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Solutions with internal jump for an autonomous elliptic system of FitzHugh-Nagumo type

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Abstract. Systems of elliptic partial differential equations which are coupled in a noncooperative way, such as the FitzHugh-Nagumo type studied in this paper, in general do not satisfy order preserving properties. This not only results in technical complications but also yields a richer solution structure. We prove the existence of multiple nontrivial solutions. In particular we show that there exists a solution with boundary layer type behaviour, and we will give evidence that this autonomous system for a certain range of parameters has a solution with both a boundary and an internal layer. The analysis uses results from bifurcation theory, variational methods, as well as some pointwise a priori estimates. The final section contains some numerically obtained results.

1. Introduction

We consider the following system of semilinear elliptic partial differential equations

$$(\mathrm{P}_{\varepsilon,\delta}) \qquad \begin{cases} -\varepsilon^2 \Delta u &= f(u) - v & \text{in } \Omega; \\ -\Delta v &= \delta u - \gamma v & \text{in } \Omega; \\ u &= v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in R^N, $N \geq 2$, with a C^3-boundary, and where the function $f(u) = u(1-u)(u-a)$, with $a \in \left(0, \frac{1}{2}\right)$, is the generic nonlinearity. The parameters δ and ε are assumed to be positive, while γ is some fixed number larger than $-\lambda_1$. Here λ_1 is the first eigenvalue of $-\Delta$ on Ω subjected to homogeneous Dirichlet boundary conditions. By a solution to $(\mathrm{P}_{\varepsilon,\delta})$ we mean a quadruple $(\varepsilon, \delta, u, v)$ with $u, v \in C^2\left(\overline{\Omega}\right)$ satisfying the equation and boundary conditions in a classical sense. For fixed $\varepsilon > 0$ we shall write (δ, u, v) instead of $(\varepsilon, \delta, u, v)$.

The nonlinearity in $(\mathrm{P}_{\varepsilon,\delta})$ appears in the one-dimensional FitzHugh-Nagumo system, which serves as a model for nerve conduction, and also in the Bonhoffer-van der Pol equation, a prototype for excitable media ([14]). In these systems the following type

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of parabolic system is considered.

$$
(1.1) \qquad \left\{ \begin{array}{rcll} U_t & = & \varepsilon^2 \Delta U + f(U) - V & \text{in } \mathrm{R}^+ \times \Omega; \\ \tau V_t & = & \Delta V + \delta U - \gamma V & \text{in } \mathrm{R}^+ \times \Omega; \\ U & = & V \;=\; 0 & \text{on } \mathrm{R}^+ \times \partial\Omega; \\ U & = & u_0; \quad V \;=\; v_0 & \text{on } \{t=0\} \times \Omega, \end{array} \right.
$$

with $\tau \geq 0$. We observe that solutions to $(\mathrm{P}_{\varepsilon,\delta})$ are equilibria to problem (1.1).

Systems similar to $(\mathrm{P}_{\varepsilon,\delta})$, also in multiple dimensions, were studied among others by Klaasen, Troy, De Figueiredo and Mitidieri, [10], [11], [7]. More recent results on the system above by the present authors are found in [18], [17] and [16]. In these references the existence and qualitative properties of solutions were investigated.

We observe that when $\delta = 0$, then system $(\mathrm{P}_{\varepsilon,\delta})$ reduces to the scalar equation

$$
(1.2) \qquad \left\{ \begin{array}{rcll} -\varepsilon^2 \Delta u & = & f(u) & \text{in } \Omega; \\ u & = & 0 & \text{on } \partial\Omega. \end{array} \right.
$$

The nontrivial solutions of this scalar equation are an energy minimizing solution, which has a boundary layer and is almost constant in the interior, and a mountain pass solution, see [5] and [6]. Although the methods used in the studies of the full system are much more intricate than for the scalar equation one could say that the results proven up to now are in a sense analogies of the results for the scalar equation. In particular the existence of two nontrivial solutions, one a minimizer of an associated energy functional and the other of mountain pass type has been proven. The minimizer is a solution with a boundary layer. The other solution, a 'peak', is the mountain pass between the minimizer and the trivial solution. The first part of the present paper is involved with showing the existence of a boundary layer type solution for the system.

