A. Dall'Acqua · G. Sweers

The Clamped Plate Equation for the Limaçon

Received: date / in final form: date

Abstract. Hadamard claimed in 1907 that the clamped plate equation is positivity preserving for domains which are bounded by a Limaçon de Pascal. We will show that this claim is false in its full generality. However, we will also prove that there are nonconvex limaçons for which the clamped plate equation has the sign preserving property. In fact we will give an explicit bound for the parameter of the limaçon where sign change may occur.

Mathematics Subject Classification (2000): 35J40, 35Q72, 35B50

Key words: clamped plate–biharmonic–positivity–nonconvex

1. Introduction

Hadamard in [10] states that the clamped plate equation for plates having the shape of a *Limaçon de Pascal* is positivity preserving. Positivity preserving for this (linear) equation on $\Omega \subset \mathbb{R}^2$ means that in the fourth order boundary value problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \frac{\partial}{\partial \nu} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

the sign of f is preserved by u. Here f is the force (density) and u the deflection of the plate of shape Ω. So the statement reads as, say for $f \in L^1(\Omega)$:

$$f \geq 0 \text{ implies } u \geq 0. \tag{2}$$

For a precise citation of Hadamard let $\Gamma_A^B = G_\Omega(A, B)$ be the corresponding Green function, that is, $u(x) = \int_\Omega G_\Omega(x, y) f(y)\, dy$ solves (1). Concerning Γ_A^B Hadamard in [10] writes:

> *M. Boggio, qui a, le premier, noté la signification physique de Γ_A^B, en a déduit l'hypothése que Γ_A^B était toujours positif. Malgré l'absence de*

Dept. of Applied Mathematical Analysis,
ITS Faculty, Delft University of Technology
PObox 5031, 2600GA Delft
The Netherlands
e-mail: `A.Dallacqua@ITS.TUDelft.nl`
e-mail: `G.H.Sweers@ITS.TUDelft.nl`

démonstration rigoureuse, l'exactitude de cette proposition ne paraît pas douteuse pour les aires convexes. Mais il était l'intéressant d'examiner si elle est vraie pour le cas du Limaçon de Pascal, qui est concave. La réponse est affirmative.

Let us focus on Hadamard's two claims separately.

Claim. There is no doubt that Γ_A^B is positive for convex domains.

This conjecture stood for a long time and only in 1949 a first counterexample, with Ω a long rectangle, was established by Duffin in [3]. This counterexample was soon to be followed by numerous others. A short survey can be found in the introduction of [13]. So by now it is well known that convexity is not a sufficient condition.

Let us remind the reader that around 1905 Boggio [2] did prove that (2) holds in case of a disk. In fact some believed that the disk might be the only domain where (2) holds. However in [5] it is shown that (2) also holds in domains that are small perturbations of the disk. Since smallness of these perturbations is defined by a C^2-norm non-convex domains are not included.

Claim. Γ_A^B is positive for some non-convex domains, namely for the Limaçons de Pascal.

Hadamard in [10] starts his proof of this claim by observing that:

> *... on constate aisément que, si l'un de ces deux points est très voisin du contour, la partie principale de Γ_A^B est positive.*

Although we are not certain what he meant by '*partie principale*' we know by now that Γ_A^B can be negative when one point is near the boundary. In fact we will show that if the Green function (on a limaçon) is negative somewhere it will be negative for some A and B near the boundary. Hadamard continues his proof by referring to the results in [9]. In this paper he gives an explicit formula for the Green function for (1) in case of a limaçon. This formula will allow us to show the theorem below. Since there is no explicit proof that his formula indeed gives the Green function we will supply such a proof in the appendix.

The domains under consideration are defined for $a \in \left[0, \frac{1}{2}\right]$ by

$$\Omega_a = \left\{ (\rho \cos\varphi, \rho \sin\varphi) \in \mathbb{R}^2; 0 \le \rho < 1 + 2a\cos\varphi \right\}.$$

For $0 \le a \le \frac{1}{2}$ the curve $\rho = 1 + 2a\cos\varphi$ is a non self-intersecting limaçon.

We will show the following:

Theorem 1. *The clamped plate problem on Ω_a with $a \in \left[0, \frac{1}{2}\right]$ is positivity preserving if and only if $a \in \left[0, \frac{1}{6}\sqrt{6}\right]$.*

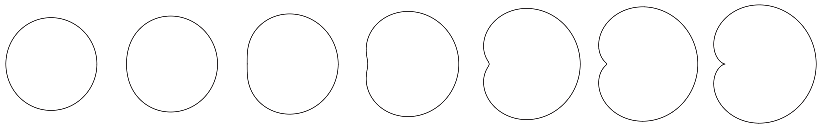

Fig. 1. Limaçons for resp. $a = .1, .175, .25, .325, \frac{1}{6}\sqrt{6}, .45, .5$. The fifth one with $a = \frac{1}{6}\sqrt{6}$ is critical for positivity.

Remark 1. The limaçon is convex precisely if $0 \leq a \leq \frac{1}{4}$. Notice that $\frac{1}{4} < \frac{1}{6}\sqrt{6}$. So Hadamard is right in the sense that convexity is not a necessary condition. He is wrong in claiming the positivity preserving property for all limaçons.

Remark 2. A related question is if the first eigenfunction has a fixed sign for all limaçons (compare the Boggio-Hadamard-conjecture versus the Szegö-conjecture in [14], see also [13]). Since one cannot expect an explicit formula for the eigenfunction this seems a much harder question. One does know that the number a where positivity of the first eigenfunction breaks down is strictly larger than the number where (2) fails. See [7].

Finally we would like to mention some papers that consider explicit solutions for the clamped plate equation. Schot constructed in [11], see also Boggio in [2], an explicit Green function on the disk and on the half-plane. Dube in [4] gives a series solution for the Green function on a limaçon.

2. Proofs

Any limaçon can be seen as the image of a circle through the conformal map $z \to z^2$ combined with two shifts. It will be convenient in the following to use complex notation for the unit disk: $B = \{z \in \mathbb{C}\,; |z| < 1\}$. The appropriate conformal map from $B \subset \mathbb{C}$ to $\Omega_a \subset \mathbb{R}^2$ is then given by

$$\begin{aligned} h_a :\ & B \to \Omega_a, \\ & \eta \mapsto x = \Big(\mathrm{Re}\left(\eta + a\eta^2\right), \mathrm{Im}\left(\eta + a\eta^2\right)\Big). \end{aligned} \tag{3}$$

The fact that this conformal map is quadratic, and hence that $\partial\Omega$ is a quartic curve, seems to allow an explicit Green function. This makes the limaçon a special case. For the clamped plate equation with constant f on domains bounded by quartic curves see [12].

