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Closed-form Solution for a Moving Boundary Problem

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Abstract This paper considers a moving boundary problem with Neumann boundary conditions and a general initial value, which occurs in an unsaturated flow with extraction. The closed-form solution for the moving boundary problem is obtained using a Laplace transform boost. This method has been successfully applied to solve moving boundary problems with Dirichelet boundary conditions, but not to the case with Neumann boundary conditions.

Key words moving boundary problems; Laplace transform boost; unsaturated flow

Introduction

This paper considers the moving boundary problem (MBP):

$$\text{(MBP)}\begin{cases}\phi_\tau = \phi_{yy}, & 0 < y < y_0 + \delta\tau, \tau > 0 \\ \phi_y(0,\tau) = \phi_y(y_0 + \delta\tau, \tau) = 0, & \tau > 0 \\ \phi(y,0) = \phi_0(y), & 0 < y < y_0\end{cases}$$

where y_0 and δ are positive constants. This problem arises from an unsaturated flow with extraction. For details, see Ref. [1].

The primary concern here is to find a closed-form solution to the MBP. Note that the problem is linear. Thus it is natural to expect that the solution can be given by the Laplace transform. However, the normal Laplace transform method can not handle a moving boundary condition. In order to avoid this difficulty, the Laplace transform boost is used as suggested by King[2]. The Laplace transform boost has been successfully applied to solve moving boundary problems with Dirichlet boundary conditions and specific initial values, see Refs. [2-4]. As far as we know, the Laplace transform boost method has never been applied to problems with Neumann boundary conditions on the moving boundary and with general initial values. It is shown below that this is possible. A closed-form solution is obtained for MBP via the Laplace transform boost method.

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1 Formal Solution Using the Laplace Transform Boost

The Laplace transform boost method is used to solve MBP. That is, a search is made for the analytic solution $\phi(y,\tau)$ which satisfies:

$$\phi_\tau = \phi_{yy},\ 0 < y < y_0 + \delta\tau,\ \tau > 0 \tag{1}$$

$$\phi_y = 0,\ \text{at } y = 0 \tag{2}$$

$$\phi_y = 0,\ \text{at } y = y_0 + \delta\tau \tag{3}$$

$$\phi = \phi_0(y),\ \text{at } \tau = 0 \tag{4}$$

The Laplace transform $\tilde{\phi}(y,p)$ of $\phi(y,\tau)$ satisfies

$$\tilde{\phi}_{yy}(y,p) - p\tilde{\phi}(y,p) + \phi_0(y) = 0 \tag{5}$$

for which the general solution is given by

$$\tilde{\phi}(y,p) = A\mathrm{e}^{-\sqrt{p}\,y} + B\mathrm{e}^{\sqrt{p}\,y} + \int_0^y \phi_0(\eta)\frac{\mathrm{e}^{\sqrt{p}(\eta-y)} - \mathrm{e}^{\sqrt{p}(y-\eta)}}{2\sqrt{p}}\mathrm{d}\eta \tag{6}$$

where A and B are functions depending only on p. The boundary condition (2) implies that $A = B$. Thus, the solution for the moving boundary problem is given by

$$\tilde{\phi}(y,p) = A(\sqrt{p})(\mathrm{e}^{-\sqrt{p}\,y} + \mathrm{e}^{\sqrt{p}\,y}) + \int_0^y \phi_0(\eta)\frac{\mathrm{e}^{\sqrt{p}(\eta-y)} - \mathrm{e}^{\sqrt{p}(y-\eta)}}{2\sqrt{p}}\mathrm{d}\eta \tag{7}$$

where $A(\sqrt{p})$ is a function of $\sqrt{p}$, determined by the boundary condition (3). For convenience, set

$$M(y,p)=\int_0^y \phi_0(\eta)\frac{e^{\sqrt{p}(\eta-y)}-e^{\sqrt{p}(y-\eta)}}{2\sqrt{p}}d\eta \quad (8)$$