One reason for the system behaving similarly to the scalar equation is that the v component may be too small to really have an influence. Also, in system $(\mathrm{P}_{\varepsilon,\delta})$ the v component 'diffuses' much faster than the u component. In this respect the system differs from the systems studied in the references above where the diffusion rates of the u and v components were of comparable order.

However, the main purpose of this paper is to show that the system can have solutions which do not correspond to solutions of the scalar equation. To pinpoint the question:

Can the autonomous system $(\mathrm{P}_{\varepsilon,\delta})$ *have stable solutions with a more complicated structure than the boundary layer type solution?*

By a stable solution (u, v) for $(\mathrm{P}_{\varepsilon,\delta})$ we mean that if the initial values of problem (1.1) are in a suitable neighborhood of that (u, v), then as time evolves the solution to the parabolic problem tends to this equilibrium. With a 'more complicated structure' we have a solution in mind for which the u component exhibits a sharp interior layer.

The question of existence of such solutions is an interesting one because it is well known that for the autonomous scalar equations (1.2) such solutions do not exist. On the other hand, if one allows spatial inhomogeneity and consider the nonautonomous equation

$$
\left\{ \begin{array}{rcll} -\varepsilon^2 \Delta u & = & f(x, u) & \text{in } \Omega; \\ u & = & 0 & \text{on } \partial\Omega, \end{array} \right.
$$

it is known that such that pattern solutions may exist, see for example [2].

The idea behind our study is that, although the system is autonomous, the function v can serve as such a spatial inhomogeneity for the first equation. This feature shows a major difference between scalar equations and systems. We observe that the coupling between the two species also forms the main difficulty finding such solutions. A fixed v can lead to a pattern like u, but this u again influences the v and moreover, in the opposite direction. So in order to have such solutions there need to be some kind of balance between u and v.

What remains, as in the case of the scalar equation, is that system $(\mathrm{P}_{\varepsilon,\delta})$ has a variational structure. Our main result will show that for a certain range of δ and ε the minimizing solution for the system does have a more complicated behavior than the energy minimizing solution of the scalar equation. Although we cannot prove the actual behavior of that solution we do include numerical results that give some evidence on how these solutions may look.

2. Assumptions and notation

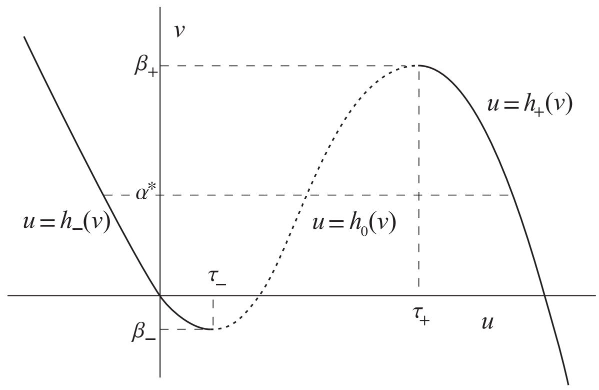
We will make the following assumptions on f, see also Figure 1.