2.1. Behaviour of the Green function

In [9, Supplement] one finds the explicit formula of the Green function for (1), which we will denote with G_a. For $x, y \in \Omega_a$ we may rewrite this function as follows

$$G_a(x,y) = \tfrac{1}{2}a^2 s^2 r^2 \left[\log\left(\tfrac{r^2}{r_1^2}\right) + \tfrac{r_1^2}{r^2} - 1 - \tfrac{a^2}{1-2a^2}\tfrac{r^2}{s^2}\left(\tfrac{r_1^2}{r^2}-1\right)^2\right], \tag{4}$$

where, with $\eta, \xi \in B$ such that $x = h_a(\eta)$ and $y = h_a(\xi)$, the r, r_1 and s are given by

$$r^2 = |\eta - \xi|^2, \quad r_1^2 = \left|1 - \eta\bar{\xi}\right|^2, \quad s^2 = \left|\eta + \xi + \tfrac{1}{a}\right|^2. \tag{5}$$

We want to study when the function G_a is of fixed sign in $\Omega_a \times \Omega_a$. For establishing this positivity we will need to consider the function

$$F(\beta, q) := \log\left(\frac{1}{q}\right) + q - 1 - \beta\frac{(q-1)^2}{q}. \tag{6}$$

Note that $q = r_1^2/r^2 \geq 1$.

Lemma 1. *Set $I_\beta := \{q \geq 1 : F(\beta,q) \leq 0\}$. It holds that:*

- $I_\beta = \{1\}$ *for* $\beta \in \left[0, \frac{1}{2}\right]$;
- $I_\beta = [1, q_\beta]$ *with* $q_\beta > 1$ *for* $\beta \in \left(\frac{1}{2}, 1\right)$;
- $I_\beta = [1, \infty)$ *for* $\beta \in [1, \infty)$.

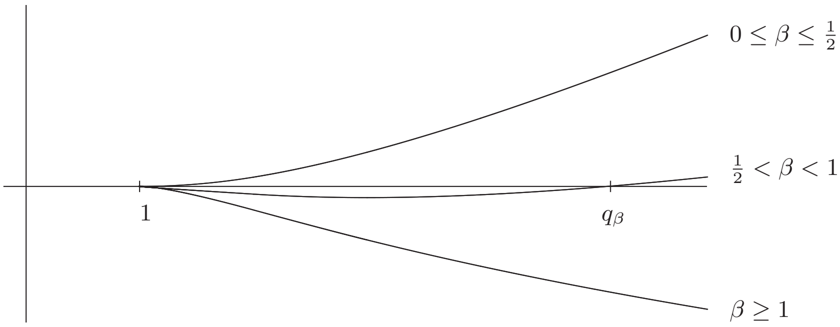

Fig. 2. Graphs of $q \mapsto F(\beta, q)$.

Remark 3. Note that $\beta \mapsto F(\beta, q)$ is decreasing and hence that $\beta \mapsto q_\beta$ is increasing.

It will be convenient to work with functions defined in the disk. If f is a function defined on Ω_a, then $\tilde{f}_a$ will denote the function $\tilde{f}_a := f \circ h_a$ defined on the disk.

We fix the auxiliary function

$$\tilde{H}_a(\eta, \xi) := \tfrac{a^2}{1-2a^2}\frac{r_1^2}{s^2} = \tfrac{a^2}{1-2a^2}\frac{|1-\eta\bar{\xi}|^2}{|\eta + \xi + \frac{1}{a}|^2}, \tag{7}$$

and hence the Green function in (4) becomes

$$\begin{aligned}\tilde{G}_a(\eta,\xi) &:= \tfrac{1}{2}a^2s^2r^2\ F\left(\tilde{H}_a(\eta,\xi), \tfrac{r_1^2}{r^2}\right)\\ &= \tfrac{1}{2}a^2s^2r^2\ F\left(\tilde{H}_a(\eta,\xi), \tfrac{|1-\eta\bar{\xi}|^2}{|\eta-\xi|^2}\right).\end{aligned} \tag{8}$$

The preceding Lemma 1 gives that if

$$\sup_{\eta,\xi\in B} \tilde{H}_a(\eta,\xi) \le \frac{1}{2}, \tag{9}$$

then F and hence G_a are positive. Note that (9) gives a condition on the parameter a which is a sufficient condition for the positivity of the function. In the following we will see that this condition is also necessary.

First we will reduce the dimension of the problem. The following lemma states that it is sufficient to study the behaviour of $\tilde{H}_a$ for couples of conjugate points.

Lemma 2. *Let $a < \frac{1}{2}$ and define the sets $\ell_{\eta,\xi}$ for $(\eta,\xi) \in B \times B$ by*

$$\ell_{\eta,\xi} = \left\{\chi = \chi_1 + i\chi_2 \in B : \chi_1 = \tfrac{\eta_1+\xi_1}{2}, |\chi| \ge \max\{|\eta|, |\xi|\}\right\}, \tag{10}$$

where $\eta = \eta_1 + i\eta_2$ and $\xi = \xi_1 + i\xi_2$.

If $\tilde{H}_a(\eta,\xi) > \frac{1}{2}$, then for every $\chi \in \ell_{\eta,\xi}$ it holds that $\tilde{H}_a(\chi,\bar{\chi}) > \frac{1}{2}$.

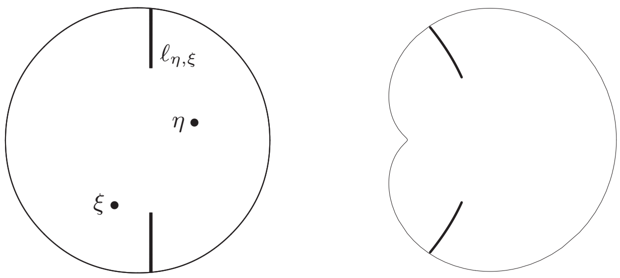

Fig. 3. A set $\ell_{\eta,\xi}$ and its image within a limaçon.

