A comparison result for perturbed radial p-Laplacians

Raul Manásevich
and
Guido Sweers

Directory

Last Revision Date: April 10, 2003

Table of Contents

A comparison result for perturbed radial p-Laplacians

Raul Manásevich & Guido Sweers

Abstract

Consider the radially symmetric p-Laplacian for $p \geq 2$ under zero Dirichlet boundary conditions. The main result of the present paper is that under appropriate conditions a solution of a perturbed (radially symmetric) p-Laplacian can be compared with the solution of the unperturbed one. As a consequence one obtains a sign preserving result for a system of p-Laplacians which are coupled in a non-quasimonotone way.

1. Introduction and main result

One of the goals when studying the Schrödinger equation $-\Delta u + Vu = f$ is to find comparison results, that is, when considering the problem for V_1 and V_2, what are the conditions such that for the same f the corresponding solutions u_1 and u_2 can be compared (see [7]). Zhao and collaborators (see [11], [2] and the references therein) obtained such comparison results on bounded domains $\Omega \subset \mathbb{R}^n$, for u satisfying zero Dirichlet boundary conditions, by estimating the iterated Green function with the Green function itself. The difficulties arise both by the singularity on the diagonal of the solution operator (the Green function $G_\Omega(x, y)$ in dimension $n \geq 2$ has a singularity when $x = y$) and as well as by the zero boundary condition. The main tool in their proofs are the Harnack inequalities both in the interior and at the boundary.

With the estimates of Zhao one may even show that for $\varepsilon > 0$ but small the nonlocal V defined by $(Vu)(x) = \varepsilon \int_\Omega G_\Omega(x, y)\, u(y)\, dy$ is in this class. As a consequence one obtains a maximum principle for a system of elliptic equations with a noncooperative coupling.

In this paper we will show a first step in transferring such a comparison result to a nonlinear equation, namely one containing the p-Laplacian with the assumption of radial symmetry. Of course the potential should have the same type of nonlinearity and that leads us to consider comparison principles for

$$\begin{cases} -\Delta_p u + \lambda V_p(u) = f & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1}$$

with $B = \{x \in \mathbb{R}^n; |x| < 1\}$, $n \geq 2$, $p \in (2, \infty)$, $\lambda > 0$ some small parameter and where V_p is an operator having the same homogeneity as the p-Laplacian. This operator may be nonlocal but is assumed to preserve radial symmetry. For $\lambda = 0$ it is well known that $f \geq 0$ implies $u \geq 0$ even in a much more general setting (see [10]).

Since we will restrict ourselves to the radial symmetric case we have following expression for the p-Laplacian:

$$\Delta_p u = r^{1-n} \left(r^{n-1} |u'|^{p-2} u' \right)'.$$

As a consequence we will find a sign preserving result for a system of p-Laplace operators which are coupled in a noncooperative way. We recall that the boundary value problem (1) is called sign preserving if every solution u is positive whenever the source term f is positive. In contrary to the 'original' maximum principle, that is, '*u cannot have a negative minimum*', such a sign preserving property may depend on nonlocal arguments. In the present paper there will not be a maximum principle in this original sense but we

supported by Fondap M.A. and Milenio grant-P01-34

Keywords: p-laplacian system, positivity, noncooperative coupling

AMS-MSC: 35B50, 35J60, 34C10

will show that for positive f solutions u of the perturbed ($\lambda \neq 0$ small) and unperturbed ($\lambda = 0$) problem can be compared. Hence a sign preserving property will hold for the perturbed problem, whenever λ is small enough, for all $f > 0$.

For easy reference we fix the following:

Notation 1.1

- $\phi_p(u) = |u|^{p-2} u$ *and its inverse is being denoted by* $\phi_p^{inv}(u) = |u|^{\frac{2-p}{p-1}} u$ *;*
- *The solution operator* G_p *for the radial* p*-Laplacian with Dirichlet boundary condition:*
$$(G_p f)(r) = \int_r^1 \phi_p^{inv} \left(\int_0^t \left(\frac{s}{t} \right)^{n-1} f(s) ds \right) dt; \tag{2}$$
- $f > 0$ *denotes* $f(r) \geq 0$ *for all* $r \in [0,1]$ *and* $f \not\equiv 0$*;*
- $f \gg 0$ *means that there is* $c > 0$ *such that* $f(r) \geq c(1-r)$ *for all* $r \in [0,1]$*.*

1.1. A non-quasimonotone system

A special case that we consider is the following non-quasimonotone (if $\lambda > 0$) nonlinear elliptic system
$$\begin{cases} -\Delta_p u &= f - \lambda \phi_p(v) & \text{in } B, \\ -\Delta_p v &= \phi_p(u) & \text{in } B, \\ u = v &= 0 & \text{on } \partial B. \end{cases} \tag{3}$$
Remember that the system $-\Delta_p u = F_1(u,v)$, $-\Delta_p v = F_2(u,v)$ is quasimonotone iff $\frac{\partial}{\partial u} F_2 \geq 0$ and $\frac{\partial}{\partial v} F_1 \geq 0$, and that in a quasimonotone setting the maximum principle can be used similar as for one equation. In a linear setting quasimonotone is also known as cooperative.

Using the solution operator G_p, the Green operator defined in (2) for the radial case, the system coincides with
$$\begin{cases} -\Delta_p u + \lambda (\phi_p \circ G_p \circ \phi_p)(u) &= f \quad \text{in } B, \\ u &= 0 \quad \text{on } \partial B, \end{cases} \tag{4}$$
Notice that $(\phi_p \circ G_p \circ \phi_p)(tu) = \phi_p(t)\, (\phi_p \circ G_p \circ \phi_p)(u)$ for all $t \in \mathbb{R}$ and hence satisfies the appropriate homogeneity condition.

The 1-dimensional case has been studied in [5]. In a reaction to that paper W. Walter raised the question what would happen in the higher dimensional case. This paper is a first step in that direction.

The linear case, $p = 2$, of (3) was studied in [6] even for general (nonradial) functions on smooth domains. A earlier result for the ball can be found in [8]. The crucial result that was used in that paper was the so-called 3G-theorem which originates from Zhao [11]. The nonlinear nature of (3) makes the general system much harder. By restricting ourselves to the radial case we are able to prove a positivity preserving property for this noncooperative system and in doing so we encounter some critical dimensions. For the linear case the Green function becomes unbounded for $n \geq 2$. Similarly, the k^{th} iterated Green function is bounded if and only if $2k > n$. For the p-Laplacian pointwise boundedness of the k^{th}-iterated homogenized Green operator, defined by $(G_p \circ \phi_p)^k = G_p \circ \phi_p \circ (G_p \circ \phi_p)^{k-1}$ for $k \geq 1$, is related to $pk > n$. These numbers reappear as a restriction in the results down below.

For sign preserving results for cooperative systems with the p-Laplacian we refer to [3]. Positivity preserving properties of (1) for $p = 2$ and linear, possibly nonlocal V, have been studied in [4].

1.2. The main condition and the theorem

The basic conditions that we will use to show that a perturbation by V_p does not destroy the positivity preserving property for λ sufficiently small is the following.