Condition 2.1. We assume that

1. $f \in C^1(\mathrm{R})$ with $f(0) \geq 0$.
2. There exist closed intervals $J_- = (-\infty, \tau_-]$, $J_0 = [\tau_-, \tau_+]$ and $J_+ = [\tau_+, \infty)$ such that $f'(s) < 0$ for $s \in (-\infty, \tau_-) \cup (\tau_+, \infty)$ and f strictly increasing on J_0. We shall denote by
   - $I_\sigma := f(J_\sigma)$ and $h_\sigma : I_\sigma \to \mathrm{R}$ is the inverse of f restricted to J_σ for $\sigma \in \{-, 0, +\}$;
   - $\beta_\sigma = f(\tau_\sigma)$ for $\sigma \in \{-, +\}$;
3. It holds that $0 \in I_0 = I_- \cap I_+ = [\beta_-, \beta_+]$;
4. There exists a unique zero $0 < \alpha^* \in I_0$ of the function $j : I_0 \to \mathrm{R}$ defined by
$$j(\alpha) = \int_{h_-(\alpha)}^{h_+(\alpha)} (f(u) - \alpha)\, du.$$
   Moreover $j'(\alpha^*) < 0$.

Remark 2.2. Note that if f satisfies the conditions above, then for every α in the interior of I_0 the function $u \longmapsto f(u) - \alpha$ has three zeroes $\{h_-(\alpha), h_0(\alpha), h_+(\alpha)\}$. For $\sigma \in \{-, 0, +\}$ we let
$$\rho_\sigma = h_\sigma(0).$$

Figure 1: The nonlinearity f



Also we have for $\alpha \in I_0$ and $\alpha < \alpha^*$ then for all $\omega \in (h_-(\alpha), h_+(\alpha))$ it holds that

$$\int_{\omega}^{h_+(\alpha)} (f(u) - \alpha)\, du > 0. \tag{2.1}$$

Remark 2.3. Since we are only interested in solutions $(\varepsilon, \delta, u, v)$ for which it holds that $h_-(\beta_+) \le u \le h_+(\beta_-)$, we can modify f outside this interval in such a way that f and f' are bounded. Observe that because of the second linear equation in $(\mathrm{P}_{\varepsilon,\delta})$, such an L^∞ bound on u implies an L^∞ bound, dependent on δ, for v. Solutions to the system with the modified f and with $u \in [h_-(\beta_+), h_+(\beta_-)]$ are solutions to the original problem.

Example 2.4. A typical example of a nonlinearity satisfying Condition 2.1 is

$$f(u) = u(u-1)(a-u)$$

with $0 < a < 1/2$. In this case $\beta_\pm = f\left(\frac{a+1}{3} \pm \frac{1}{3}\sqrt{a^2 - a + 1}\right)$, $\rho_- = 0$, $\rho_0 = a$, $\rho_+ = 1$ and $\alpha^* = \frac{a+1}{3}$.

Definition 2.5.

1. For every $\nu > 0$ we define

$$\Omega_{(\nu)} = \{x \in \Omega\,;\ \mathrm{dist}(x, \partial\Omega) > \nu\}.$$

2. Let $u, g \in C(\bar{\Omega})$ and $\Sigma \subset \Omega$. We say $-\Delta w \le g$ in $\mathcal{D}'(\Sigma)$-sense, if for every $\varphi \in C_0^\infty(\Sigma)$ with $\varphi \ge 0$ it holds that

$$-\int_\Sigma w \Delta\varphi\, dx \le \int_\Sigma g\varphi\, dx.$$

3. A function $\underline{u} \in C(\bar{\Omega})$ is called a subsolution to the problem

$$\begin{cases} -\Delta u &= g(x,u) & \text{in } \Omega; \\ u &= 0 & \text{on } \partial\Omega, \end{cases}$$

if $-\Delta \underline{u} \leq g(x, \underline{u})$ in $\mathcal{D}'(\Omega)$-sense and $\underline{u} \leq 0$ on $\partial\Omega$. A supersolution is defined by reversing the inequality signs.