Proof. By hypothesis one has:

$$\tilde{H}(\eta,\xi) = \frac{a^2}{1-2a^2}\frac{(1-\eta_1\xi_1-\eta_2\xi_2)^2 + (\eta_1\xi_2-\eta_2\xi_1)^2}{\left(\eta_1+\xi_1+\frac{1}{a}\right)^2 + (\eta_2+\xi_2)^2} > \frac{1}{2},$$

which is equivalent to

$$2a^2\left(1+\eta_1^2\xi_1^2+\eta_2^2\xi_2^2-2\eta_1\xi_1-2\eta_2\xi_2+\eta_1^2\xi_2^2+\eta_2^2\xi_1^2\right) > \\ \left(1-2a^2\right)\left(\eta_1^2+\xi_1^2+\tfrac{1}{a^2}+2\eta_1\xi_1+\tfrac{2}{a}\eta_1+\tfrac{2}{a}\xi_1+\eta_2^2+\xi_2^2+2\eta_2\xi_2\right),$$

or similarly

$$2a^2(1+|\eta|^2)(1+|\xi|^2) > \\ (\eta_1+\xi_1)^2+(\eta_2+\xi_2)^2+\tfrac{1}{a^2}+\tfrac{2}{a}(\eta_1+\xi_1)-2-4a(\eta_1+\xi_1). \quad (11)$$

For $\chi \in \ell_{\eta,\xi}$, we have

$$\tilde{H}\left(\chi,\bar{\chi}\right)-\tfrac{1}{2}=\tfrac{a^2}{1-2a^2}\frac{\left(1-\chi_1^2+\chi_2^2\right)^2+4\chi_1^2\chi_2^2}{\left(2\chi_1+\frac{1}{a}\right)^2}-\frac{2\chi_1^2+\frac{1}{2a^2}+\frac{2}{a}\chi_1}{\left(2\chi_1+\frac{1}{a}\right)^2}$$

$$=\tfrac{a^2}{1-2a^2}\tfrac{1+\chi_1^4+\chi_2^4-2\chi_1^2+2\chi_2^2-2\chi_1^2\chi_2^2+4\chi_1^2\chi_2^2-\frac{2}{a^2}\chi_1^2-\frac{1}{2a^4}-\frac{2}{a^3}\chi_1+4\chi_1^2+\frac{1}{a^2}+\frac{4}{a}\chi_1}{\left(2\chi_1+\frac{1}{a}\right)^2}$$

$$=\tfrac{1}{1-2a^2}\tfrac{1}{\left(2\chi_1+\frac{1}{a}\right)^2}\left(a^2(1+|\chi|^2)^2-(2\chi_1^2+\tfrac{1}{2a^2}+\tfrac{2}{a}\chi_1-1-4a\chi_1)\right). \quad (12)$$

By the definition of $\ell_{\eta,\xi}$ and (11) it follows that the last term is positive:

$$(1+|\chi|^2)^2 \geq (1+|\eta|^2)(1+|\xi|^2) > \tfrac{1}{a^2}\left(2\chi_1^2+\tfrac{1}{2a^2}+\tfrac{2}{a}\chi_1-1-4a\chi_1\right).$$

□

Remark 4. Note that (12) implies: $\tilde{H}_a\left(\chi,\bar{\chi}\right)$ is increasing in $|\chi_2|$.

We are now able to prove that (9) also gives a necessary condition for the positivity of F and hence of G_a.

Lemma 3. *Let* $a<\frac{1}{2}$.

i. If $\tilde{H}_a(v,\bar{v})>\frac{1}{2}$ *then there is* $\chi\in\ell_{v,\bar{v}}$ *such that*

$$F\left(\tilde{H}_a\left(\chi,\bar{\chi}\right),\tfrac{\left|1-\chi^2\right|^2}{|\chi-\bar{\chi}|^2}\right)<0. \quad (13)$$

ii. If (13) holds, then $F\left(\tilde{H}_a\left(z,\bar{z}\right),\frac{\left|1-z^2\right|^2}{|z-\bar{z}|^2}\right)<0$ *for every* $z\in\ell_{\chi,\bar{\chi}}$.

Proof. First claim: Since the function $\beta\mapsto F\left(\beta,q\right)$ is decreasing, see (6), and the function $\tilde{H}_a\left(z,\bar{z}\right)$ is increasing in $|z_2|$, by Remark 4, one gets that

$$F\left(\tilde{H}_a\left(z,\bar{z}\right),\tfrac{\left|1-z^2\right|^2}{|z-\bar{z}|^2}\right)\leq F\left(\tilde{H}_a\left(v,\bar{v}\right),\tfrac{\left|1-z^2\right|^2}{|z-\bar{z}|^2}\right) \text{ for every } z\in\ell_{v,\bar{v}}. \quad (14)$$

In $F\left(\tilde{H}_a\left(v,\bar{v}\right),\frac{\left|1-z^2\right|^2}{|z-\bar{z}|^2}\right)$ the first argument does not depend on z; it is a fixed coefficient which is larger then 1/2 by hypothesis. Hence, applying Lemma 1, one has that there exists a $q_{\tilde{H}_a(v,\bar{v})}>1$ such that

$$F\left(\tilde{H}_a\left(v,\bar{v}\right),\tfrac{\left|1-z^2\right|^2}{|z-\bar{z}|^2}\right)<0,\ \forall z\in\ell_{v,\bar{v}} \text{ with } \tfrac{\left|1-z^2\right|^2}{|z-\bar{z}|^2}<q_{\tilde{H}_a(v,\bar{v})}. \quad (15)$$

Note that the function $|z_2| \mapsto \frac{|1-z^2|^2}{|z-\bar{z}|^2}$ is decreasing, since

$$\frac{\partial}{\partial z_2} \frac{|1-z^2|^2}{|z-\bar{z}|^2} = -\frac{1}{2z_2^3}(1-|z|^2+2z_2^2)(1-|z|^2). \tag{16}$$

Hence, since $\frac{|1-z^2|^2}{|z-\bar{z}|^2}$ is equal to 1 at the boundary, it follows that there exists $\chi \in \ell_{v,\bar{v}}$ such that

$$\frac{|1-\chi^2|^2}{|\chi-\bar{\chi}|^2} < q_{\tilde{H}_a(v,\bar{v})}. \tag{17}$$

Combining (14), (15) and (17) the first claim follows.