Condition 1.1 *The operator V_p is as follows:*

i. $V_p(t\,u) = t^{p-1}V_p(u)$ *for* $t \geq 0$;

ii. V_p *is continuous from* $C^1[0,1]$ *to* $C[0,1]$ *and moreover there is* $C_V^p > 0$ *such that*

$$\|V_p(u)\|_{C[0,1]} \leq C_V^p \ \|u\|_{C^1[0,1]}^{p-1} \ \text{for all } u \in C^1[0,1];$$

iii. there is $C_{V,p,n} > 0$ *such that* $G_p(|V_p \circ G_p(f)|)(r) \leq C_{V,p,n}\, G_p\,|f|\,(r)$ *for* $r \in [0,1]$ *and for all* $f \in C[0,1]$.

Remark 1.1.1 *For* $V_p = \phi_p$ *the third item in the condition above implies a nonlinear 3G-type result:*

$$(G_p \circ \phi_p \circ G_p f)(r) \leq C_{V,p,n}\, G_p f\,(r) \ \text{ for } r \in [0,1] \text{ and } 0 < f \in C[0,1]. \tag{5}$$

For $V_p = \phi_p$ *we are able to show that iii. is satisfied when both* $p \geq 2$ *and* $p > \frac{1}{2}n$ *hold. See Lemma 3.1 below.*

Remark 1.1.2 *Notice that if* V_p *and* $\tilde{V}_p$ *satisfy Condition 1.1 then so does* $V_p + \tilde{V}_p$. *Only the third condition needs some reflection. Set* $v = (V_p \circ G_p)(f)$ *and* $\tilde{v} = (\tilde{V}_p \circ G_p)(f)$ *and one obtains by* $\phi_p^{inv}(a+b) \leq \phi_p^{inv}(a) + \phi_p^{inv}(b)$ *for* $a, b \geq 0$ *that*

$$\begin{aligned}(G_p\,|v+\tilde{v}|)(r) &\leq \int_r^1 \phi_p^{inv}\left(\int_0^t \left(\frac{s}{t}\right)^{n-1}(|v(s)| + |\tilde{v}(s)|)\,ds\right)dt \\ &\leq (G_p\,|v|)(r) + (G_p\,|\tilde{v}|)(r).\end{aligned}$$

Theorem 1.2 ***(the main result)*** *Fix* $p > 2$ *and suppose that the operator* V_p *satisfies Condition 1.1 above. Then there exists* λ_p *such that for all* $f \in C[0,1]$ *with* $f > 0$ *and* $\lambda \in [0, \lambda_p]$ *the following holds:*

i. there exists a solution $u \in C^{1,\frac{1}{p-1}}[0,1]$ *of (1) with* $\phi_p(u') \in C^1[0,1]$;

ii. every solution u *of (1) satisfies*

$$\frac{1}{2}G_p f(r) \leq u(r) \leq \frac{3}{2}G_p f(r) \ \text{ for } r \in [0,1],$$

and hence every solution is positive.

The proof will be postponed to the following sections.

Remark 1.2.1 *Notice that we do not state uniqueness of the solution for the perturbed problem. As can be seen from the case* $n = 1$ *in [5] uniqueness is not obvious in general.*

For the non-quasimonotone system (3) we have the following result.

Corollary 1.3 *If $p > \frac{1}{3}n$ then the radially symmetric case of the non-quasimonotone system in (3) is positivity preserving for λ sufficiently small. That is, there exists $\lambda_p > 0$ such that for every $\lambda \in [0, \lambda_p]$ and $f \in C\left(\bar{B}\right)$ with $f = f\left(|x|\right)$ and $f > 0$ there exists a radially symmetric solution u of (3) and every radially symmetric solution is strictly positive.*

Proof. It is sufficient to show that V_p satisfies Condition 1.1 where $V_p\left(u\right) = \phi_p \circ G_p \circ \phi_p\left(u\right)$. Indeed Corollary 3.5 implies that this holds whenever $p \geq 2$ and $p > \frac{1}{3}n$. □

The approach of this paper is to get estimates from above for the perturbation in terms of a function that itself gives a uniform estimate from below for the Green operator. A strong restriction of this approach is that one needs to catch the positive function f in one number α_f such that, for some uniform constant C, the following holds:

$$\left|\left(G_p \circ V_p f\right)(r)\right| \leq C\, \alpha_f\, \left(1-r\right) \text{ and } \alpha_f\, \left(1-r\right) \leq \left(G_p f\right)(r).$$

For $p > n$ we will use $\alpha_f = \left(G_p f\right)(0)$ and for $p < n$ the number $\alpha_f = \sup_{[0,1]} r^{\frac{p-n}{p-1}} \left(G_p f\right)(r)$. Needless to say that we do not expect this to give the best possible result. The next paragraph contains an explanation for the case $p = 2$.

- **Optimal bounds in the linear case that use one parameter**

For $p = 2$ a uniform estimate using only one parameter would coincide with obtaining an estimate from below for the Green function $G\left(r, s\right)$ by a multiple of the product $g_1(r)g_2(s)$ where g_1, g_2 are positive functions. For the linear Dirichlet problem such an estimate is almost never optimal since this would mean that the Green function could be estimated from below and from above by multiples of the same product. Only for the 1-dimensional Neumann problem this is possible.

In the linear case the radial symmetric Green operator reduces to an integral operator $G_2(f) = \int_0^1 G(\cdot, s) f(s) ds$ with the following kernel:

$$G\left(r,s\right) = \begin{cases} \frac{1}{n-2} s^{n-1}\left(s^{2-n} - 1\right) & \text{if } s \geq r, \\ \frac{1}{n-2} s^{n-1}\left(r^{2-n} - 1\right) & \text{if } s < r, \end{cases} \quad \text{for } n > 2,$$

$$G\left(r,s\right) = -s \log\left(\max\left(r,s\right)\right) \qquad \text{for } n = 2.$$

For $n > 2$ these can be estimated in terms of powers of s and r and distances to the boundary $1-r$ and $1-s$, by

$$c_1 s^{n-1} \min\left(\frac{1-r}{r^{n-2}}, \frac{1-s}{s^{n-2}}\right) \leq G\left(r,s\right) \leq c_2 s^{n-1} \min\left(\frac{1-r}{r^{n-2}}, \frac{1-s}{s^{n-2}}\right), \tag{6}$$

Hence optimal estimates in product form are from below

$$G\left(r,s\right) \geq c_n\, s^{n-1} \left(1-s\right)\left(1-r\right), \tag{7}$$

and for the estimate from above one cannot go beyond

$$G\left(r,s\right) \leq C_n\, s^{n-1} \left(\frac{1-s}{s^{n-2}}\right)^{\theta} \left(\frac{1-r}{r^{n-2}}\right)^{1-\theta}$$

with $\theta \in [0,1]$. Optimal two-sided estimates for the Green function on general domains are due to Zhao [11]. See also [6] or [9]. As just explained, the sharp expressions that are used in these papers cannot be of the form $g_1(r) \cdot g_2(s)$.

1.3. Other examples

Our main interest focuses on the non-quasimonotone system in (3). Other examples of operators V_p satisfying the main condition (Condition 1.1) are $V_p = \phi_p \circ A$ with A as follows:

i. $(Au)(r) = a(r)u(r) + b(r)u'(r)$, for $p > n$ and $a, b \in C[0,1]$. If $b = 0$ then we may allow $p > \frac{1}{2}n$. This result follows from Lemma's 3.1 and 3.3 and the remark following Condition 1.1. For $a \leq 0$ this example is not so interesting since with this local perturbation one may proceed by the local arguments of the maximum principle.

ii. $(Au)(r) = \int_0^1 a(r,s)\, u(s)\, s^{n-1} ds$, with appropriate kernel a. For the precise condition see Lemma 3.6. Also kernels like

$$\begin{aligned} (Au)(r) &= \left(\int_0^1 a(r,s)\, (u(s))^\alpha\, ds \right)^{\frac{1}{\alpha}} \quad \text{or} \\ (Au)(r) &= \int_0^1 \left(\int_0^1 a(r,t,s)\, (u(s))^\alpha\, ds \right)^{\frac{1}{\alpha}} dt \end{aligned}$$

satisfy Condition 1.1 for appropriate restrictions relating α with n and p.