We shall work in the following abstract setting. Denote by $e_1 \in C^2(\bar{\Omega})$ the eigenfunction corresponding to the principal eigenvalue λ_1 of $-\Delta$, with homogeneous Dirichlet boundary conditions, normalized such that $\max e_1 = 1$. That is,

$$\begin{cases} -\Delta e_1 &= \lambda_1 e_1 & \text{in } \Omega; \\ e_1 &= 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

and $\max e_1 = 1$. Let Y be the space

$$Y = \left\{ u \in C(\bar{\Omega}) \ ; |u| \leq \mu e_1 \text{ for some } \mu \geq 0 \right\} \tag{2.3}$$

equipped with the norm

$$\|u\| = \inf \{ \mu \geq 0 \text{ such that } |u| \leq \mu e_1 \}.$$

With this norm Y is a Banach lattice, see for example [1]. Finally we let

$$X = Y \times Y. \tag{2.4}$$

3. Main results

Our first result concerns the existence of solutions. The important aspect of this theorem is that it gives the existence of solutions where the maximum of the v-component can be larger than the value α^*. As we shall see, it is this property which will cause solutions to the system to differ from the solutions for the scalar case.

In the construction of solutions to $(P_{\varepsilon,\delta})$ we shall fix ε small enough and use δ as a parameter. We denote by $\mathcal{S}_\varepsilon = \{(\delta, u, v)\}$ the solution set of $(P_{\varepsilon,\delta})$ in $\mathbb{R} \times X$. As was observed earlier, system $(P_{\varepsilon,0})$ corresponds to the scalar equation (1.2). It was proven by Clément c.s., [5] that there exists for ε small enough a nondegenerate nontrivial solution, say u_ε° to this problem. Using the nondegeneracy of the solution $(0, u_\varepsilon^\circ, 0)$ and a continuation argument we shall prove the following theorem.

Theorem 3.1. *Let for ε small enough u_ε° be the unique solution to* (1.2) *with maximum close to ρ_+ and let $\mathcal{C}_\varepsilon^+$ be the component of $(0, u_\varepsilon^*, 0)$ in $\mathcal{S}_\varepsilon \cap \mathbb{R}^+ \times X$, that is, $\mathcal{C}_\varepsilon^+$ is the maximal connected subset in $\mathcal{S}_\varepsilon \cap \mathbb{R}^+ \times X$ containing the element $(0, u_\varepsilon^\circ, 0)$. Given $\overline{\alpha} \in (\alpha^*, \beta_+)$, there exists $\varepsilon_{\overline{\alpha}}$ such that for all $\varepsilon < \varepsilon_{\overline{\alpha}}$ there exists an element $(\delta_\varepsilon, u_\varepsilon, v_\varepsilon) \in \mathcal{C}_\varepsilon^+$, such that $\max v_\varepsilon = \overline{\alpha}$. Moreover there exist $0 < \underline{\delta} < \overline{\delta}$ depending only on $\overline{\alpha}$ such that $\delta_\varepsilon \in [\underline{\delta}, \overline{\delta}]$.*

Next we consider the behaviour of the solutions obtained in the previous theorem. To do this we shall use the so called sweeping principle of McNabb, see [5] and also [12] and [20].

Denote, for $\alpha \in [0, \beta_+)$, by $(\widehat{\delta}_\alpha, \widehat{v}_\alpha)$ the unique solution to

$$\begin{cases} -\Delta v + \gamma v &= \delta h_+(v) \quad \text{in } \Omega; \\ v &= 0 \quad \text{on } \partial\Omega; \\ \max v &= \alpha, \end{cases}$$

see Lemma 5.2. The following theorem holds.

Theorem 3.2. *Let $\{(\delta_\varepsilon, u_\varepsilon, v_\varepsilon)\ ;\ \varepsilon < \varepsilon_{\overline{\alpha}}\}$ be as in Theorem 3.1 and let $(\widehat{\delta}_{\overline{\alpha}}, \widehat{v}_{\overline{\alpha}})$ be as above (see also Lemma 5.2). It holds true that*

1. $\lim_{\varepsilon\to 0^+} \delta_\varepsilon = \widehat{\delta}_{\overline{\alpha}}$;
2. $\lim_{\varepsilon\to 0^+} v_\varepsilon = \widehat{v}_{\overline{\alpha}}$ *in* $C^1(\overline{\Omega})$;
3. $\lim_{\varepsilon\to 0^+} u_\varepsilon = h_+(\widehat{v}_{\overline{\alpha}})$ *uniformly on compact subsets of* Ω.