Second claim: If $F\left(\tilde{H}_a(\chi,\bar{\chi}), \frac{|1-\chi^2|^2}{|\chi-\bar{\chi}|^2}\right) < 0$ we can deduce from Lemma 1 that

$$\tilde{H}_a(\chi,\bar{\chi}) > \tfrac{1}{2} \text{ and } \frac{|1-\chi^2|^2}{|\chi-\bar{\chi}|^2} < q_{\tilde{H}_a(\chi,\bar{\chi})}. \tag{18}$$

Since $\tilde{H}_a(z,\bar{z})$ is increasing in $|z_2|$ (Remark 4) and the function $|z_2| \mapsto \frac{|1-z^2|^2}{|z-\bar{z}|^2}$ is decreasing, see (16), from (18) one gets that

$$\tilde{H}_a(z,\bar{z}) > \tfrac{1}{2} \text{ and } \frac{|1-z^2|^2}{|z-\bar{z}|^2} < q_{\tilde{H}_a(\chi,\bar{\chi})} \text{ for every } z \in \ell_{\chi,\bar{\chi}}. \tag{19}$$

Since $\beta \mapsto q_\beta$ is increasing (Remark 3), from (19) we have that

$$\frac{|1-z^2|^2}{|z-\bar{z}|^2} < q_{\tilde{H}_a(z,\bar{z})} \text{ for every } z \in \ell_{\chi,\bar{\chi}}. \tag{20}$$

By (19), (20) and Lemma 1 it follows that $F\left(\tilde{H}_a(z,\bar{z}), \frac{|1-z^2|^2}{|z-\bar{z}|^2}\right) < 0$ for every $z \in \ell_{\chi,\bar{\chi}}$. □

The previous results show that if the function $G_a(x,y)$ is negative for some $x,y \in \Omega_a$ then G_a will be negative somewhere near opposite boundary points. To be precise:

Corollary 1. *Suppose that $G_a(x,y) < 0$ for some $x,y \in \Omega_a$, then for all $\varepsilon > 0$ there is $x^\varepsilon \in \Omega_a$ with $d_{\Omega_a}(x^\varepsilon) < \varepsilon$ such that:*

$$G_a((x_1^\varepsilon, x_2^\varepsilon), (x_1^\varepsilon, -x_2^\varepsilon)) < 0.$$

By $d_\Omega(x)$ we denote the distance of x to the boundary of Ω:

$$d_\Omega(x) = \inf\{|x-x^*|; x^* \in \partial\Omega\}. \tag{21}$$

Proof. If $\tilde{G}_a(\eta,\xi) < 0$, Lemma 1 gives that necessarily $\tilde{H}_a(\eta,\xi) > \frac{1}{2}$. Hence, one has from Lemma 2 that $\tilde{H}_a(z,\bar{z}) > \frac{1}{2}$ for every $z \in \ell_{\eta,\xi}$. The claim follows directly from Lemma 3. □

2.2. Positivity of the Green function

Using the results of the previous section, we have seen that the function $\tilde{H}_a$ in (7) plays a crucial role for the positivity of the Green function. Let us collect this result.

Corollary 2. *The Green function for the clamped plate equation on a limaçon is positive if and only if*

$$\sup_{\eta,\xi\in B} \tilde{H}_a(\eta,\xi) = \frac{a^2}{1-2a^2} \sup_{\eta,\xi\in B} \frac{\left|1-\eta\bar{\xi}\right|^2}{|\eta+\xi+1|^2} \le \frac{1}{2}. \tag{22}$$

Condition (22) gives an upper bound for the parameter a. In the following Lemma we give the explicit value of this upper bound.

Lemma 4. *Inequality (22) is satisfied if and only if* $a \le \frac{1}{6}\sqrt{6}$.

Proof. Lemma 2 implies that it is sufficient to verify (22) for couples of conjugate points, that is:

$$\sup_{\chi\in B} \tilde{H}_a(\chi,\bar{\chi}) = \tfrac{a^2}{1-2a^2} \sup_{\chi\in B} \frac{\left|1-\chi^2\right|^2}{|\chi+\bar{\chi}+1|^2} \le \frac{1}{2}.$$

By (12) we find

$$\tilde{H}_a(\chi,\bar{\chi}) - \tfrac{1}{2} = \tfrac{1}{1-2a^2} \frac{a^2(1+|\chi|^2)^2+1+4a\chi_1}{\left(2\chi_1+\frac{1}{a}\right)^2} - \tfrac{1}{1-2a^2}\tfrac{1}{2},$$

which gives

$$\sup_{\chi\in B} \tilde{H}_a(\chi,\bar{\chi}) - \frac{1}{2} = \tfrac{1}{1-2a^2} \sup_{\chi\in B} \frac{4a^2+1+4a\chi_1}{\left(2\chi_1+\frac{1}{a}\right)^2} - \tfrac{1}{1-2a^2}\tfrac{1}{2}. \tag{23}$$

A straightforward computation shows that the maximum in (23) is attained for $\chi_1 = -2a$ (and $|\chi| = 1$). We obtain

$$\sup_{\chi\in B} \tilde{H}_a(\chi,\bar{\chi}) - \tfrac{1}{2} = \tfrac{a^2}{1-2a^2} \frac{1}{-4a^2+1} - \tfrac{1}{1-2a^2}\tfrac{1}{2} = \tfrac{1}{1-2a^2} \frac{6a^2-1}{2\left(1-4a^2\right)},$$

which is non-negative for $a > \frac{1}{6}\sqrt{6}$. □

2.3. Sharp estimates for the Green function

The Green function for the biharmonic problem in two dimensions does not have a singularity in L^∞-sense: $(x,y) \mapsto G(x,y)$ is uniformly bounded. However, a natural solution space concerning the Dirichlet boundary condition ($u = \frac{\partial}{\partial \nu} u = 0$), see [1], is the Banach lattice (with the natural ordering):

$$C_e(\bar{\Omega}) = \{u \in C(\bar{\Omega}); \|u\|_e := \sup_{x\in\Omega} \left| \frac{u(x)}{d_\Omega^2(x)} \right| < \infty\},$$

where $d_\Omega(.)$ is as in (21). However $(x \mapsto G(x,.))$ from $\bar{\Omega}$ into $C_e(\bar{\Omega})$ does show 'a singularity' when $x \to \partial\Omega$. Precise information for the singularity of polyharmonic Dirichlet Green functions on balls in $\mathbb{R}^n$, where the Green function is known to be positive, can be found in [8].

In the next theorem one finds how the estimate of G_a from below changes depending on a. It is interesting to see that although the Green function becomes negative, no 'boundary-singularity' from below appears.

Note that Theorem 1 is a direct consequence of Theorem 2.