If we set $\alpha = q-1$ and $a(r,t,s) = \chi_{[t>r]}\chi_{[s<t]}\,(s/t)^{n-1}$ we find that u is a solution of the non-quasimonotone nonlinear elliptic system (3). For $p = q$ and $n = 1$, but with f not necessarily symmetric, this system was studied in [5].

iii. $A(u) = (G_q \circ \phi_q)$. For this operator A, which corresponds with the system

$$\left\{ \begin{array}{rcll} -\Delta_p u &=& f - \lambda \phi_p(v) & \text{in } B, \\ -\Delta_q v &=& \phi_q(u) & \text{in } B, \\ u = v &=& 0 & \text{on } \partial B \end{array} \right. \tag{8}$$

we find that Condition 1.1 is satisfied when $n, p, q \geq 2$ are such that $n < 2p + q\frac{p-1}{q-1}$. See Lemma 3.4. A similar condition can be found when using four different powers as long as the p-Laplacians have $p \geq 2$ and the homogeneity fits.

2. On the solution operator

2.1. Elementary properties of G_p

The solution operator for (1) with $\lambda = 0$ is G_p. First note that $f \in C[0,1]$ implies that $G_p f \in C^{1,\frac{1}{p-1}}[0,1]$. Moreover, if $f > 0$ set $t_0 = \inf\{t \in [0,1]; f(t) > 0\}$ and we find that

$$r \mapsto \phi_p^{inv}\left(\int_0^r \left(\frac{s}{r}\right)^{n-1} f(s) ds \right) \in C^1([0,t_0) \cup (t_0,1]) \cap C^{\frac{1}{p-1}}[0,1]. \tag{9}$$

By integrating we find that $G_p f \in C^2([0,1] \setminus \{t_0\}) \cap C^{1,\frac{1}{p-1}}[0,1]$. Also(9) immediately shows that $(G_p f)'(r) = 0$ for $r \leq t_0$ and $(G_p f)'(r) < 0$ for $t_0 < r \leq 1$.

2.2. Comparing with a sort of fundamental solution

We start by studying the outcome of this operator applied on some special distributions for the right hand side, namely $d_s(r) = s^{1-n}\delta_s(r)$, with δ the Dirac delta function at $s \in (0,1)$ with weight s^{1-n}. This weight is the appropriate normalization for the radial symmetric nature of the problem. We will see that $p = n$ is critical in the following sense. If $n < p$ then (and only then) the functions $r \mapsto (G_p d_s)(r)$ are uniformly bounded with respect to s.

Lemma 2.1 *Set* $d_s(r) = s^{1-n}\delta_s(r)$.

i. If $p > n$, *then* $(G_p d_s)(r) = \frac{p-1}{p-n}\left(1 - \max(r,s)^{\frac{p-n}{p-1}}\right)$.

ii. If $p = n$, *then* $(G_p d_s)(r) = -\log(\max(r,s))$.

iii. If $p < n$, *then* $(G_p d_s)(r) = \frac{p-1}{n-p}\left(\max(r,s)^{\frac{p-n}{p-1}} - 1\right)$.

Proof. The result follows from a direct computation. □

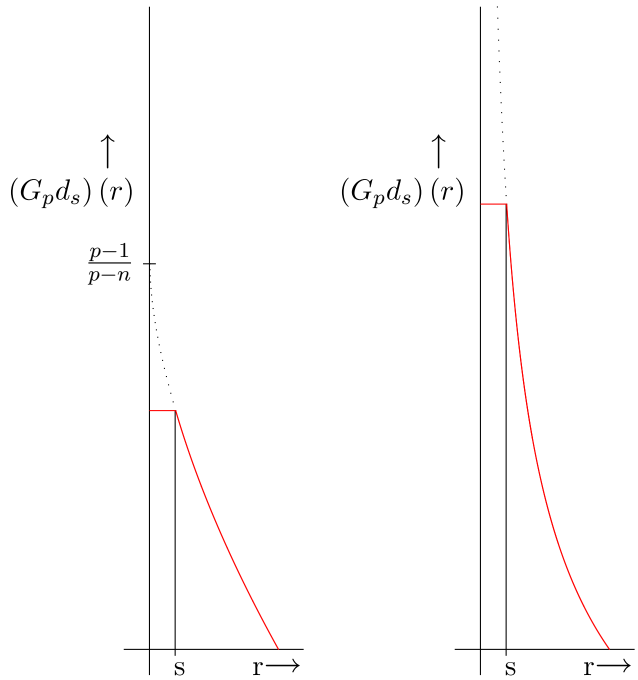

Figure 1: $p > n$ respectively $p \leq n$

With the solutions for the delta function we will have the following version of a comparison principle.

Lemma 2.2 *Let* $u = G_p f$ *with* $0 < f \in C[0,1]$. *Then for every* $r \in [0,1]$ *and* $s \in (0,1)$ *one has*

$$u(r) \geq \frac{u(s)}{(G_p d_s)(s)} (G_p d_s)(r). \tag{10}$$

Proof. Since $u(r) = \int_r^1 \phi_p^{inv}\left(\int_0^t \left(\frac{\sigma}{t}\right)^{n-1} f(\sigma) d\sigma\right) dt$ it is immediate for $f \geq 0$ that $0 \leq r_1 \leq r_2 \leq 1$ implies $u(r_1) \geq u(r_2)$ and hence (10) holds on $[0,s]$.

For $t \in [s,1]$ we proceed by contradiction. Set $v(r) = \frac{u(s)}{(G_p d_s)(s)} (G_p d_s)(r)$ and suppose that $v(\tau) > u(\tau)$ for some $\tau \in (s,1)$. Then there exist $\tau_1, \tau_2 \in [s,1]$ such that $s \leq \tau_1 \leq \tau \leq \tau_2 \leq 1$ with

$$\frac{u(\tau_1)}{v(\tau_1)} = 1 \text{ and } \frac{u(r)}{v(r)} < 1 \text{ for } r \in (\tau_1, \tau_2),$$

and with either

$$\tau_2 = 1 \text{ or } \frac{u(\tau_2)}{v(\tau_2)} = 1.$$

It follows by an elementary argument that

$$\frac{u'(\tau_1)}{v'(\tau_1)} \geq 1 \geq \frac{u'(\tau_2)}{v'(\tau_2)}. \tag{11}$$

The differential equations for u and v on $(s,1)$ give, using $\phi_p(v') < 0$, that

$$\left(\phi_p\left(\frac{u'}{v'}\right)\right)' = \left(\frac{r^{n-1}\phi_p(u')}{r^{n-1}\phi_p(v')}\right)' = \frac{\left(r^{n-1}\phi_p(u')\right)'}{r^{n-1}\phi_p(v')} \geq 0.$$

It follows, after integrating and applying ϕ_p^{inv} and by using (11), that for any $\tau \in (\tau_1, \tau_2)$

$$1 \geq \frac{u'(\tau_2)}{v'(\tau_2)} \geq \frac{u'(\tau)}{v'(\tau)} \geq \frac{u'(\tau_1)}{v'(\tau_1)} \geq 1.$$

Hence $u' \equiv v'$ on $[\tau_1, \tau_2]$ which implies $u \equiv v$ on $[\tau_1, \tau_2]$, a contradiction. □

2.3. Two-sided estimates for G_p

In the next three lemmata we will prove a relation between the upper and lower estimates of $G_p f$.

