_n} = n+1$ and $h_{D_n} = 2n-2$)
   $$\begin{aligned} &h_{A_1} = 2,\ h_{A_2} = 3,\ \ldots\ ,\ h_{A_{15}} = 16,\ h_{A_{16}} = 17,\\ &h_{D_4} = 6,\ h_{D_5} = 8,\ \ldots\ ,\ h_{D_{15}} = 28,\ h_{D_{16}} = 30,\\ &h_{E_6} = 12,\ h_{E_7} = 18,\ h_{E_8} = 30. \end{aligned}$$
   Note that all of these Coxeter numbers are ≤ 30, so none of the ADE lattices with Coxeter number strictly smaller than 30 can appear in the decomposition. That only leaves the possibilities E_8 and D_{16}. The condition $h_1 r_1 + \ldots + h_t r_t = 480$ can then only be satisfied with
   $$R = E_8 \perp E_8 \quad \text{or} \quad R = D_{16}.$$

[1] This is of course up to isometry, but we will forego the distinction by identifying the two.

In the former case, we already have an even, self-dual lattice and we obtain our first possibility

$$\boxed{L = E_8 \perp E_8.}$$

So we consider the second possibility $R = D_{16}$, which is not self-dual. We are seeking the even, self-dual lattice L within the inclusions

$$D_{16} \leq L = L^\sharp \leq D_{16}^\sharp.$$

Since $D_{16}^\sharp/D_{16} \simeq (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ (with nontrivial elements corresponding to the cosets of s, c, v) and since $|L/D_{16}| = |D_{16}^\sharp/L^\sharp|$ we need to have $|L/D_{16}| = 2$ and L (if it exists) should be obtained by adding one coset to D_{16}. We have $D_{16} \cup (D_{16} + v) = \mathbb{Z}^{16}$, which is self-dual but odd. The remaining two options $D_{16}^+ = D_{16} \cup (D_{16} + s)$ and $D_{16} \cup (D_{16} + c)$ are isometric and do in fact yield even, self-dual lattices. So up to isometry our only other option for L is

$$\boxed{L = D_{16}^+.}$$

3. If L is an n-dimensional positive-definite lattice that is self-dual (which is necessarily integral), its theta function Θ_L is an element of $M_{\frac{n}{2}}(\Gamma_\vartheta, \epsilon)$ as stated in the problem.[2] Also as stated in the problem, the algebra of such modular forms is generated by $\theta_{00} \in M_{\frac{1}{2}}(\Gamma_\vartheta, \epsilon)$ and $E_4 \in M_4(\Gamma_\vartheta, \epsilon) \leq M_4(\mathrm{SL}_2(\mathbb{Z}))$. Note that we have the q-expansions

$$\theta_{00}(\tau) = 1 + 2q^{1/2} + 2q^2 + 2q^{9/2} + 2q^8 + \dots$$

and

$$E_4(\tau) = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + 30240q^5 + 60480q^6 + 82560q^7 + 140400q^8 + \dots.$$

In expanding elements of $M_{\frac{n}{2}}(\Gamma_\vartheta, \epsilon)$, we can replace $E_4(\tau)$ with

$$\Delta_+(\tau) := \frac{1}{16}\big(\theta_{00}(\tau)^8 - E_4(\tau)\big) = q^{1/2} - 8q + 28q^{3/2} - 64q^2 + 126q^{5/2} - 224q^3 + 344q^{7/2} - 512q^4 + \dots$$

- If $1 \leq n \leq 7$, then the requirement that the q-expansion of Θ_L starts with the constant term 1 implies that
$$\Theta_L(\tau) = \theta_{00}(\tau)^n = 1 + 2n\, q^{1/2} + 2n(n-1)q + \dots,$$
which necessarily has norm 1 vectors ($2n$ of them).
- If $8 \leq n \leq 15$, then we have
$$\Theta_L(\tau) = \theta_{00}(\tau)^n + c\, \theta_{00}(\tau)^{n-8}\, \Delta_+(\tau) = 1 + (c + 2n)q^{1/2} + [2c(n-12) + 2n(n-1)]\, q + \dots$$
for some constant c. The absence of norm 1 vectors require $c = -2n$ and if we plug this in we find
$$\Theta_L(\tau) = 1 + 2n(23 - n)q + \dots$$
So such an L would necessarily have norm 2 vectors.
- If $16 \leq n \leq 23$, then we have
$$\begin{aligned}\Theta_L(\tau) &= \theta_{00}(\tau)^n + c_1\, \theta_{00}(\tau)^{n-8}\, \Delta_+(\tau) + c_2\, \theta_{00}(\tau)^{n-16}\, \Delta_+(\tau)^2 \\ &= 1 + (c_1 + 2n)q^{1/2} + [c_2 + 2c_1(n-12) + 2n(n-1)]\, q + \dots\end{aligned}$$
The absence of norm 1 and norm 2 vectors requires setting $c_1 = -2n$ and $c_2 = -2n(23 - n)$.

[2] This is including the case where L is even, for which $n \equiv 0 \pmod 8$.

So we focus on the $16 \leq n \leq 23$ case and check the corresponding theta function of the shadow

$$\Theta_{S(L)}(\tau) = (-i\tau)^{-n/2}\Theta_L(1 - 1/\tau)$$

for these values, which needs to have nonnegative and integer Fourier coefficients.[3] Since

$$(-i\tau)^{-4}\, E_4\left(1 - \frac{1}{\tau}\right) = E_4(\tau) \quad \text{ and } \quad (-i\tau)^{-1/2}\, \theta_{00}\left(1 - \frac{1}{\tau}\right) = \theta_{10}(\tau)$$

We find that

$$\Theta_{S(L)}(\tau) = \theta_{10}(\tau)^n + c_1\, \theta_{10}(\tau)^{n-8}\, \widetilde{\Delta}_+(\tau) + c_2\, \theta_{10}(\tau)^{n-16}\, \widetilde{\Delta}_+(\tau)^2,$$

where

$$\widetilde{\Delta}_+(\tau) := \frac{1}{16}\big(\theta_{10}(\tau)^8 - E_4(\tau)\big)\,.$$

Plugging in the Fourier expansions for $c_1 = -2n$ and $c_2 = -2n(23 - n)$ we find

$$\Theta_{S(L)}(\tau) = \frac{q^{\frac{r-16}{8}}}{2^{23-n}}\,\big(-n(23-n) + n(n^2 - 71n + 5200)q + \ldots\big)\,.$$

For $n < 23$, there are negative Fourier coefficients due to the leading term.[4] This proves that there are no positive definite, odd, self-dual lattices with no norm 1 and no norm 2 vectors for dimensions $n < 23$.

Remark. Our arguments do not exclude the case $n = 23$ and in fact there is such a lattice in 23 dimensions as stated in the problem. Plugging in the values $c_1 = -2n$ and $c_2 = -2n(23 - n)$ for $n = 23$, its theta function is fixed to be

$$\Theta_L(\tau) = 1 + 4600q^{3/2} + 93150q^2 + 953856q^{5/2} + 6476800q^3 + \ldots$$

with the theta function for the shadow given by

$$\Theta_{S(L)}(\tau) = 94208q^{15/8} + 8294400q^{23/8} + 190771200q^{31/8} + \ldots.$$

[3]If L is even, this relation still holds with $S(L)$ defined to be L itself.

[4]Moreover for $n < 22$, this leading term is not an integer.