To state our final results, we introduce the energy functional $\Phi_\varepsilon : W_0^{1,2}(\Omega) \to \mathrm{R}$ by

$$\Phi_\varepsilon(u) = \frac{\varepsilon^2}{2}\int_\Omega |\nabla u|^2\, dx + \frac{\delta_\varepsilon}{2}\int_\Omega u G_\gamma u\, dx - \int_\Omega F(u)\, dx,$$

where δ_ε is as in Theorem 3.1, $G_\gamma = (-\Delta + \gamma)_0^{-1}$ is the Green operator for $(-\Delta + \gamma)$ with respect to zero Dirichlet boundary condition and $F(u) = \int_0^u f(s)\, ds$. Without loss of generality we assume that f has been modified in such a way that $\Phi_{\cdot\varepsilon}$ is well defined and differentiable on $W_0^{1,2}(\Omega)$ as well as bounded from below. Moreover, the functional is coercive and lower semicontinuous, see [11], and hence it attains its infimum. That is, there exists a function that minimizes Φ_ϵ and is a solution to $(\mathrm{P}_{\varepsilon,\delta})$. This gives another way of proving existence of solutions. By explicitly constructing a function and using a careful comparison of energies, we prove the following multiplicity result.

Theorem 3.3. *There exists ε_* such that for all $\varepsilon < \varepsilon_*$ the function u_ε in Theorem 3.1 is not the global minimizer of the energy functional Φ_ε. In particular, if $\Phi_\varepsilon(u_\varepsilon) < 0$, the global minimizer is nontrivial and there exist at least two nontrivial solutions.*

We shall also give conditions under which we can deduce that $\Phi_\varepsilon(u_\varepsilon) < 0$ and hence have the existence of multiple nontrivial solutions.

Based on our construction of a function with lower energy than u_ε we make the following conjecture on the shape of the energy minimizing solution. We also give some numerical evidence in support of this conjecture.

Conjecture 3.4. *For fixed ε small enough there exists a smooth curve of solutions (δ, u, v) to $(\mathrm{P}_{\varepsilon,\delta})$ and values $\delta_1 < \delta_2 < \delta_3$ such that the curve has the following branches*

1. *for $\delta \in [0, \delta_2)$ a branch consisting of stable boundary layer solutions.*
2. *for $\delta \in (\delta_1, \delta_3)$ a branch consisting of stable boundary and interior layer solutions*
3. *for $\delta \in (\delta_1, \delta_3)$ a branch of unstable solutions with the u component having a boundary layer and a downward peak.*
4. *for $\delta \in [0, \delta_3)$ a branch of unstable solutions with the u component having an upward peak.*

4. Construction of solutions

Our method of proving the existence of solutions to problem $(\mathrm{P}_{\varepsilon,\delta})$ relies on the construction of appropriate order intervals $\mathcal{U}_\varepsilon = [\underline{u}_\varepsilon, \overline{u}_\varepsilon]$ and $\mathcal{V}_\delta = [\delta \underline{v}, \delta \overline{v}]$, in which the solution will be obtained. We want the functions $\underline{u}_\varepsilon$, $\overline{u}_\varepsilon$, $\underline{v}$ and $\overline{v}$ to have the following uniform properties.

- For every $v \in \mathcal{V}_\delta$ it holds that $\underline{u}_\varepsilon$, respectively $\overline{u}_\varepsilon$, is a subsolution, respectively supersolution, to the scalar equation
$$\begin{cases} -\varepsilon^2 \Delta u = f(\underline{u}_\varepsilon) - v & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$
- If $u \in \mathcal{U}_\varepsilon$ and
$$\begin{cases} -\Delta v = \delta u - \gamma v & \text{in } \Omega; \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$
then it follows that $v \in \mathcal{V}_\delta$.