Lemma 2.3 *If $p > n$, then for every $f \in C[0,1]$ with $f > 0$ one has*

$$\frac{p-n}{p-1}(1-r)(G_p f)(0) \leq (G_p f)(r) \leq (G_p f)(0) \text{ for all } r \in [0,1]. \tag{12}$$

Proof. The estimate from above is obvious by the definition of G_p. For the estimate from below note that Lemma 2.2 implies that for every $\varepsilon > 0$

$$(G_p f)(r) \geq \frac{(G_p f)(\varepsilon)}{(G_p d_\varepsilon)(\varepsilon)} (G_p d_\varepsilon)(r) \text{ for } r \in [0,1].$$

Letting $\varepsilon \downarrow 0$ one finds

$$(G_p f)(r) \geq \left(1 - r^{\frac{p-n}{p-1}}\right)(G_p f)(0) \geq \tfrac{p-n}{p-1}(1-r)(G_p f)(0).$$ □

Lemma 2.4 *Suppose that $p = n$ and let $f \in C[0,1]$ with $f > 0$. Set*

$$\tilde{\theta}_f = \sup_{0<r\leq 1} \frac{(G_p f)(r)}{1 - \log r}.$$

Then one has

$$\tilde{\theta}_f (1-r) \leq (G_p f)(r) \leq \tilde{\theta}_f (1 - \log r). \tag{13}$$

Proof. Let r_0 be the number such that

$$\tilde{\theta}_f (1 - \log r_0) = (G_p f)(r_0).$$

Then

$$\begin{aligned}
(G_p f)(r) &\geq \frac{(G_p f)(r_0)}{(G_p \delta_{r_0})(r_0)} (G_p \delta_{r_0})(r) \\
&= \frac{\tilde{\theta}_f (1 - \log r_0)}{-\log r_0} (-\log(\max(r_0, r))) \\
&\geq \tilde{\theta}_f \frac{1 - \log r_0}{-\log r_0} \min(-\log r_0, 1)(1-r) \\
&= \tilde{\theta}_f \min\left(1 - \log r_0, 1 + \frac{1}{-\log r_0}\right)(1-r) \geq \tilde{\theta}_f (1-r).
\end{aligned}$$

□

Lemma 2.5 *Suppose that* $p < n$ *and let* $f \in C[0,1]$ *with* $f > 0$. *Set*

$$\theta_f = \sup_{0<r\leq 1} r^{\frac{n-p}{p-1}} \left(G_p f\right)(r).$$

Then one has

$$\theta_f \min\left(1, \frac{n-p}{p-1}\right)(1-r) \leq \left(G_p f\right)(r) \leq \theta_f\, r^{\frac{p-n}{p-1}}. \tag{14}$$

Remark 2.5.1 *The number* θ_f *is a weighted* L_∞*-norm for the function* $G_p f$.

Proof. The estimate from above follows by the definition of θ_f. For the estimate from below consider $g(r) = r^{\frac{n-p}{p-1}} \left(G_p f\right)(r)$. Since $G_p f \in C[0,1]$ we find $g(0) = g(1) = 0$. Hence g has an global maximum inside, say in r_0, and $\theta_f = g(r_0)$. By Lemma 2.2 we have

$$\begin{aligned} \left(G_p f\right)(r) &\geq \frac{\left(G_p f\right)(r_0)}{\left(G_p d_{r_0}\right)(r_0)} \left(G_p d_{r_0}\right)(r) \\ &= \frac{\theta_f\, r_0^{\frac{p-n}{p-1}}}{r_0^{\frac{p-n}{p-1}} - 1} \left(\max\left(r_0, r\right)^{\frac{p-n}{p-1}} - 1\right) \end{aligned} \tag{15}$$

and using that $r^{\frac{p-n}{p-1}} - 1 \geq \frac{n-p}{p-1}(1-r)$, we continue (15) by

$$\begin{aligned} &\geq \frac{\theta_f\, r_0^{\frac{p-n}{p-1}}}{r_0^{\frac{p-n}{p-1}} - 1} \min\left(r_0^{\frac{p-n}{p-1}} - 1, \frac{n-p}{p-1}(1-r)\right) \\ &= \theta_f \min\left(r_0^{\frac{p-n}{p-1}}, \frac{n-p}{p-1} \frac{r_0^{\frac{p-n}{p-1}}}{r_0^{\frac{p-n}{p-1}} - 1}(1-r)\right) \\ &\geq \theta_f \min\left(1, \frac{n-p}{p-1}(1-r)\right) = \theta_f \min\left(1, \frac{n-p}{p-1}\right)(1-r). \end{aligned}$$

□

Remark 2.5.2 *It is crucial in this proof that we are able to give an estimate independent of* r_0. *Note that any larger exponent, say* $\alpha > \frac{p-n}{p-1}$ *and* $\tilde{\theta}_f = \sup_{0<r\leq 1} r^{-\alpha} \left(G_p f\right)(r)$ *fails to give a uniform estimate from below.*

2.4. Estimates for G_p applied to a singular function

Finally we will show the following two estimates that will be used later.

Lemma 2.6 *Let* $\alpha \in [0, n)$ *and set* $g_\alpha(r) = r^{-\alpha}$ *if* $\alpha > 0$ *and* $g_0(r) = 1 - \log r$, *then*

$$\left(G_p g_\alpha\right)(r) \leq c_{n,p,\alpha} \begin{cases} 1 - r & \text{if } \alpha < p, \\ -\log r & \text{if } \alpha = p, \\ r^{\frac{p-\alpha}{p-1}} - 1 & \text{if } \alpha > p. \end{cases} \tag{16}$$

Proof. Let $\alpha \in (0, n)$. Since $\alpha < n$ holds, a straightforward computation yields

$$\left(G_p g_\alpha\right)(r) = \int_r^1 \left(\int_0^t \left(\frac{s}{t}\right)^{n-1} s^{-\alpha} ds\right)^{\frac{1}{p-1}} dt =$$

$$= (n-\alpha)^{\frac{-1}{p-1}} \int_r^1 t^{\frac{1-\alpha}{p-1}} dt = \tilde{c}_{n,p,\alpha} \begin{cases} 1 - r^{\frac{p-\alpha}{p-1}} & \text{if } \alpha < p, \\ -\log r & \text{if } \alpha = p, \\ r^{\frac{p-\alpha}{p-1}} - 1 & \text{if } \alpha > p, \end{cases} \tag{17}$$

implying (16). For $\alpha = 0$ a similar computation shows that for some $c_{n,p} > 0$ it holds that $(G_p g_0)(r) \leq c_{n,p}(1-r)$. □

3. Verification of the main condition

First we will show a comparison between the solution operator G_p and the iterated $G_p \circ \phi_p \circ G_p$.