To implement the boundary condition on the moving boundary, introduce a moving coordinate:

$$\eta = y-\delta\tau \quad (9)$$

Let $\psi(y,\tau)=\phi(y,\tau)$. Equation (1) is equivalent to

$$\psi_{\eta\eta}+\delta\psi_\eta-\psi_\tau=0,\ -\delta\tau<\eta<y_0,\ \tau>0 \quad (10)$$

with the initial condition

$$\psi=\phi_0,\ \text{at}\ \tau=0$$

and the boundary conditions:

$$\psi_\eta=0,\ \text{at}\ \eta=y_0 \quad (11)$$

$$\psi_\eta=0,\ \text{at}\ \eta=-\delta\tau \quad (12)$$

The Laplace transform $\tilde{\psi}(\eta,p)$ of $\psi(\eta,\tau)$ satisfies

$$\tilde{\psi}_{\eta\eta}(\eta,p)+\delta\tilde{\psi}_\eta(\eta,p)-p\tilde{\psi}(\eta,p)+\phi_0(\eta)=0 \quad (13)$$

with the general solution given by

$$\tilde{\psi}(\eta,p)=C_p e^{-\left(\frac{\delta}{2}+\sqrt{\delta^2/4+p}\right)\eta}+D_p e^{\left(-\frac{\delta}{2}+\sqrt{\delta^2/4+p}\right)\eta}+N(\eta,p) \quad (14)$$

where C_p and D_p are functions depending only on p, $N(\eta,p)$ is a special solution to be determined. Now apply a Laplace transform boost to represent Eq. (7) in the moving coordinate system (η,τ). From Theorem 2 of King[2], the result is

$$\tilde{\psi}(\eta,p)=\left[1+\frac{\delta}{2}\Big/\sqrt{\delta^2/4+p}\right]A\left[\frac{\delta}{2}+\sqrt{\delta^2/4+p}\right]\cdot e^{-\left(\frac{\delta}{2}+\sqrt{\delta^2/4+p}\right)\eta}\left[1-\frac{\delta}{2}\Big/\sqrt{\delta^2/4+p}\right]A\cdot\left[-\frac{\delta}{2}+\sqrt{\delta^2/4+p}\right]e^{\left(-\frac{\delta}{2}+\sqrt{\delta^2/4+p}\right)\eta}+\exp\left(-\delta\frac{\partial}{\partial\eta}\frac{\partial}{\partial p}\right)M(\eta,p) \quad (15)$$

where $\exp\left(-\delta\frac{\partial}{\partial\eta}\frac{\partial}{\partial p}\right)$ is the boost operator. Comparing Eqs. (14) and (15) yields following formulae:

$$C_p=\left[1+\frac{\delta}{2}\Big/\sqrt{\delta^2/4+p}\right]A\left[\frac{\delta}{2}+\sqrt{\delta^2/4+p}\right] \quad (16)$$

$$D_p=\left[1-\frac{\delta}{2}\Big/\sqrt{\delta^2/4+p}\right]A\left[-\frac{\delta}{2}+\sqrt{\delta^2/4+p}\right] \quad (17)$$

and

$$N(\eta,p)=\exp\left(-\delta\frac{\partial}{\partial\eta}\frac{\partial}{\partial p}\right)M(\eta,p) \quad (18)$$

Substituting Eq. (8) into Eq. (18) yields

$$N(\eta,p)=\int_0^\eta \phi_0(\zeta)\cdot\frac{e^{\left(\frac{\delta}{2}+\sqrt{\delta^2/4+p}\right)(\zeta-\eta)}-e^{\left(-\frac{\delta}{2}+\sqrt{\delta^2/4+p}\right)(\eta-\zeta)}}{2\sqrt{\delta^2/4+p}}d\zeta \quad (19)$$