4.1. The order interval for the u component

We keep in mind that because of the homogeneous boundary condition the v-component of a solution $(\varepsilon, \delta, u, v)$ to $(\mathrm{P}_{\varepsilon,\delta})$ will be small in some neighbourhood of the boundary of Ω. First we obtain a function which we shall use to construct the subsolutions near the boundary.

Proposition 4.1. *Suppose that f satisfies Condition 2.1 and let $\underline{\alpha} \in (f(0), \alpha^*)$ be fixed. Then there exist $\varepsilon_0 > 0$ and a function $U \in C^2\left(\overline{B}(0,1)\right)$ such that*

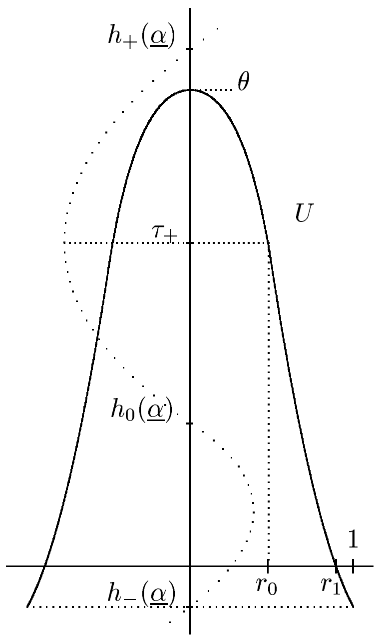
1. *U satisfies*
$$\begin{cases} -\varepsilon_0^2 \Delta U = f(U) - \underline{\alpha} & \text{in } B(0,1); \\ U = h_-(\underline{\alpha}) \leq 0 & \text{on } \partial B(0,1). \end{cases} \tag{4.1}$$
2. *U is radially symmetric with respect to the origin;*
3. *$U(0) > h_+(\alpha^*)$, $U'(0) = 0$ and $U'(r) < 0$ for all $r \in (0,1]$.*

Proof. Let $g(s) = f(s + h_-(\underline{\alpha})) - \underline{\alpha}$. By our choice of $\underline{\alpha}$ we have that $\underline{\alpha} \in I_0$ and hence g has three zeroes, $h_\sigma(\underline{\alpha}) - h_-(\underline{\alpha})$ for $\sigma \in \{-, 0, +\}$. Moreover, for all $\omega \in [0, h_+(\underline{\alpha}) - h_-(\underline{\alpha}))$ we have that
$$\int_\omega^{h_+(\underline{\alpha}) - h_-(\underline{\alpha})} g(s)\, ds = \int_{\omega + h_-(\underline{\alpha})}^{h_+(\underline{\alpha})} (f(s) - \underline{\alpha})\, ds > 0.$$
Using a result in [5], there exists for all ε small enough a solution w_ε to the problem
$$\begin{cases} -\varepsilon^2 \Delta w = g(w) & \text{in } B(0,1); \\ w = 0 & \text{on } \partial B(0,1), \end{cases}$$

with the properties that w is positive, radially symmetric, radially decreasing and

$$\max w_\varepsilon \to h_+(\underline{\alpha}) - h_-(\underline{\alpha}) \quad \text{as } \varepsilon \to 0^+.$$

Choose ε_0 such that $\max w_{\varepsilon_0} = w_{\varepsilon_0}(0) > h_+(\alpha^*) - h_-(\underline{\alpha})$ and let $U(x) = w_{\varepsilon_0} + h_-(\underline{\alpha})$. Then U satisfies (4.1) and $\max U > h_+(\alpha^*)$. □



For the function $U(x) = U(|x|)$, constructed in the proposition above for $\underline{\alpha}$ fixed we introduce the following notations.