Lemma 3.1 *Suppose that $p \geq 2$ satisfies $p > \frac{1}{2}n$. Let $a \in C[0,1]$ with $a \geq 0$. Then there is a constant $C_{a,p,n}$ such that for all $f \in C[0,1]$:*

$$(G_p(|\phi_p \circ (a \cdot G_p)(f)|))(r) \leq C_{a,p,n}\, G_p |f|(r). \tag{18}$$

Remark 3.1.1 *The major restriction here is $p > \frac{1}{2}n$. Although we do expect this lemma to hold for all $p \geq 2$ a proof will be much more involved. The reason is the following. For $p > \frac{1}{2}n$ we are able to characterize $G_p f$ by one number $G_p f(0)$ (for $p > n$, see (12)), $\tilde{\theta}_f$ (for $p = n$, see (13)) or θ_f (for $p < n$, see (14)). Indeed this number is used to find uniform estimates from below for $G_p f$ and from above for $(G_p \circ \phi_p \circ G_p)(f)$. Such estimates in terms of one number follow from Lemma 2.6 only if $p > \frac{1}{2}n$. Whenever $p \in \left(2, \frac{1}{2}n\right]$ such characterization by one number does not seem to be sufficient and consequently it will be necessary to capture the behavior of $G_p f$ in a more elaborate way.*

Proof. Since $(G_p(|\phi_p \circ (a.G_p)(f)|))(r) \leq \|a\|_\infty (G_p \circ \phi_p \circ G_p)|f|(r)$ it will be sufficient to consider

$$H_p f := (G_p \circ \phi_p \circ G_p)(f)$$

for $0 < f \in C[0,1]$. Let us denote by $\mathbf{1}$ the function $\mathbf{1}(x) \equiv 1$. If $p > n$ then the estimates in Lemma 2.3 imply that

$$\begin{aligned} H_p f(r) &\leq (G_p f)(0)\,(G_p \circ \phi_p)(\mathbf{1})(r) \\ &\leq (G_p f)(0) \int_r^1 \left(\int_0^t \left(\frac{s}{t}\right)^{n-1} \mathbf{1}\, ds \right)^{\frac{1}{p-1}} dt \\ &\leq (G_p f)(0)\,(1-r) \leq \frac{p-1}{p-n} (G_p f)(r). \end{aligned} \tag{19}$$

For $p \in \left(\frac{1}{2}n, n\right)$ we have

$$\begin{aligned} H_p f(r) &\leq (G_p \circ \phi_p)\left(\theta_f (\cdot)^{\frac{p-n}{p-1}}\right)(r) \\ &\leq \theta_f \int_r^1 \left(\int_0^t \left(\frac{s}{t}\right)^{n-1} s^{(p-n)} ds \right)^{\frac{1}{p-1}} dt \\ &= \frac{1}{p^{\frac{1}{p-1}}} \theta_f \int_r^1 t^{\frac{p-n+1}{p-1}} dt = \frac{p-1}{p^{\frac{1}{p-1}}(2p-n)} \theta_f \left(1 - r^{\frac{2p-n}{p-1}}\right) \\ &\leq \frac{p-1}{p^{\frac{1}{p-1}}(2p-n)} \max\left(1, \tfrac{2p-n}{p-1}\right) \theta_f\,(1-r) \\ &\leq \frac{p-1}{p^{\frac{1}{p-1}}(2p-n)} \frac{\max\left(1, \frac{2p-n}{p-1}\right)}{\min\left(1, \frac{n-p}{p-1}\right)} (G_p f)(r). \end{aligned}$$

Finally the case $p = n$. It follows that

$$\begin{aligned} H_p f(r) &\leq (G_p \circ \phi_p)(\theta_f (1 - \log \cdot))(r) \\ &\leq \theta_f \int_r^1 \left(\int_0^t \left(\frac{s}{t}\right)^{n-1} (1 - \log s)^{n-1} ds \right)^{\frac{1}{n-1}} dt \\ &\leq c_n \theta_f \int_r^1 t^{\frac{1}{n-1}} (1 - \log t)\, dt \\ &\leq \tilde{c}_n \theta_f (1 - r) \leq \tilde{c}_n (G_p f)(r). \end{aligned}$$

□

For such multiplication by a it is obvious that it is sufficient to consider the positive operator where a is replaced by $|a|$. For more general operators let us introduce a splitting in a positive and a negative part.

Lemma 3.2 *Suppose that V_p satisfies Condition 1.1. Defining for $f \in C[0,1]$*

$$V_G^+(f)(r) = \max\{0, (V_p \circ G_p(f))(r)\} \tag{20}$$

$$V_G^-(f)(r) = -\min\{0, (V_p \circ G_p(f))(r)\} \tag{21}$$

we find that $V_G^\pm$ are continuous from $C[0,1]$ to $C[0,1]$. Moreover, there exist $c_V', C_V' > 0$ such that

$$\left\|V_G^\pm(f)\right\|_{C[0,1]} \leq c_V' \|f\|_{C[0,1]} \text{ for all } f \in C[0,1]; \tag{22}$$

$$G_p \circ V_G^\pm(f)(r) \leq C_V' G_p(|f|)(r) \text{ for all } f \in C[0,1]. \tag{23}$$

Proof. The continuity is straightforward. By Condition 1.1.ii one finds

$$\left\|V_G^\pm(f)\right\|_\infty \leq \|V_p \circ G_p f\|_\infty \leq c_V^p \|G_p f\|_{C^1[0,1]}^{p-1} \leq c_V^p 2^{p-1} \|f\|_\infty.$$

By Condition 1.1.iii

$$G_p \circ V_G^\pm(f)(r) \leq G_p(|V_p \circ G_p f|)(r) \leq C_{V,p,n} G_p(|f|)(r).$$

□

Next we address the perturbation by a derivative.

Lemma 3.3 *Suppose that $p \geq 2$ satisfies $p > n$. Let $b \in C[0,1]$. Then there is a constant $C_{b,p,n}$ such that for all $f \in C[0,1]$:*

$$\left(G_p \left| \phi_p \circ \left(b.\frac{d}{dr} \circ G_p \right) \right| \right) f(r) \leq C_{b,p,n} G_p |f|(r).$$

Hence for all p, n, b as above V_p defined by $V_p(u)(r) = \phi_p(b(r) u'(r))$ satisfies Condition 1.1.

Proof. Note that

$$\left(\frac{d}{dr} \circ G_p \right) f(r) = -\phi_p^{inv} \left(\int_0^r \left(\frac{s}{r}\right)^{n-1} f(s)\, ds \right)$$

implies that

$$\begin{aligned} V_G^{\pm} f(r) &= \phi_p^{inv}\left(b^{\mp}(r)\right)\left(\int_0^r \left(\frac{s}{r}\right)^{n-1} f(s)\, ds\right)^{\pm} + \phi_p^{inv}\left(b^{\pm}(r)\right)\left(\int_0^r \left(\frac{s}{r}\right)^{n-1} f(s)\, ds\right)^{\mp} \\ &\leq \phi_p^{inv}\left(|b(r)|\right)\int_0^r \left(\frac{s}{r}\right)^{n-1} |f|(s)\, ds \end{aligned}$$

Since $p > n$ we may proceed similar as (19) for $f > 0$ starting with

$$G_p \circ V_G^{+} f(r) \leq \|b\|_\infty\, G_p(f)(0)\, \left(G_p \circ V_G^{+}\right)(\mathbf{1})(r).$$

□

For the coupled system we have to deal with $G_p \circ \phi_p \circ G_p \circ \phi_p \circ G_p$. It will not be much more trouble to have a p-Laplacian with another exponent (say q) in the second equation as long as the homogeneity fits. In that case we would have to consider $G_p \circ \phi_p \circ G_q \circ \phi_q \circ G_p$, with G_q and ϕ_q defined in Notation 1.1 with the obvious replacement of p by q.

