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A sign-changing global minimizer on a convex domain

Introduction: Recently one has established the existence of stable sign-changing solutions for the elliptic problem

$$(1) \qquad \begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In [5] there is an example of a sign changing stable solution on a convex domain with $f(0) \neq 0$. Matano [2] shows the existence of a sign-changing stable solution even with $f(0) = 0$. A next question will be: does a global minimizer have a fixed sign? It has been guessed that the answer is positive if the domain is convex.

In this note we will recall a proof for the ball and give a counterexample for a triangle.

We will assume that $f \in C^{0,1}$, Ω is bounded with $\partial\Omega \in C^{0,1}$ and that (1) has a solution u that minimizes the energy functional J. This functional $J : H_0^1(\Omega) \to \mathbb{R}$ is defined by

$$(2) \qquad J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega F(v)\, dx,$$

where $F(s) = \int_0^s f(t)\, dt$. It is classical that if $J(u) = \min \{ J(v);\ v \in H_0^1(\Omega) \}$, then u is a $C^2(\Omega) \cap C(\bar{\Omega})$ – solution,

$$(3) \qquad J'(u)v := \int_\Omega \nabla u . \nabla v\, dx - \int_\Omega f(u)\, v\, dx = 0 \quad \text{for all } v \in H_0^1(\Omega)$$

and

$$(4) \qquad J''(u)(v) := \int_\Omega |\nabla v|^2\, dx - \int_\Omega f'(u) v^2\, dx \geq 0 \quad \text{for all } v \in H_0^1(\Omega).$$

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Proposition 1: *If* f *is antisymmetric or* Ω *is a ball in* $\mathbb{R}^n$, *then the global minimizer* u *has a fixed sign.*

Remark 1: The result that a (local) minimizer cannot change sign on a ball is due to Lin and Ni, [1]. In their unpublished preprint they also prove the result for Ω being the difference of two balls with the same center. We will sketch their proof.

Proof: i) If f is antisymmetric, $f(s) = -f(-s)$, then $J(|u|) = J(u)$. Hence $|u|$ is a minimizing solution and $|u| \in C^2(\Omega)$. It follows from $x \in \Omega$ and $u(x) = 0$, that $\nabla u(x) = 0$. Then the strong maximum principle shows that $u \equiv 0$. Hence u has a fixed sign.

ii) Suppose Ω is a ball with center 0. Then differentiate the solution u in a tangential direction, that is, apply $\frac{d}{d\theta} = x_i \frac{d}{dx_j} - x_j \frac{d}{dx_i}$. Since $\frac{d}{d\theta}$ and Δ commute, the function $\varphi = \frac{d}{d\theta} u$ satisfies $-\Delta\varphi = f'(u)\varphi$ in Ω. Moreover $\varphi = 0$ on $\partial\Omega$. Then either $\varphi = 0$ or φ is an eigenfunction (with eigenvalue 0) of

$$(5) \qquad \begin{cases} -\Delta v - f'(u)v = \lambda v & \text{in } \Omega\,, \\ v = 0 & \text{on } \partial\Omega\,. \end{cases}$$

From (4) one finds that all eigenvalues, except maybe the first, are strictly positive. Hence φ is a multiple of the first eigenfunction. If φ is nonzero this shows φ has a fixed sign, which contradicts $\int_{\theta=0}^{2\pi} \varphi\, d\theta = 0$.

Since this holds for all i and j, u is radially symmetric. Now suppose $u = u(r)$ changes sign; then there is a positive number r_0 such that $u_r(r_0) = 0$. Set

$$v(r) = \begin{cases} u_r(r) & \text{for } r < r_0\,, \\ 0 & \text{for } r \geq r_0\,. \end{cases}$$

Then $v \in H_0^1(\Omega)$ and

$$0 \leq J''(u)(v) = \int_{|x|<r_0} \left(|\nabla u_r|^2 - f'(u)\, u_r^2 \right) dx =$$

$$= \int_{|x|<r_0} u_r(-\Delta u_r - f'(u)u_r)\, dx = -(n-1) \int_{|x|<r_0} r^{-2} u_r^2 dx\,,$$

which gives a contradiction for nonconstant u. □

Proposition 2: *There is* $f \in C^{0,1}(\mathbb{R})$, *with* $f(0) = 0$, *and* $\Omega \subset \mathbb{R}^2$, *bounded and convex, such that the global minimizer changes sign.*

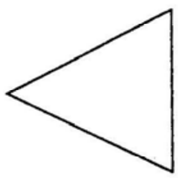
Remark 2: In this note we will construct just one example. A forthcoming paper of Matano will certainly have a more rigorous approach to sign–changing stable solutions. However, it is not clear if this considers global minimizers.