- Let r_0 and r_1 be unique points in $(0,1)$ for which

(4.2)
$$\begin{aligned} U(r_0) &= \tau_+ = h_+(\beta_+) \quad \text{and} \\ U(r_1) &= 0. \end{aligned}$$

- $\theta := U(0)$.
- Let

(4.3)
$$\begin{aligned} c_0 &= \inf_{r\in[0,r_1)} \tfrac{U(r)}{r_1 - r} \quad \text{and} \\ r_2 &= r_0 + r_1. \end{aligned}$$

Observe that $U'(r_1) < 0$ implies $c_0 > 0$.

By the regularity assumption on the boundary of Ω, it satisfies an interior sphere condition. That is, there exists $\varepsilon_\Omega > 0$ such that for every $\varepsilon < \varepsilon_\Omega$ it holds that

$$\Omega = \bigcup \left\{ B(x,\varepsilon)\ ;\ x \in \Omega_{(\varepsilon)} \right\}.$$

We shall assume from now on that $\varepsilon < \varepsilon_1$ with

(4.4)
$$\varepsilon_1 = \frac{\varepsilon_0 \varepsilon_\Omega}{r_1}.$$

We define the following translations and rescalings of the function U:

$$Z_y(x) = U\left((\varepsilon_0/\varepsilon)(y - x)\right) \quad \text{for} \quad x \in B(y, r_1\varepsilon/\varepsilon_0).$$

It holds that

$$\begin{cases} -\varepsilon^2 \Delta Z_\varepsilon^{(y)} &= f(Z_\varepsilon^{(y)}) - \underline{\alpha} \qquad \text{in } B(y, r_1\varepsilon/\varepsilon_0); \\ Z_\varepsilon^{(y)} &= 0 \qquad \text{on } \partial B(y, r_1\varepsilon/\varepsilon_0). \end{cases}$$

Next we define for $x \in \{x \in \Omega\,;\, 0 < \operatorname{dist}(x, \partial\Omega) < 2r_1\varepsilon/\varepsilon_0\}$

$$\underline{U}_\varepsilon(x) = \sup\left\{ Z_\varepsilon^{(y)}(x)\ ;\ y \in \Omega \text{ such that } \operatorname{dist}(y, \partial\Omega) = r_1\varepsilon/\varepsilon_0 \right\}.$$

It holds that $\underline{U}_{\varepsilon} \in C\left(\bar{\Omega}\backslash\Omega_{(r_1\varepsilon/\varepsilon_0)}\right)$ and in $\mathcal{D}'\left(x \in \Omega\,;\, 0 < \operatorname{dist}\left(x, \partial\Omega\right) < 2r_1\varepsilon/\varepsilon_0\right)$-sense we have that

$$-\varepsilon^2 \Delta \underline{U}_{\varepsilon}\left(x\right) \leq f\left(\underline{U}_{\varepsilon}\left(x\right)\right) - \underline{\alpha}, \tag{4.5}$$

see for example [5]. Also

$$\begin{array}{rcll} \underline{U}_{\varepsilon}\left(x\right) & = & 0 & \text{if} \quad x \in \partial\Omega; \\ \underline{U}_{\varepsilon}\left(x\right) & = & \theta & \text{if} \quad \operatorname{dist}\left(x, \partial\Omega\right) = r_1\varepsilon/\varepsilon_0; \\ \underline{U}_{\varepsilon}\left(x\right) & = & \tau_+ & \text{if} \quad \operatorname{dist}\left(x, \partial\Omega\right) = r_2\varepsilon/\varepsilon_0. \end{array}$$

We extend $\underline{U}_{\varepsilon}$ to the whole of $\bar{\Omega}$ in the following way:

$$\underline{u}_{\varepsilon}\left(x\right) = \begin{cases} \underline{U}_{\varepsilon}\left(x\right) & \text{if } \bar{\Omega}\backslash\Omega_{(\varepsilon r_2/\varepsilon_0)} \\ \tau_+ & \text{if } \Omega_{(\varepsilon r_2/\varepsilon_0)}. \end{cases} \tag{4.6}$$