Lemma 3.4 *Suppose that $n, p, q \geq 2$ are such that $n < 2p + q\frac{p-1}{q-1}$. Then there is a constant $C_{p,q,n}$ such that for all $f \in C[0,1]$ with $f \geq 0$:*

$$\left(G_p \circ \phi_p \circ G_q \circ \phi_q \circ G_p\right) f(r) \leq C_{p,n}\, G_p f(r).$$

In other words, for n, p, q as above Condition 1.1 is satisfied for $V_p = \phi_p \circ G_q \circ \phi_q \circ G_p \circ \phi_p$.

Corollary 3.5 *Suppose that $p > 2$ and $n < 3p$. Then $V_p = \phi_p \circ G_p \circ \phi_p \circ G_p \circ \phi_p$ satisfies Condition 1.1.*

Proof. (of Lemma 3.4) Let us denote $H_{p,q} f = \left(G_p \circ \phi_p \circ G_q \circ \phi_q \circ G_p\right) f$. If $n < p$ then the estimates in Lemma 2.3 imply that

$$\begin{aligned} H_{p,q} f(r) &\leq (G_p f)(0)\left(G_p \circ \phi_p \circ G_q \circ \phi_q\right)(\mathbf{1})(r) \\ &\leq (G_p f)(0)\,(1-r) \leq \frac{p-1}{p-n}(G_p f)(r). \end{aligned}$$

For $n = p$ Lemma 2.6 implies that

$$\begin{aligned} H_{p,q} f(r) &\leq \tilde{\theta}_f\, \left(G_p \circ \phi_p \circ G_q \circ \phi_q\right)(1 - \log(\cdot))(r) \\ &\leq c_{n,q}\tilde{\theta}_f\, \left(G_p \circ \phi_p (1 - \cdot)\right)(r) \leq c_{n,p,q}(1-r). \end{aligned}$$

For $n > p$ we have by Lemma 2.4 and Lemma 2.6 that with $H_p f$ as in a Lemma 3.1

$$\begin{aligned} H_p f(r) &\leq \left(G_q \circ \phi_q\right)\left(\theta_f (\cdot)^{\frac{p-n}{p-1}}\right)(r) = \theta_f\left(G_q (\cdot)^{\frac{q-1}{p-1}(p-n)}\right)(r) \leq \\ &\leq c_{p,q,n}\,\theta_f \begin{cases} 1-r & \text{if } 1 + \frac{1}{q-1} + \frac{p-n}{p-1} > 0, \\ -\log r & \text{if } 1 + \frac{1}{q-1} + \frac{p-n}{p-1} = 0, \\ r^{1+\frac{1}{q-1}+\frac{p-n}{p-1}} - 1 & \text{if } 1 + \frac{1}{q-1} + \frac{p-n}{p-1} < 0. \end{cases} \end{aligned}$$

and hence again by Lemma 2.6, depending on the sign of

$$\gamma_{p,q,n} = \frac{p + (p-1)\left(1 + \frac{1}{q-1} + \frac{p-n}{p-1}\right)}{p-1} = 2 + \frac{1}{p-1} + \frac{1}{q-1} + \frac{p-n}{p-1}$$

then

$$H_{p,q}f\left(r\right)\leq\tilde{c}_{p,q,n}\,\theta_{f}\begin{cases}1-r & \text{if } \gamma_{p,q,n}>0,\\ -\log r & \text{if } \gamma_{p,q,n}=0,\\ r^{\gamma_{p,q,n}}-1 & \text{if } \gamma_{p,q,n}<0.\end{cases}$$

Since $G_pf\left(r\right)\geq c_{p,n}\theta_f\left(1-r\right)$ we find the result of the lemma whenever $\gamma_{p,q,n}>0$, that is, for $n<2p+q\frac{p-1}{q-1}$. □

Lemma 3.6 *Let $p,q\geq 2$ and suppose that $Au\left(r\right)=\int_0^1 a\left(r,s\right)u\left(s\right)s^{n-1}ds$ is such that for some $\gamma<1+\frac{1}{p-1}$*

$$\begin{cases}\textit{if } p>n \textit{ then} & \int_0^1 \left|a\left(r,s\right)\right|\,s^{n-1}ds\leq C\,r^{-\gamma}\\ \textit{if } p=n \textit{ then} & \int_0^1 \left|a\left(r,s\right)\right|\,\left(1-\log s\right)s^{n-1}ds\leq C\,r^{-\gamma}\\ \textit{if } p<n \textit{ then} & \int_0^1 \left|a\left(r,s\right)\right|\,s^{n-\frac{n-1}{p-1}}ds\leq C\,r^{-\gamma}\end{cases}\tag{24}$$

Then operator $V_p=\phi_p\circ A$ satisfies Condition 1.1.

Proof. Set $a\left(r,s\right)=a^{+}\left(r,s\right)-a^{-}\left(r,s\right)$ with $a^{+},a^{-}\geq 0$ and denote $g_\alpha\left(r\right)=r^{-\alpha}$ First suppose that $p>n$. We find by Lemma 2.6 if $\left(p-1\right)\alpha<p$ that

$$\begin{aligned}&\left(G_p\circ V_G^{+}f\right)\left(r\right)\leq\left(G_pf\right)\left(0\right)\;G_p\circ\phi_p\left(\int_0^1 a^{+}\left(\cdot,s\right)s^{n-1}ds\right)\left(r\right)\leq\\ \leq\;\;&C\,\left(G_pf\right)\left(0\right)\;\left(G_p\circ\phi_p\circ g_\alpha\right)\left(r\right)\;=\;C\,\left(G_pf\right)\left(0\right)\;G_p\circ g_{\left(p-1\right)\alpha}\left(r\right)\leq\\ \leq\;\;&C_{p,n,\alpha}\,\left(G_pf\right)\left(0\right)\,\left(1-r\right)\end{aligned}$$

The condition $\left(p-1\right)\alpha<p$ coincides with $\alpha<1+\frac{1}{p-1}$.

For $p<n$ we proceed for $\left(n-p\right)\alpha<p$ by

$$\left(G_p\circ V_G^{+}f\right)\left(r\right)\leq G_p\circ\phi_p\left(\int_0^1 a^{+}\left(\cdot,s\right)\,\left(G_pf\right)\left(s\right)\,s^{n-1}ds\right)\left(r\right)\leq$$

$$\leq\theta_f\;G_p\circ\phi_p\left(\int_0^1 a^{+}\left(\cdot,s\right)\,s^{\frac{p-n}{p-1}}\,s^{n-1}ds\right)\left(r\right)\leq C_{p,n,\alpha}\,\theta_f\,\left(1-r\right).$$ □

4. Main proofs

4.1. Comparison results for G_p.

In this section we compare the Green operator for the perturbed and the unperturbed right hand side. First we need an elementary estimate:

Lemma 4.1 *Suppose that $p\geq 2$. For $a,b\geq 0$ it holds that*

$$\phi_p^{inv}\left(a-b\right)-\phi_p^{inv}\left(a\right)\geq -2\,\phi_p^{inv}\left(b\right).$$

Proof. For $a\geq b$ we may use Minkowski's inequality:

$$\phi_p^{inv}\left(a-b\right)-\phi_p^{inv}\left(a\right)=\phi_p^{inv}\left(a-b\right)-\phi_p^{inv}\left(a-b+b\right)\geq$$

$$\geq\left(a-b\right)^{\frac{1}{p-1}}-\left(\left(a-b\right)^{\frac{1}{p-1}}+b^{\frac{1}{p-1}}\right)=-\,b^{\frac{1}{p-1}}.$$