To implement boundary condition (11), it is necessary to have

$$-C_p\left(\frac{\delta}{2}+\sqrt{\delta^2/4+p}\right)e^{-\left(\frac{\delta}{2}+\sqrt{\delta^2/4+p}\right)\eta}+D_p\left(-\frac{\delta}{2}+\sqrt{\delta^2/4+p}\right)e^{\left(-\frac{\delta}{2}+\sqrt{\delta^2/4+p}\right)\eta}+\int_0^\eta\phi_0(\zeta)\left[-\left(\frac{\delta}{2}+\sqrt{\delta^2/4+p}\right)e^{\left(\frac{\delta}{2}+\sqrt{\delta^2/4+p}\right)(\zeta-\eta)}-\left(-\frac{\delta}{2}+\sqrt{\delta^2/4+p}\right)e^{\left(-\frac{\delta}{2}+\sqrt{\delta^2/4+p}\right)(\eta-\zeta)}\right]\Big/ 2\sqrt{\delta^2/4+p}\,d\zeta=0 \quad (20)$$

at $\eta=y_0$. Combining Eqs. (16), (17), and (20) yields

$$-H(\xi+\delta)(\xi+\delta)e^{-(\xi+\delta)y_0}+H(\xi)\xi e^{-\xi y_0}=\frac{1}{2}\int_0^\eta\phi_0(\zeta)\left[(\xi+\delta)e^{(\xi+\delta)(\zeta-y_0)}+\xi e^{\xi(y_0-\zeta)}\right]d\zeta \quad (21)$$

where $H(\xi)=\xi A(\xi)$ and $\xi=-\frac{\delta}{2}+\sqrt{\delta^2/4+p}$. Solving the difference equation (21) yields a particular solution

$$H(\xi)=\frac{1}{2}\int_0^{y_0}\phi_0(\eta)e^{-\xi\eta}d\eta+\frac{1}{2}\int_0^{y_0}\phi_0(\eta)+\sum_{n=1}^{\infty}\frac{\xi+n\delta}{\xi}e^{-(2n\xi+n^2\delta)y_0}\left[e^{-(\xi+n\delta)\eta}+e^{(\xi+n\delta)\eta}\right]d\eta \quad (22)$$

Hence, from Eqs. (7) and (22), we have

$$\tilde{\phi}(y,p)=H(\sqrt{p})(e^{-\sqrt{p}y}+e^{\sqrt{p}y})/\sqrt{p}+\int_0^y\phi_0(\eta)\frac{e^{\sqrt{p}(\eta-y)}-e^{\sqrt{p}(y-\eta)}}{2\sqrt{p}}d\eta=$$

$$\int_0^y\phi_0(\eta)\frac{e^{\sqrt{p}(\eta-y)}-e^{\sqrt{p}(y-\eta)}}{2\sqrt{p}}d\eta+\int_0^{y_0}\phi_0(\eta)\frac{e^{-\sqrt{p}(y+n)}+e^{\sqrt{p}(y-\eta)}}{2\sqrt{p}}d\eta+$$

$$\frac{1}{2}\int_0^{y_0}\phi_0(\eta)\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{p}}+\frac{n\delta}{p}\right)\cdot\left[e^{-\sqrt{p}(2ny_0+\eta)-(n\delta\eta+n^2\delta y_0)}+e^{\sqrt{p}(\eta-2ny_0)+(n\delta\eta-n^2\delta y_0)}\right]\cdot\left[e^{-\sqrt{p}y}+e^{\sqrt{p}y}\right]d\eta \quad (23)$$

From tabulated inverse Laplace transforms[5], the following is formally obtained:

$$\phi(y,\tau)=\int_0^{y_0}\phi_0(\eta)\frac{e^{-(y-\eta)^2/4\tau}+e^{-(y+\eta)^2/4\tau}}{2\sqrt{\pi\tau}}d\eta+$$

$$\frac{1}{2}\int_0^{y_0}\phi_0(\eta)\sum_{n=1}^{\infty}e^{-(n\delta\eta+n^2\delta y_0)}\cdot$$

$$\left\{e^{-(2ny_0+\eta+y)^2/4\tau}\left[\frac{1}{\sqrt{\pi\tau}}+n\delta f\left(\frac{2ny_0+\eta+y}{2\sqrt{\tau}}\right)\right]+\right.$$