Remark 3: Without the condition $f(0) = 0$ one can modify the example in [5] to obtain the result of proposition 2.

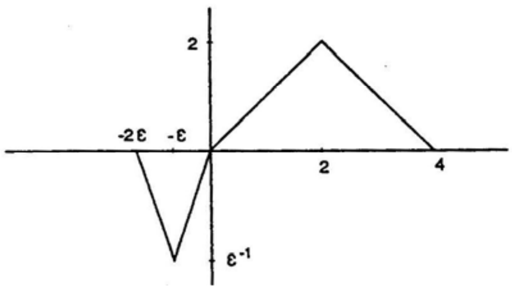
Proof: Set $\Omega = \{(x_1,x_2) \in \mathbb{R}^2;\ 2|x_2| < x_1 < 1\}$ and define the Lipschitz–continuous functions f_ε for $\varepsilon > 0$ by

$f_\varepsilon(s) = 0$	on $(-\infty,-2\varepsilon]$,
$f_\varepsilon(s) = -\varepsilon^{-2}(s+2\varepsilon)$	on $(-2\varepsilon,-\varepsilon]$,
$f_\varepsilon(s) = \varepsilon^{-2}s$	on $(-\varepsilon,0]$,
$f_\varepsilon(s) = s$	on $(0,2]$,
$f_\varepsilon(s) = 4 - s$	on $(2,4]$,
$f_\varepsilon(s) = 0$	on $(4,\infty)$.

Ω:



f_ε:



Note that $f(s) = -2\varepsilon\, f(-\frac{1}{2}\varepsilon s)$ for $s > 0$.

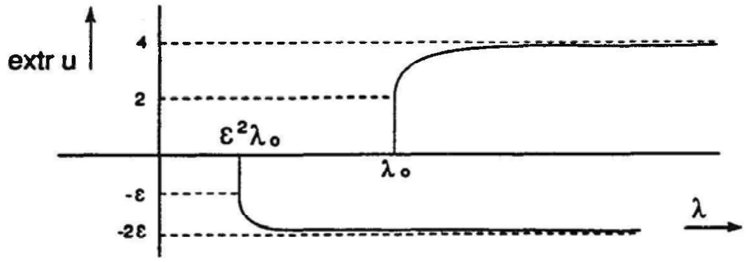
Let λ_0 denote the first eigenvalue of

(6) $$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

then the bifurcation picture for solutions with fixed sign of

(7) $$\begin{cases} -\Delta u = \lambda f_\varepsilon(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

looks as follows.



Since $s^{-1}f_\varepsilon(s)$ is decreasing on [0,4] (and strictly on [2,4]) it is known that there is a unique positive solution for every $\lambda > \lambda_0$. See [4].
There is no positive solution for $\lambda < \lambda_0$. Similar arguments hold for negative solutions. Let U_λ and V_λ^ε denote the positive, respectively the negative solution of (7) for $\lambda > \lambda_0$.
Let $J_\varepsilon(\lambda,u)$ denote the energy functional for (7), that is

$$J_\varepsilon(\lambda,u) = \int_\Omega \Big(\tfrac{1}{2}|\nabla u|^2 - \lambda \int_0^u f_\varepsilon(s)\, ds \Big)\, dx.$$

Lemma 3: $\lim\limits_{\lambda\to\infty} \lambda^{-1} J_\varepsilon(\lambda,U_\lambda) = -4|\Omega|$ *and* $\lim\limits_{\lambda\to\infty} \lambda^{-1} J_\varepsilon(\lambda,V_\lambda^\varepsilon) = -|\Omega|$, *uniformly for* $\varepsilon \in [0,1]$, *where* $|\Omega|$ *is the Lebesgue measure of* Ω.