If $a \leq b$ we proceed by:

$$\phi_p^{inv}(a-b) - \phi_p^{inv}(a) = -\left((b-a)^{\frac{1}{p-1}} + a^{\frac{1}{p-1}}\right) \geq$$

$$\geq -\left(b^{\frac{1}{p-1}} + b^{\frac{1}{p-1}}\right) = -2\, b^{\frac{1}{p-1}}. \qquad \square$$

We will also need the following order result:

Lemma 4.2 *Let $g_1, g_2 \in C[0,1]$ with $g_1 \leq g_2$. Then for all $r \in [0,1]$:*

$$-G_p(g_1)'(r) \leq -G_p(g_2)'(r), \tag{25}$$

$$G_p(g_1)(r) \leq G_p(g_2)(r). \tag{26}$$

Proof. Directly from (2):$-(G_p f)'(r) = r^{\frac{1-n}{p-1}} \phi_p^{inv}\left(\int_0^r s^{n-1} f(s) ds\right).$ $\square$

Lemma 4.3 *Let $p \geq 2$ and let $f, g \in C[0,1]$ with $f \geq 0$. Then for all $r \in [0,1]$ one finds:*

$$\left|(G_p(f+g))'(r) - (G_p f)'(r)\right| \leq 2\left|(G_p|g|)'(r)\right|. \tag{27}$$

If moreover $|g(s)| \leq \theta\, r^{-\alpha}$ for some $\alpha \in [0,n)$ and $\theta > 0$, then the following estimate holds with $C = 2(n-\alpha)^{\frac{-1}{p-1}}$:

$$\left|(G_p(f+g))'(r) - (G_p f)'(r)\right| \leq C\, \theta^{\frac{1}{p-1}}\, r^{\frac{1-\alpha}{p-1}}. \tag{28}$$

Remark 4.3.1 *Note that $\left|(G_p|g|)'(r)\right| \leq r^{\frac{1-n}{p-1}} \|g\|_{L_1}^{\frac{1}{p-1}}$ where $\|g\|_{L_1} = \int_0^1 |g(s)|\, s^{n-1} ds$.*

Proof. By Lemma 4.2 we find

$$-(G_p(f+g))'(r) + (G_p(f))'(r) \leq -(G_p(f+|g|))'(r) + (G_p(f))'(r).$$

Using that $\frac{1}{p-1} \in (0,1]$ implies $|a+b|^{\frac{1}{p-1}} \leq |a|^{\frac{1}{p-1}} + |b|^{\frac{1}{p-1}}$, we find with

$$a = \int_0^r s^{n-1} f(s) ds \text{ and } b = \int_0^r s^{n-1} |g(s)|\, ds$$

that

$$-(G_p(f+|g|))'(r) + (G_p(f))'(r) \leq -(G_p(|g|))'(r)$$

Lemma 4.1 shows

$$-(G_p(f+g))'(r) + (G_p(f))'(r) \geq$$

$$\geq -(G_p(f-|g|))'(r) + (G_p(f))'(r) \geq 2(G_p(|g|))'(r).$$

The estimate of (28) follows by

$$r^{\frac{1-n}{p-1}} \left(\int_0^r s^{-\alpha}\, s^{n-1} ds\right)^{\frac{1}{p-1}} = r^{\frac{1-n}{p-1}} \left(\tfrac{1}{n-\alpha} r^{n-\alpha}\right)^{\frac{1}{p-1}}. \qquad \square$$

Corollary 4.4 *Let $p \geq 2$ and let $f, g \in C[0,1]$ with $f \geq 0$. Then for all $r \in [0,1]$ one finds:*

$$(G_p f)(r) - 2\,(G_p|g|)(r) \leq (G_p(f+g))(r) \leq (G_p f)(r) + 2\,(G_p|g|)(r). \tag{29}$$

Proof. The result follows by (27) and an integration from $r = 1$. $\square$

4.2. A fixed point argument

For (1) one might obtain a solution when λ is small by the following iteration procedure. Defining $S_{\lambda,p} : C[0,1] \times C^1[0,1] \to C^1[0,1]$ by

$$S_{\lambda,p}(f;u) := G_p(f - \lambda V_p(u)). \tag{30}$$

one considers the iteration $u_0 = G_p f$, and

$$u_{n+1} = G_p(f - \lambda V_p(u_n)) \text{ for } n \in \mathbb{N}.$$

Since the present problem does not satisfy an order preservation such an iteration might result in a sequence that does have a converging subsequence, but that is not converging itself. For example it could happen that $u_{2n} \to \overline{u}$ and $u_{2n+1} \to \underline{u} \neq \overline{u}$. The functions $\overline{u}$ and $\underline{u}$ do satisfy an 4^{th}-order system but do not necessarily satisfy (1). Instead of using such an iteration we will use a fixed point argument for existence of a solution to

$$u \mapsto S_{\lambda,p}(f;u).$$

For a survey on fixed point methods see [1].

Proposition 4.5 *Let $p \geq 2$ and $\lambda > 0$. Suppose that V_p satisfies Condition 1.1. Then, with $C'_V > 0$ as in Lemma 3.2, the following holds for all $f, g \in C[0,1]$ with $f > 0$*

$$|S_{\lambda,p}(f;G_p(g))(r) - G_p(f)(r)| \leq 2C'_{V,p,n}\,\lambda^{\frac{1}{p-1}}\,G_p(|g|)(r).$$

Proof. By Lemma 4.3 and Lemma 4.2 one finds

$$\begin{aligned}(S_{\lambda,p}(f;G_p(g)))(r) - (G_p f)(r) &= G_p(f - \lambda V_p(G_p(g)))(r) - (G_p f)(r) \leq \\ &\leq G_p\left(f + \lambda V_G^-(g)\right)(r) - (G_p f)(r) \leq 2\left(G_p\left(\lambda V_G^-(g)\right)\right)(r)\end{aligned}$$

and similarly

$$\begin{aligned}(S_{\lambda,p}(f;G_p(g)))(r) - (G_p f)(r) &\geq G_p\left(f - \lambda V_G^+(g)\right)(r) - (G_p f)(r) \geq \\ &\geq -2\left(G_p\left(\lambda V_G^+(g)\right)\right)(r).\end{aligned}$$

Condition 1.1 and Lemma 3.2 imply that for all $r \in [0,1]$

$$\begin{aligned}\left(G_p\left(\lambda V_G^-(g)\right)\right)(r) &\leq 2C'_V\,\lambda^{\frac{1}{p-1}}G_p(|g|)(r),\\ \left(G_p\left(\lambda V_G^+(g)\right)\right)(r) &\leq 2C'_V\,\lambda^{\frac{1}{p-1}}G_p(|g|)(r),\end{aligned}$$

completing the estimate. □

Corollary 4.6 *With p, λ, V_p and $C'_{V,p,n}$ as in Proposition 4.5, setting*

$$\hat{\lambda}_p = \left(\tfrac{1}{8}C'_{V,p,n}\right)^{p-1}, \tag{31}$$

it follows that for all $\lambda \in \left[0, \hat{\lambda}_p\right]$, $f, g \in C[0,1]$ with $f > 0$ and

$$G_p(|g|)(r) \leq 2G_p(f)(r) \text{ for all } r \in [0,1],$$

that $S_{\lambda,p}(f;G_p(g))$ satisfies

$$0 < \tfrac{1}{2}G_p f(r) \leq S_{\lambda,p}(f;G_p(g))(r) \leq \tfrac{3}{2}G_p f(r) \text{ for all } r \in [0,1].$$