$$
\mathrm{e}^{-(2ny_0+\eta-y)^2/4\tau}\left[\frac{1}{\sqrt{\pi\tau}}+n\delta f\left(\frac{2ny_0+\eta-y}{2\sqrt{\tau}}\right)\right]\Bigg\}\mathrm{d}\eta+
$$

$$
\frac{1}{2}\int_0^{y_0}\phi_0(\eta)\sum_{n=1}^{\infty}\mathrm{e}^{(n\delta n-n^2\delta y_0)}\cdot
$$

$$
\Bigg\{\mathrm{e}^{-(2ny_0-\eta+y)^2/4\tau}\left[\frac{1}{\sqrt{\pi\tau}}+n\delta f\left(\frac{2ny_0-\eta+y}{2\sqrt{\tau}}\right)\right]+
$$

$$
\mathrm{e}^{-(2ny_0-\eta-y)^2/4\tau}\left[\frac{1}{\sqrt{\pi\tau}}+n\delta f\left(\frac{2ny_0-\eta-y}{2\sqrt{\tau}}\right)\right]\Bigg\}\mathrm{d}\eta \tag{24}
$$

where the function f is defined by $f(x)=\mathrm{e}^{x^2}\cdot\mathrm{erfc}(x)$.

It should be noted that the inverse Laplace transform of $\mathrm{e}^{a\sqrt{p}}$ with $a>0$ does not exist. But terms $\mathrm{e}^{a\sqrt{p}}$ with $a>0$ are treated the same as $\mathrm{e}^{a\sqrt{p}}$ with $a<0$ in inverting Eq. (23). Therefore, it need to be verified that Eq. (24) is a solution of the MBP. This is done in the next section.

2 Verification of the Solution

In the last section, a formal solution of the moving boundary problem is obtained. Is it the real solution of the moving boundary problem? It is shown below that Eq. (24) is indeed a solution.

Since the function f is bounded, it can be seen, by comparison with $\sum \mathrm{e}^{-n^2}$, that the above series converges for all $\tau>0$. Moreover, the function $\phi(y,\tau)$ can be differentiated explicitly. Note that $(\mathrm{erfc}(x))'=-\frac{2}{\sqrt{\pi}}\mathrm{e}^{-x^2}$. By a straight-forward computation:

$$
\phi_y(y,\tau)=\int_0^{y_0}\phi_0(\eta)\cdot
$$

$$
\frac{(\eta-y)\mathrm{e}^{-(y-n)^2/4\tau}-(y+\eta)\mathrm{e}^{-(y+\eta)^2/4\tau}}{4\tau\sqrt{\pi\tau}}\mathrm{d}\eta+
$$

$$
\frac{1}{2}\int_0^{y_0}\phi_0(\eta)\sum_{n=1}^{\infty}\mathrm{e}^{-(n\delta\eta+n^2\delta y_0)}\cdot
$$

$$
\Bigg\{-\frac{1}{2\tau\sqrt{\pi\tau}}\mathrm{e}^{-(2ny_0+\eta+y)^2/4\tau}[2ny_0+\eta+y+
$$

$$
2n\delta\tau]+\frac{1}{2\tau\sqrt{\pi\tau}}\mathrm{e}^{-(2ny_0+\eta-y)^2/4\tau}[2ny_0+
$$

$$
\eta-y+2n\delta\tau]\Bigg\}\mathrm{d}\eta+\frac{1}{2}\int_0^{y_0}\phi_0(\eta)\sum_{n=1}^{\infty}\mathrm{e}^{(n\delta\eta-n^2\delta y_0)}\cdot
$$

$$
\Bigg\{-\frac{1}{2\tau\sqrt{\pi\tau}}\mathrm{e}^{-(2ny_0+\eta+y)^2/4\tau}[2ny_0-\eta+y+
$$

$$
2n\delta\tau]+\frac{1}{2\tau\sqrt{\pi\tau}}\mathrm{e}^{-(2ny_0-\eta-y)^2/4\tau}[2ny_0-\eta-y+
$$

$$
2n\delta\tau]\Bigg\}\mathrm{d}\eta=\frac{1}{2}\int_0^{y_0}\phi_0(\eta)\sum_{n=-\infty}^{\infty}\mathrm{e}^{-(n\delta\eta+n^2\delta y_0)}\cdot
$$