Proof: We will show the second statement. Since V_λ^ε is the only stable solution of

$$\begin{cases} -\Delta u = \lambda \min\big(f_\varepsilon(u),0\big) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

the function minimizes

$$J_\varepsilon^-(\lambda,u) = \int_\Omega \Big(\tfrac{1}{2}|\nabla u|^2 - \lambda \int_0^u \min(f_\varepsilon(s),0)\, ds \Big)\, dx \quad \text{for } \lambda > \lambda_0.$$

Since we can estimate $J^-_\varepsilon(\lambda,u)$ from below by $-\lambda|\Omega|$:

$$J^-_\varepsilon(\lambda,u) \geq -\lambda \int_\Omega \int_0^u \min(f_\varepsilon(s),0)\, ds\, dx \geq -\lambda \int_\Omega 1\, dx\, ;$$

it is sufficient to show that for all $\sigma > 0$ there is $\varphi_\varepsilon \in H^1_0(\Omega)$ such that uniformly for $\varepsilon \in [0,1]$

$$\lim_{\lambda\to\infty} \lambda^{-1} J^-_\varepsilon(\lambda,\varphi_\varepsilon) < -|\Omega| + \sigma.$$

Take $\varphi \in C^\infty_0(\Omega)$ with $\varphi = -2$ in a closed subset of Ω with measure larger than $|\Omega| - \frac{1}{2}\sigma$. The result follows for λ large since

$$\lambda^{-1}J^-_\varepsilon(\lambda,\varepsilon\varphi) < \lambda^{-1} \varepsilon^2 \int_\Omega \tfrac{1}{2}|\nabla\varphi|^2 dx - |\Omega| + \tfrac{1}{2}\sigma \leq$$
$$\leq \lambda^{-1} \int_\Omega \tfrac{1}{2}|\nabla\varphi|^2 dx - |\Omega| + \tfrac{1}{2}\sigma\, . \qquad \square$$

Because of Lemma 3 there is $\lambda_1 > \lambda_0$ such that

$$J_\varepsilon(\lambda,U_\lambda) < J_\varepsilon(\lambda,V^\varepsilon_\lambda) < -\tfrac{1}{2}|\Omega| \qquad \text{for all } \lambda \geq \lambda_1 \text{ and } \varepsilon \in [0,1].$$

Lemma 4: $U_{\lambda_1}(x_1,x_2) < \frac{1}{3}\lambda_1(x_1^2 - 4x_2^2)$.

Proof: For t large enough

(8) $$U_{\lambda_1}(x_1,x_2) < \tfrac{1}{3}\lambda_1((x_1+t)^2 - 4x_2^2) \quad \text{in } \Omega.$$

Since $-\Delta\lambda_1((x_1+t)^2 - 4x_2^2) = 6\lambda_1 > \lambda_1 \max f_\varepsilon$ and since $\lambda_1((x_1+t)^2 - 4x_2^2) > 0$ in $\bar{\Omega}$ for $t > 0$, this function is a supersolution for $t \geq 0$.
By the Sweeping Principle [3, Theorem 9] one finds (8) for all $t \geq 0$. $\square$

Finally we will show, for $\varepsilon > 0$ but small enough, that U_{λ_1} does not minimize $J_\varepsilon(\lambda_1,\cdot)$. We will modify U_{λ_1} near (0,0) to obtain a $H^1_0(\Omega)$–function with lower energy. Hence the solution of (7) for $\lambda = \lambda_1$ that minimizes $J_\varepsilon(\lambda_1,\cdot)$ is not U_{λ_1} or $V^\varepsilon_{\lambda_1}$, which are the only stable solutions with fixed sign.

Set
$$\Omega^1_\delta = \{(x_1,x_2) \in \Omega \,;\, x_1 < \delta\}\,,$$
$$\Omega^2_\delta = \{(x_1,x_2) \in \Omega \,;\, \delta < x_1 < 2\delta\} \text{ and}$$
$$\Omega^3_\delta = \{(x_1,x_2) \in \Omega \,;\, x_1 < 2\delta\}\,.$$

Then $|\Omega^3_\delta| = 2\delta^2$.

Moreover define $z \in C^{0,1}(\mathbb{R})$ by

$$\begin{cases} z(s) = 0 & \text{for } s \le 1\,, \\ z(s) = s-1 & \text{for } 1 < s \le 2\,, \\ z(s) = 1 & \text{for } 2 < s\,, \end{cases}$$

and set

$$u_\delta(x_1,x_2) = z(\delta^{-1}x_1)\, U_{\lambda_1}(x_1,x_2).$$

Then $u_\delta \in H^1_0(\Omega)$ and

$$\nabla u_\delta(x_1,x_2) = \delta^{-1} U_{\lambda_1}(x_1,x_2)(1,0) + z(\delta^{-1}x_1)\, \nabla U_{\lambda_1}(x_1,x_2) \qquad \text{in } \Omega^2_\delta.$$