Theorem 2. *For every $(\eta,\xi) \in B \times B$, the following estimates hold:*

i. for all $a \in \left[0, \frac{1}{2}\right]$ there exists $c_1 > 0$ such that

$$\tilde{G}_a(\eta,\xi) \le c_1 \; d_B(\eta)\, d_B(\xi) \min\left\{1, \frac{d_B(\eta)\, d_B(\xi)}{|\eta-\xi|^2}\right\}, \tag{24}$$

ii. for all $a \in \left[0, \frac{1}{6}\sqrt{6}\right]$, there exists $c_2 > 0$ such that

$$\tilde{G}_a(\eta,\xi) \ge c_2 \; \left(\tfrac{1}{6}\sqrt{6} - a\right) d_B(\eta)\, d_B(\xi) \min\left\{1, \frac{d_B(\eta)\, d_B(\xi)}{|\eta-\xi|^2}\right\}, \tag{25}$$

iii. for $a \in \left(\frac{1}{6}\sqrt{6}, \frac{1}{2}\right]$ there exists $(\eta^*,\xi^*) \in B \times B$ such that

$$\tilde{G}_a(\eta^*,\xi^*) < 0.$$

iv. for all $a \in \left(\frac{1}{6}\sqrt{6}, \frac{1}{2}\right]$, there exists $c_3 > 0$ such that

$$\tilde{G}_a(\eta,\xi) \ge -c_3 \; \left(a - \tfrac{1}{6}\sqrt{6}\right) d_B(\eta)^2\, d_B(\xi)^2, \tag{26}$$

where the constants c_1 and c_2 do not depend on a.

Remark 5. Let us observe that for every $\varepsilon > 0$ there exists two constants m_ε, M such that for every η, $\xi \in B$ and $a \in [0, \frac{1}{2} - \varepsilon]$ it holds:

$$\begin{aligned} m_\varepsilon \,.\, |\eta-\xi| \le |h_a(\eta) - h_a(\xi)| \le M \,.\, |\eta-\xi|, \\ m_\varepsilon \,.\, d(\eta,\partial B) \le \; d(h_a(\eta), \partial\Omega_a) \; \le M \,.\, d(\eta,\partial B). \end{aligned} \tag{27}$$

Using (27) one can prove estimates for G_a similar to the one proven for $\tilde{G}_a$ in Theorem 2. Near the cusp (when $a \to \frac{1}{2}$) the estimate from below in (27) breaks down.

Remark 6. One may derive that for $a \in \left[0, \frac{1}{6}\sqrt{6}\right]$ there exist constants c_4, c_5, independent on a, such that

$$c_4 \ \left(\tfrac{1}{6}\sqrt{6} - a\right) D(x,y) \le G_a\left(x,y\right) \le c_5 \ D(x,y),$$

where $D(x,y) = d\left(x\right) d\left(y\right) \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\}$ and $d(\cdot) = d_{\Omega_a}(\cdot)$.

Remark 7. Note that the Green function is positive on the diagonal. This follows from the eigenfunction expansion and taking $x = y$:

$$G(x,y) = \sum_i \frac{1}{\lambda_i}\varphi_i(x)\varphi_i(y).$$

Here λ_i, φ_i are the eigenvalues/functions of the corresponding eigenvalue problem. Note that $\lambda_i > 0$ holds for all i.

Proof. We will prove the statements separately.

i. One has from (4) that

$$\begin{aligned}\tilde{G}_a\left(\eta,\xi\right) &\le \tfrac{1}{2}a^2 s^2\left[-r^2\log\left(\frac{r_1^2}{r^2}\right) + r_1^2 - r^2\right]\\ &\le 2\left[-r^2\log\left(\frac{r_1^2}{r^2}\right) + r_1^2 - r^2\right]. \qquad (28)\end{aligned}$$

The term inside the brackets in the right hand side of (28) is the Green function for the clamped plate equation on the disk. Inequality (24) follows using the estimate in [6, Prop.2.3(iii)].

ii. Let $a_0 = \frac{1}{6}\sqrt{6}$ and $s_0 = |\eta + \xi + \frac{1}{a_0}|$. Using that s is decreasing in a for all $\eta, \xi \in B$ when $a < \frac{1}{2}$, one finds for $a \in \left[0, \frac{1}{6}\sqrt{6}\right]$ that

$$\begin{aligned}\tilde{G}_a\left(\eta,\xi\right) &\ge \tfrac{1}{2}a^2\left(s_0^2 r^2\log\left(\frac{r^2}{r_1^2}\right) + s_0^2(r_1^2 - r^2) - \frac{a^2}{1-2a^2}(r_1^2 - r^2)^2\right)\\ &= \tfrac{a^4}{1-2a^2}\tfrac{1-2a_0^2}{a_0^4}\tilde{G}_{a_0}\left(\eta,\xi\right)\\ &\quad + \tfrac{1}{2}a^2 s_0^2\left(1 - \tfrac{a^2}{1-2a^2}\tfrac{1-2a_0^2}{a_0^2}\right)\left[-r^2\log\left(\frac{r_1^2}{r^2}\right) + r_1^2 - r^2\right]\\ &\ge \tfrac{1}{2}a^2 s_0^2\left(1 - 4\tfrac{a^2}{1-2a^2}\right)\left[-r^2\log\left(\frac{r_1^2}{r^2}\right) + r_1^2 - r^2\right],\end{aligned}$$

since $\tilde{G}_{a_0}\left(\eta,\xi\right) \ge 0$, see Corollary 2 and Lemma 4. For $a \in \left[0, \frac{1}{6}\sqrt{6}\right]$ one has $\frac{1}{2}a^2 s_0^2(1 - 4\frac{a^2}{1-2a^2}) \ge \frac{1}{10}(\frac{1}{6}\sqrt{6} - a)$, hence by using [6, Prop.2.3(iii)] one gets

$$\tilde{G}_a\left(\eta,\xi\right) \ge c_2 \ \left(\tfrac{1}{6}\sqrt{6} - a\right) d_B\left(\eta\right) d_B\left(\xi\right) \min\left\{1, \frac{d_B\left(\eta\right) d_B\left(\xi\right)}{\left|\eta - \xi\right|^2}\right\}.$$

iii. This claim follows from Corollary 2 and Lemma 4.
iv. Let $a_0 = \frac{1}{6}\sqrt{6}$ and $s_0 = |\eta + \xi + \frac{1}{a_0}|$. We have

$$\begin{aligned}\tilde{G}_a(\eta,\xi) &= \frac{1}{2}a^2\frac{s^2}{s_0^2}\left(r^2 s_0^2 \log\left(\frac{r^2}{r_1^2}\right) + s_0^2(r_1^2 - r^2) - \frac{a_0^2}{1-2a_0^2}\left(r_1^2 - r^2\right)^2\right)\\ &\quad + \frac{1}{2}a^2\left(\frac{s^2}{s_0^2}\frac{a_0^2}{1-2a_0^2} - \frac{a^2}{1-2a^2}\right)\left(r_1^2 - r^2\right)^2\\ &\geq \frac{1}{2}a^2\left(\frac{s^2}{s_0^2}\frac{1}{4} - \frac{a^2}{1-2a^2}\right)\left(r_1^2 - r^2\right)^2, \qquad (29)\end{aligned}$$