$$
\Bigg\{-\frac{1}{2\tau\sqrt{\pi\tau}}\mathrm{e}^{-(2ny_0+\eta+y)^2/4\tau}[2ny_0+\eta+y+2n\delta\tau]+
$$

$$
\frac{1}{2\tau\sqrt{\pi\tau}}\mathrm{e}^{-(2ny_0+\eta-y)^2/4\tau}[2ny_0+\eta-y+2n\delta\tau]\Bigg\}\mathrm{d}\eta \tag{25}
$$

One can readily verify that $\phi(y,\tau)$ defined by Eq. (24) satisfies $\phi_\tau=\phi_{yy}$ and $\phi(y,0)=\phi_0(y)$. By Eq. (25), $\phi_y(y,\tau)=0$. Furthermore, it follows from Eq. (25) that

$$
\phi_y(y_0+\delta\tau,\tau)=\frac{1}{2}\int_0^{y_0}\phi_0(\eta)\sum_{n=-\infty}^{\infty}\mathrm{e}^{-(n\delta\eta+n^2\delta y_0)}\cdot\Bigg\{-\frac{1}{2\tau\sqrt{\pi\tau}}\mathrm{e}^{-[(2n+1)y_0+\eta+\delta\tau]^2/4\tau}\cdot
$$

$$
[(2n+1)y_0+\eta+(2n+1)\delta\tau]+\frac{1}{2\tau\sqrt{\pi\tau}}\mathrm{e}^{-[(2n-1)y_0+\eta-\delta\tau]^2/4\tau}\cdot
$$

$$
[(2n-1)y_0+\eta+(2n-1)\delta\tau]\Bigg\}\mathrm{d}\eta=\frac{1}{2}\int_0^{y_0}\phi_0(\eta)\sum_{n=-\infty}^{\infty}\mathrm{e}^{-(n\delta\eta+n^2\delta y_0)}\cdot
$$

$$
\Bigg\{-\frac{1}{2\tau\sqrt{\pi\tau}}\mathrm{e}^{\{-[(2n+1)y_0+\eta]^2-(\delta\tau)^2\}/4\tau}\mathrm{e}^{-\frac{1}{2}(2n+1)\delta y_0-\frac{1}{2}\delta\eta}[(2n+1)y_0+\eta+(2n+1)\delta\tau]+
$$

$$
\frac{1}{2\tau\sqrt{\pi\tau}}\mathrm{e}^{\{-[(2n-1)y_0+\eta]^2-(\delta\tau)^2\}/4\tau}\mathrm{e}^{\frac{1}{2}(2n-1)\delta y_0+\frac{1}{2}\delta y}[(2n-1)y_0+\eta+(2n-1)\delta\tau]\Bigg\}\mathrm{d}\eta=
$$

$$
\frac{1}{2}\int_0^{y_0}\phi_0(\eta)\sum_{n=-\infty}^{\infty}\frac{1}{2\tau\sqrt{\pi\tau}}\cdot
$$

$$
\Bigg\{-\mathrm{e}^{\{-[(2n+1)y_0+\eta]^2-(\delta\tau)^2\}/4\tau}\mathrm{e}^{-\left[\frac{1}{4}(2n+1)^2\delta y_0+\frac{1}{4}\delta y_0+\frac{1}{2}(2n+1)\delta\eta\right]}[(2n+1)y_0+\eta+(2n+1)\delta\tau]+
$$

$$
\mathrm{e}^{\{-[2n-1)y_0+\eta]^2-(\delta\tau)^2\}/4\tau}\mathrm{e}^{-\left[\frac{1}{4}(2n-1)^2\delta y_0+\frac{1}{4}\delta y_0+\frac{1}{2}(2n-1)\delta\eta\right]}[(2n-1)y_0+\eta+(2n-1)\delta\tau]\Bigg\}\mathrm{d}\eta=0 \tag{26}
$$

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Therefore, the formal solution (24) is a classical solution of the moving boundary problem (MBP).

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