By using lemma 4 we can estimate the difference in energy as follows:

$$(9)\qquad J_\varepsilon(\lambda_1,u_\delta) - J_\varepsilon(\lambda_1,U_{\lambda_1}) \le \frac{1}{2}\int_{\Omega^3_\delta} \left(|\nabla u_\delta|^2 - |\nabla U_{\lambda_1}|^2\right) dx + \lambda_1 \int_{\Omega^3_\delta} \frac{1}{2} U^2_{\lambda_1}\, dx \le$$

$$\le \int_{\Omega^2_\delta} \left(\frac{1}{2}\delta^{-2}\, U^2_{\lambda_1} dx + \delta^{-1}\, U_{\lambda_1} z(\delta^{-1}x_1) \frac{d}{dx_1} U_{\lambda_1} \right) dx + \lambda_1 \int_{\Omega^3_\delta} \frac{1}{2} U^2_{\lambda_1}\, dx \le$$

$$\le |\Omega^3_\delta| \left(\frac{1}{2}\delta^{-2} \left(\tfrac{1}{3}\lambda_1 4\delta^2\right)^2 + \delta^{-1} \left(\tfrac{1}{3}\lambda_1 4\delta^2\right) \|\nabla U_{\lambda_1}\|_\infty + \frac{1}{2}\lambda_1 \left(\tfrac{1}{3}\lambda_1 4\delta^2\right)^2 \right) \le$$

$$\le 2\delta^2 \left(\frac{8}{9}\lambda_1^2\delta^2 + \frac{4}{3}\lambda_1\delta\, \|\nabla U_{\lambda_1}\|_\infty + \frac{8}{9}\lambda_1^3\delta^4 \right) \le C(\lambda_1)\,\delta^3 \qquad \text{for } 2\delta < 1.$$

The function v_δ defined by

$$v_\delta(x_1,x_2) = -\frac{1}{2}\delta\, U_{\lambda_1}(\delta^{-1}x_1, \delta^{-1}x_2)$$

satisfies:

$$-\Delta v_\delta(x) = \tfrac{1}{2}\delta^{-1}(\Delta U_{\lambda_1})(\delta^{-1}x) =$$

$$= -\tfrac{1}{2}\delta^{-1}\lambda_1 f_\delta\big(U_{\lambda_1}(\delta^{-1}x)\big) =$$

$$= \tfrac{1}{2}\delta^{-1}\lambda_1 2\delta f_\delta\big(-\tfrac{1}{2}\delta U_{\lambda_1}(\delta^{-1}x)\big) =$$

$$= \lambda_1 f_\delta(v_\delta(x)) .$$

Hence v_δ is a solution of (7) with $\varepsilon = \delta$ and Ω replaced by Ω^1_δ.

After extending v_δ by 0 outside of Ω^1_δ we obtain:

$$(10) \qquad J_\delta(\lambda_1, v_\delta) = \int_{\Omega^1_\delta} \Big(\tfrac{1}{2}|\nabla v_\delta|^2 - \lambda_1 \int_0^{v_\delta} f_\delta(s)\, ds\Big)\, dx =$$

$$= \tfrac{1}{4}\int_{\Omega^1_\delta} \Big(\tfrac{1}{2}|(\nabla U_{\lambda_1})(\delta^{-1}x)|^2 - \lambda_1 \int_0^{U_{\lambda_1}(\delta^{-1}x)} f_\delta(t)dt \Big)\, dx =$$

$$= \tfrac{1}{4}\delta^2 J_\delta(\lambda_1, U_{\lambda_1}) .$$

Finally, we set $w_\delta = u_\delta + v_\delta$ and we find, since $\operatorname{supp} u_\delta \in \bar\Omega\backslash\Omega^1_\delta$ and $\operatorname{supp} v_\delta \in \bar\Omega^1_\delta$ that by (9) and (10) for δ sufficiently small:

$$J_\delta(\lambda_1, w_\delta) = J_\delta(\lambda_1, u_\delta) + J_\delta(\lambda_1, v_\delta) \le$$

$$\le (1+\tfrac{1}{4}\delta^2)\, J_\delta(\lambda_1, U_{\lambda_1}) + C(\lambda_1)\,\delta^3 < J_\delta(\lambda_1, U_{\lambda_1}) . \qquad \square$$

The example uses a triangle for a domain and a piecewise linear right hand side. One can modify both Ω and f to have the same result on a smooth, strictly convex domain with a C^∞–function f.

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