since $\tilde{G}_{a_0}$ is positive in the entire domain. Using that $s_0^2 \geq (\sqrt{6}-2)^2$ one gets that

$$\frac{1}{2}a^2\left(\frac{s^2}{s_0^2}\frac{1}{4} - \frac{a^2}{1-2a^2}\right) = \frac{1}{8}a^2\left(\frac{1-6a^2}{1-2a^2} + \frac{s^2 - s_0^2}{s_0^2}\right)$$

$$\begin{aligned}&\geq -\frac{1}{8}a^2\left(\frac{1+\sqrt{6}a}{1-2a^2}\sqrt{6} + \frac{1}{(\sqrt{6}-2)^2}\left(\frac{1}{a} + \sqrt{6} + 4\right)\frac{\sqrt{6}}{a}\right)(a - \frac{1}{6}\sqrt{6})\\ &\geq -7(a - \frac{1}{6}\sqrt{6}), \qquad (30)\end{aligned}$$

Hence, from (29) and (30) it follows that there exists a constant $c_3 > 0$ such that

$$\tilde{G}_a(\eta,\xi) \geq -c_3(a - \frac{1}{6}\sqrt{6})d_B(\eta)^2 d_B(\xi)^2,$$

for $a \in (\frac{1}{6}\sqrt{6}, \frac{1}{2})$.

□

A. The Green function for the limaçon

As promised in the introduction this appendix will contain a proof that the function supplied by Hadamard is indeed the Green function for the limaçons. For $x, y \in \mathbb{R}^2$ let $R = |x - y|$. The function $U = R^2 \log(R)$ satisfies $\Delta^2 U(\cdot) = \delta_y(\cdot)$ in $\mathbb{R}^2$. Then writing

$$G_a(x,y) = R^2 \log(R) + J_a(x,y), \qquad (31)$$

the function

$$J_a(x,y) := -R^2 \log(a r_1 s) + \tfrac{a^2}{2}s^2\left(r_1^2 - r^2\right) - \tfrac{a^4}{2(1-2a^2)}\left(r_1^2 - r^2\right)^2, \qquad (32)$$

should be biharmonic and such that G_a satisfies the boundary condition.

Note that (31) follows from (4) using that $ars = R$. In fact one has

$$\begin{aligned}R &= \left|\left(\eta + a\eta^2\right) - \left(\xi + a\xi^2\right)\right| = a\left|\eta^2 + \tfrac{\eta}{a} - \xi^2 - \tfrac{\xi}{a}\right| =\\ &= a\ |\eta - \xi|\left|\eta + \xi + \tfrac{1}{a}\right| = ars.\end{aligned}$$

A.1. Boundary condition:

Let us rewrite (31) as

$$G_a(x,y) = \tfrac{1}{2}a^2 s^2 \left[r^2 \log\left(\tfrac{r^2}{r_1^2}\right) + r_1^2 - r^2\right] - \tfrac{a^4}{2(1-2a^2)}\left(r_1^2 - r^2\right)^2. \qquad (33)$$

When $x \in \partial\Omega_a$, then $\eta \in \partial D$ and it holds $r_1 = r$. It follows from (33) that $G_a(x,y) = 0$ at the boundary. Now we are interested in $\frac{\partial}{\partial\nu}G_a(x,y)$ on $\partial\Omega_a$. One observes that the term $\left(r_1^2 - r^2\right)^2$ gives no contribution because it is a zero of order two at the boundary. The remaining term is a product of two factors: one that is non-zero at the boundary and the other that is identically zero. Hence, when we look at the normal derivative at the boundary the only relevant term will be

$$\tfrac{\partial}{\partial\nu}\left[r^2 \log\left(\tfrac{r^2}{r_1^2}\right) + r_1^2 - r^2\right]. \qquad (34)$$

Using that the term inside the brackets in (34) is the Green function for the disk, see [2], one gets that also the second Dirichlet boundary condition is satisfied.

A.2. The function $J_a(x,y)$ is biharmonic on Ω_a.

To prove the biharmonicity of J_a it is convenient to consider separately the term with the logarithm and the remaining part.

We first observe that $\log(ar_1 s)$ is a harmonic function on Ω_a. From this, the identity $\Delta^2\left(R^2\log(ar_1 s)\right) = 0$ follows using that if v is a harmonic function then $R^2 v$ is biharmonic.

Lemma 5. *It holds that*

$$\Delta_x^2\left(s^2\left(r_1^2 - r^2\right) - \tfrac{a^2}{1-2a^2}\left(r_1^2 - r^2\right)^2\right) = 0.$$

Proof. Next to $h_a : B \subset \mathbb{C} \to \mathbb{R}^2$ we will use $\mathbf{h}_a(\eta) : \mathbb{C} \to \mathbb{C}$ defined by $\mathbf{h}_a(\eta) = \eta + a\eta^2$ with $\eta = \eta_1 + i\eta_2$.

Let us consider

$$\begin{aligned} K(x,y) := {} & \left|h_a^{-1}(x) + h_a^{-1}(y) + \tfrac{1}{a}\right|^2 (1 - \left|h_a^{-1}(x)\right|^2) \\ & - \tfrac{a^2}{1-2a^2}(1 - \left|h_a^{-1}(x)\right|^2)^2(1 - \left|h_a^{-1}(y)\right|^2), \end{aligned}$$

and then $s^2\left(r_1^2 - r^2\right) - \frac{a^2}{1-2a^2}\left(r_1^2 - r^2\right)^2 = (1 - |h_a^{-1}(y)|^2)K(x,y)$, and

$$\begin{aligned} Y(\eta,\xi) := K(h_a(\eta), h_a(\xi)) = {} & \left|\eta + \xi + \tfrac{1}{a}\right|^2 \left(1 - |\eta|^2\right) \\ & - \tfrac{a^2}{1-2a^2}\left(1 - |\eta|^2\right)^2\left(1 - |\xi|^2\right). \end{aligned}$$