- **Proof of Theorem 1.2**

An apriori bound. We will first show that for λ sufficiently small every fixed point of $u = S_{\lambda,p}(f;u)$ will necessarily lie in $\left[\frac{1}{2}G_p(f), \frac{3}{2}G_p(f)\right]$. Indeed, using Corollary 4.4 twice and Condition 1.1 we find that

$$\begin{aligned} G_p\left|V_p(u)\right|(r) &= G_p\left|V_p(S_{\lambda,p}(f;u))\right|(r) \\ &= G_p\left|V_pG_p(f-\lambda V_p(u))\right|(r) \\ &\le 2\lambda^{\frac{1}{p-1}}C_{V,p,n}\left(G_p\left|f-\lambda V_p(u)\right|\right)(r) \\ &\le 2\lambda^{\frac{1}{p-1}}C_{V,p,n}\left(G_p\left(f+\lambda\left|V_p(u)\right|\right)\right)(r) \\ &\le 2\lambda^{\frac{1}{p-1}}C_{V,p,n}G_pf(r)+4\lambda^{\frac{2}{p-1}}C_{V,p,n}G_p\left|V_p(u)\right|(r) \end{aligned}$$

Hence

$$G_p\left|V_p(u)\right|(r) \le \frac{2\lambda^{\frac{1}{p-1}}C_{V,p,n}}{1-4\lambda^{\frac{2}{p-1}}C_{V,p,n}}G_pf(r)$$

and for $\lambda \in \left[0, \lambda_p^\circ\right]$ with

$$\lambda_p^\circ = \left(8C_{V,p,n}\right)^{\frac{1-p}{2}} \tag{32}$$

we get

$$G_p\left|V_p(u)\right|(r) \le 4\lambda^{\frac{1}{p-1}}C_{V,p,n}G_pf(r). \tag{33}$$

Again using Corollary 4.4 we have

$$\begin{aligned} \left|u(r)-G_p(f)(r)\right| &= \left|S_{\lambda,p}(f;u)(r)-G_p(f)(r)\right| \\ &\le 2\lambda^{\frac{1}{p-1}}G_p\left|V_p(u)\right|(r). \end{aligned} \tag{34}$$

Combining (33) and (34) shows $u \in \left[\frac{1}{2}G_p(f), \frac{3}{2}G_p(f)\right]$ for $\lambda \in \left[0, \lambda_p^\circ\right]$.

Existence. Fix $f \in C[0,1]$ and let us consider

$$D = \left\{u \in C^1[0,1] \text{ with } \|u\|_{C^1} \le 2\left\|G_pf\right\|_{C^1}\right\}.$$

The mapping $u \mapsto S_{\lambda,p}(f;u)$ from $C^1[0,1]$ to itself is completely continuous and maps D into D. Indeed by Lemma 4.3, the properties of G_p and the assumption on V_p one finds that

$$\begin{aligned} &\left|\left(S_{\lambda,p}(f;u)\right)' - \left(G_p(f)\right)'\right| \le 2\left|\left(G_p(\lambda V_p(u))\right)'\right| \le \\ &\le 4\lambda^{\frac{1}{p-1}}\left\|V_p(u)\right\|_\infty^{1/(p-1)} \le 4\,c_V^p\,\lambda^{\frac{1}{p-1}}\|u\|_{C^1[0,1]}, \end{aligned} \tag{35}$$

and since $S_{\lambda,p}(f;u)$ and $G_p(f)$ are zero in $r=1$ it follows for $\lambda \in \left[0, \lambda_p^*\right]$, where

$$\lambda_p^* := \left(8c_V^p\right)^{1-p}, \tag{36}$$

that

$$\begin{aligned} &\left\|S_{\lambda,p}(f;u)\right\|_{C^1[0,1]} \le \left\|S_{\lambda,p}(f;u)-G_p(f)\right\|_{C^1[0,1]} + \left\|G_p(f)\right\|_{C^1[0,1]} \\ = \ &\left\|S_{\lambda,p}(f;u)-G_p(f)\right\|_\infty + \left\|S_{\lambda,p}(f;u)'-G_p(f)'\right\|_\infty + \left\|G_p(f)\right\|_{C^1[0,1]} \\ \le \ &\tfrac{1}{2}\left\|G_p(f)\right\|_\infty + 4\,c_V^p\,\lambda^{\frac{1}{p-1}}\|u\|_{C^1[0,1]} + \left\|G_p(f)\right\|_{C^1[0,1]} \le 2\left\|G_p(f)\right\|_{C^1[0,1]}. \end{aligned}$$

By Schauder's fixed point Theorem there exists $u \in D$ such that $u = S_{\lambda,p}(f;u)$.

Conclusion. There exists a solution $u \in \left[\frac{1}{2}G_pf, \frac{3}{2}G_pf\right]$ whenever $\lambda \in \left[0, \lambda_p^\diamond\right]$ with $\lambda_p^\diamond = \min\left\{\lambda_p^*, \lambda_p^\circ\right\}$ defined by (32) and (36). □

References

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev. 18 (1976), 620-709. 16

[2] Kai Lai Chung, and Zhong Xin Zhao, From Brownian motion to Schrödinger's equation. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 312. Springer-Verlag, Berlin, 1995. 3

[3] J. Fleckinger, J. Hernandez and F. de Thélin, On maximum principles and existence of positive solutions for some cooperative elliptic systems. Differential Integral Equations 8 (1995), no. 1, 69–85. 4

[4] P. Freitas and G. Sweers, Positivity results for a nonlocal elliptic equation. Proc. Roy. Soc. Edinburgh Sect. A 128 (1998), no. 4, 697–715. 4

[5] R. Manásevich and G. Sweers, A noncooperative elliptic system with p-Laplacians that preserves positivity. Nonlinear Anal. 36 (1999), no. 4, Ser. A: Theory Methods, 511–528. 4, 5, 7

[6] E. Mitidieri and G. Sweers, Weakly coupled elliptic systems and positivity, Math. Nachr. 173 (1995), 259-286. 4, 6

[7] R. G. Pinsky, Positive harmonic functions and diffusion. Cambridge Studies in Advanced Mathematics, 45. Cambridge University Press, Cambridge, 1995. 3

[8] G. Sweers, A strong maximum principle for a noncooperative elliptic system, SIAM J. Math. Anal. 20 (1989), 367-371. 4

[9] G. Sweers, Positivity for a strongly coupled elliptic system by Green function estimates. J. Geom. Anal. 4 (1994), no. 1, 121–142. 6

[10] J.L. Vázquez, A strong maximum principle for some quasilinear elliptic equations, Appl. Math.Optim 12 (1984), 191-202. 3

[11] Z. Zhao, Green functions for Schrödinger operator and conditioned Feynman-Kac gauge, J. Math. Anal. Appl. 116 (1986), 309-334. 3, 4, 6

Raul Manásevich:
Centro de Modelamiento Matemático
Universidad de Chile
Av. Blanco Encalada 2120 Piso 7
Santiago
Chile

manasevi@dim.uchile.cl

Guido Sweers:
Dept. of Applied Mathematical Analysis
Delft University of Technology
P.O. Box 5031
2600 GA Delft
The Netherlands

g.h.sweers@its.tudelft.nl