Since h is a conformal map, it holds that:

$$\begin{aligned}\Delta_\eta Y(\eta,\xi) &= |\mathbf{h}_a'(\eta)|^2 (\Delta_x K)(h_a(\eta), h_a(\xi)), \qquad (35)\\ \Delta_\eta^2 Y(\eta,\xi) &= \Delta_\eta |\mathbf{h}_a'(\eta)|^2 \Delta_x K(h_a(\eta), h_a(\xi))\\ &\quad + 2\sum_{i=1}^{2} \tfrac{\partial}{\partial \eta_i} |\mathbf{h}_a'(\eta)|^2 \tfrac{\partial}{\partial \eta_i}(\Delta_x K)(h_a(\eta), h_a(\xi))\\ &\quad + |\mathbf{h}_a'(\eta)|^4 \left(\Delta_x^2 K\right)(h_a(\eta), h_a(\xi)). \qquad (36)\end{aligned}$$

The idea is to use (36) in order to calculate $\left(\Delta_x^2 K\right)(h_a(\eta), h_a(\xi))$ in terms of $\Delta_\eta^2 Y(\eta,\xi)$. Since $\Delta_\eta = 4\frac{\partial}{\partial\bar\eta}\frac{\partial}{\partial\eta}$, one has

$$\begin{aligned}\tfrac{\partial}{\partial\eta} Y(\eta,\xi) &= \left(\bar\eta+\bar\xi+\tfrac{1}{a}\right)(1-|\eta|^2) - \bar\eta\left|\eta+\xi+\tfrac{1}{a}\right|^2\\ &\quad + \tfrac{2a^2}{1-2a^2}\bar\eta(1-|\eta|^2)(1-|\xi|^2),\\ \tfrac{\partial^2}{\partial\bar\eta\partial\eta} Y(\eta,\xi) &= (1-|\eta|^2) - \eta\left(\bar\eta+\bar\xi+\tfrac{1}{a}\right) - \left|\eta+\xi+\tfrac{1}{a}\right|^2 - \bar\eta\left(\eta+\xi+\tfrac{1}{a}\right)\\ &\quad + \tfrac{2a^2}{1-2a^2}(1-|\eta|^2)(1-|\xi|^2) - \tfrac{2a^2}{1-2a^2}\bar\eta\eta(1-|\xi|^2),\\ \tfrac{\partial^3}{\partial\eta\partial\bar\eta\partial\eta} Y(\eta,\xi) &= -2\bar\eta - 2\left(\bar\eta+\bar\xi+\tfrac{1}{a}\right) - \tfrac{4a^2}{1-2a^2}\bar\eta(1-|\xi|^2),\\ \tfrac{\partial^4}{\partial\bar\eta\partial\eta\partial\bar\eta\partial\eta} Y(\eta,\xi) &= -4 - \tfrac{4a^2}{1-2a^2}(1-|\xi|^2),\end{aligned}$$

which gives

$$\begin{aligned}\Delta_\eta Y(\eta,\xi) &= 4(1-3|\eta|^2) - 4\eta\left(\bar\xi+\tfrac{1}{a}\right) - 4\left|\eta+\xi+\tfrac{1}{a}\right|^2 - 4\bar\eta\left(\xi+\tfrac{1}{a}\right)\\ &\quad + \tfrac{8a^2}{1-2a^2}(1-2|\eta|^2)(1-|\xi|^2),\\ \Delta_\eta^2 Y(\eta,\xi) &= -64 - \tfrac{64a^2}{1-2a^2}(1-|\xi|^2).\end{aligned}$$

By the definition of the conformal map h_a in (3) and from (35) we obtain that $|\mathbf{h}_a'(\eta)|^2 = |2a\eta+1|^2$, $\Delta_\eta |\mathbf{h}_a'(\eta)|^2 = 16a^2$ and

$$\begin{aligned}(\Delta_x K)(h_a(\eta), h_a(\xi)) &= \tfrac{4}{|2a\eta+1|^2}\left((1-3|\eta|^2) - \eta\left(\bar\xi+\tfrac{1}{a}\right) - \left|\eta+\xi+\tfrac{1}{a}\right|^2\right)\\ &\quad + \tfrac{4}{|2a\eta+1|^2}\left(-\bar\eta\left(\xi+\tfrac{1}{a}\right) + \tfrac{2a^2}{1-2a^2}(1-2|\eta|^2)(1-|\xi|^2)\right).\end{aligned}$$

We find

$$\begin{aligned}&\sum_{i=1}^{2} \tfrac{\partial}{\partial\eta_i}|\mathbf{h}_a'(\eta)|^2 \tfrac{\partial}{\partial\eta_i}(\Delta_x K)(h_a(\eta), h_a(\xi))\\ &= -\tfrac{64a^2}{|2a\eta+1|^2}\left(-4\eta_1^2 - 4\eta_2^2 - 4\eta_1\xi_1 - \tfrac{4}{a}\eta_1 - 4\eta_2\xi_2 - \tfrac{1}{a^2} - \tfrac{4}{a}\xi_1 + 1 - |\xi|^2\right)\\ &\quad - \tfrac{64a^2}{|2a\eta+1|^2}\tfrac{2a^2}{1-2a^2}(1-2|\eta|^2)(1-|\xi|^2) + \tfrac{32a^2}{|2a\eta+1|^2}\left(-8\eta_2^2 - 4\xi_2\eta_2\right)\\ &\quad + \tfrac{16a}{|2a\eta+1|^2}\left(-8\eta_1 - 4\xi_1 - \tfrac{4}{a} - 16a\eta_1^2 - 8a\eta_1\xi_1 - 8\eta_1\right)\\ &\quad + \tfrac{16a}{|2a\eta+1|^2}\tfrac{2a^2}{1-2a^2}\left(-8a\eta_1^2 - 4\eta_1 - 8a\eta_2^2\right)(1-|\xi|^2)\end{aligned}$$

$$= -\tfrac{64a^2}{|2a\eta+1|^2}\left(-2\eta_1\xi_1 - 2\eta_2\xi_2 - \tfrac{1}{a}\xi_1 + 1 - |\xi|^2\right)$$
$$-\tfrac{64a^2}{|2a\eta+1|^2}\tfrac{2a^2}{1-2a^2}\left(1+\tfrac{\eta_1}{a}\right)(1-|\xi|^2).$$

Hence from (36) we get

$$\begin{aligned}
&-1 - \tfrac{a^2}{1-2a^2}(1-|\xi|^2) = \\
&= \tfrac{a^2}{|2a\eta+1|^2}\left(-4\eta_1^2 - 4\eta_2^2 - 4\eta_1\xi_1 - \tfrac{4}{a}\eta_1 - 4\eta_2\xi_2 - \tfrac{1}{a^2} - \tfrac{4}{a}\xi_1 + 1 - |\xi|^2\right) \\
&+ \tfrac{a^2}{|2a\eta+1|^2}\tfrac{2a^2}{1-2a^2}(1-2|\eta|^2)(1-|\xi|^2) - \tfrac{2a^2}{|2a\eta+1|^2}\tfrac{2a^2}{1-2a^2}\left(1+\tfrac{\eta_1}{a}\right)(1-|\xi|^2) \\
&- \tfrac{2a^2}{|2a\eta+1|^2}\left(-2\eta_1\xi_1 - 2\eta_2\xi_2 - \tfrac{1}{a}\xi_1 + 1 - |\xi|^2\right) \\
&+ |\mathbf{h}_a'(\eta)|^4\left(\Delta_x^2 K\right)(h_a(\eta), h_a(\xi)),
\end{aligned}$$

$$\begin{aligned}
&-1 - \tfrac{a^2}{1-2a^2}(1-|\xi|^2) = \\
&= -\tfrac{1}{|2a\eta+1|^2}|2a\eta+1|^2 - \tfrac{4a^2}{|2a\eta+1|^2}\left(\eta_1\xi_1 + \eta_2\xi_2 + \tfrac{1}{a}\xi_1\right) + \tfrac{a^2}{|2a\eta+1|^2}(1-|\xi|^2) \\
&+ \tfrac{a^2}{|2a\eta+1|^2}\tfrac{2a^2}{1-2a^2}(1-|\xi|^2) - \tfrac{a^2}{|2a\eta+1|^2}\tfrac{4a^2}{1-2a^2}|\eta|^2(1-|\xi|^2) \\
&- \tfrac{2a^2}{|2a\eta+1|^2}\tfrac{2a^2}{1-2a^2}(1-|\xi|^2) - \tfrac{2a^2}{|2a\eta+1|^2}\tfrac{2a}{1-2a^2}\eta_1(1-|\xi|^2) \\
&- \tfrac{2a^2}{|2a\eta+1|^2}(1-|\xi|^2) - \tfrac{2a^2}{|2a\eta+1|^2}\left(-2\eta_1\xi_1 - 2\eta_2\xi_2 - \tfrac{1}{a}\xi_1\right) \\
&+ |\mathbf{h}_a'(\eta)|^4\left(\Delta_x^2 K\right)(h_a(\eta), h_a(\xi)),
\end{aligned}$$

$$\begin{aligned}
&-\tfrac{a^2}{1-2a^2}(1-|\xi|^2) = \\
&= -\tfrac{a^2}{|2a\eta+1|^2}(1-|\xi|^2) - \tfrac{a^2}{|2a\eta+1|^2}\tfrac{1}{1-2a^2}(1-|\xi|^2)|2a\eta+1|^2 \\
&+ \tfrac{a^2}{|2a\eta+1|^2}\tfrac{1}{1-2a^2}(1-|\xi|^2) - \tfrac{a^2}{|2a\eta+1|^2}\tfrac{2a^2}{1-2a^2}(1-|\xi|^2) \\
&+ |\mathbf{h}_a'(\eta)|^4\left(\Delta_x^2 K\right)(h_a(\eta), h_a(\xi)),
\end{aligned}$$

$$\begin{aligned}
0 = &+ \tfrac{a^2}{|2a\eta+1|^2}(1-|\xi|^2)\left(-1 + \tfrac{1}{1-2a^2} - \tfrac{2a^2}{1-2a^2}\right) \\
&+ |\mathbf{h}_a'(\eta)|^4\left(\Delta_x^2 K\right)(h_a(\eta), h_a(\xi)),
\end{aligned}$$

which gives the claim. □

Acknowledgment: We would like to thank H.Ch. Grunau for carefully reading the first version of our manuscript.

References

1. Amann, H.: *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, Siam Review, **18,** (1976) 620-709.
2. Boggio, T.: *Sulle funzioni di Green d'ordine m*, Rend. Circ. Mat. Palermo, **20,** (1905) 97-135.
3. Duffin, R.J.: *On a question of Hadamard concerning super-biharmonic functions*, J. Math. Phys., **27,** (1949) 253-258.
4. Dube, R.S.: *Green's function of an elastic plate in the shape of Pascal's Limacon and clamped along the boundary*, Indian J. Pure Appl. Math., **2,** (1971) 59-68.
5. Grunau, H.C., Sweers, G.: *Positivity for perturbations of polyharmonic operators with Dirichlet boundary conditions in two dimensions*, Math. Nachr., **179,** (1996) 89-102.
6. Grunau, H.C., Sweers, G.: *Positivity for equations involving polyharmonic operators with Dirichlet boundary condition*, Math. Ann., **307,** (1997) 589-626.
7. Grunau, H.C., Sweers, G.: *Sign change for the Green function and the first eigenfunction of equations of clamped-plate type*, Arch. Ration. Mech. Anal., **150,** (1999) 179-190.
8. Grunau, H.C., Sweers, G.: *Sharp estimates for iterated Green function*, Proc. Roy. Soc. Edinburgh Sect. A, **132,** (2002) 91-120.
9. Hadamard, J.: *Memoire sur le probleme d'analyse relatif a l'equilibrie des plaques elastiques encastrees*, in: Œuvres de Jacques Hadamard, Tomes II, Éditions du Centre National de la Recherche Scientifique, Paris 1968, 515-641, reprint of: *Mèmoire couronne par l'Acadèmie des Sciences (Prix Vaillant)*, Mèm. Sav. Etrang., **33,** 1907.
10. Hadamard, J.: *Sur certains cas interessants du probleme biharmonique*, in: Œuvres de Jacques Hadamard, Tomes III, Éditions du Centre National de la Recherche Scientifique, Paris 1968, 1297-1299, reprint of: IV Congr. Math., (1908) Rome.
11. Schot, S.H.: *The Green's Functions Method for the Supported Plate Boundary Value Problem*, Z. Anal. Anwendungen, **11,** (1992) 359-370.
12. Sen, B.: *Note on the bending of thin uniformly loaded plates bounded by cardioids, lemniscates and certain other quartic curves*, Z. Angew. Math. Mech., **20,** (1940) 99–103.
13. Sweers, G.: *When is the first eigenfunction for the clamped plate equation of fixed sign?*, Electron J. Differ. Equ., **6,** (2001) 285-296.
14. Szegö, G.: *On membranes and plates*, Proc. Nat. Acad. Sci. U. S. A., **36**, (1950) 210-216.