Then $\underline{u}_{\varepsilon}\left(x\right)$ is continuous and $\underline{u}_{\varepsilon}\left(x\right) = 0$ for $x \in \partial\Omega$. Moreover, for any function $v\left(x\right)$ such that $v\left(x\right) \leq \beta_+$ on Ω and $v\left(x\right) \leq \underline{\alpha}$ on $\Omega\backslash\Omega_{(\varepsilon r_2/\varepsilon_0)}$ we have in $\mathcal{D}'\left(\Omega\right)$ -sense that

$$-\varepsilon^2 \Delta \underline{u}_{\varepsilon} \leq f\left(\underline{u}_{\varepsilon}\right) - v.$$

This follows by direct integration, using the Green identity.

We now turn to the supersolution. We choose $\omega > 0$ such that $f'\left(s\right) + \omega \geq 0$ for all s, see Remark 2.3 and denote by $\overline{u}_{\varepsilon}$ the unique solution to

$$\begin{cases} -\varepsilon^2 \Delta u + \omega u = \omega \rho_+ & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.7}$$

If u is a solution to

$$\begin{cases} -\varepsilon^2 \Delta u = f\left(u\right) - v & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $v \geq 0$ and $u \leq \rho_+$, then it also holds that $u \leq \overline{u}_{\varepsilon}$. Indeed, by the choice of ω we have in Ω that

$$\begin{array}{rcl} \left(-\varepsilon^2 \Delta + \omega\right)\left(u - \overline{u}_{\varepsilon}\right) & = & f\left(u\right) - v + \omega\left(u - \rho_+\right) \\ & \leq & f\left(u\right) - f\left(\rho_+\right) - \omega\left(u - \rho_+\right) \leq 0, \end{array}$$

and the remark follows from the maximum principle.

The order interval which we shall use for the u-components is

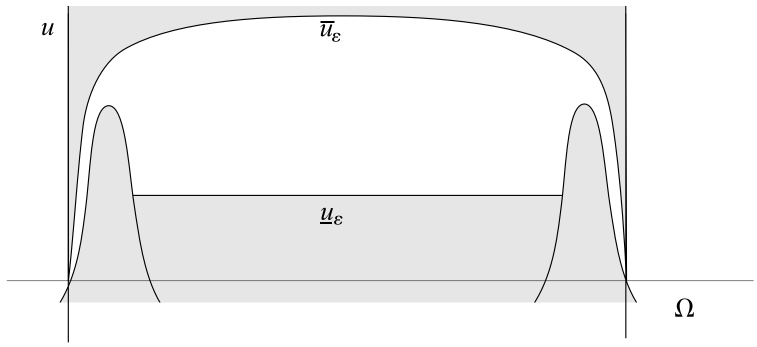
$$\mathcal{U}_{\varepsilon} = \left[\underline{u}_{\varepsilon}, \overline{u}_{\varepsilon}\right], \tag{4.8}$$

see Figure 2. For later reference we summarize the properties of $\underline{u}_{\varepsilon}$ and $\overline{u}_{\varepsilon}$ in the following lemma.

Lemma 4.2. *Let $\underline{u}_{\varepsilon}$ and $\overline{u}_{\varepsilon}$ be as defined in* (4.6) *and* (4.7). *Then for any function $0 \leq v \leq \beta_+$ and $v\left(x\right) \leq \underline{\alpha}$ on $\Omega\backslash\Omega_{(\varepsilon r_2/\varepsilon_0)}$ it holds in $\mathcal{D}'\left(\Omega\right)$-sense that*

$$-\varepsilon^2 \Delta \underline{u}_{\varepsilon} \leq f\left(\underline{u}_{\varepsilon}\right) - v \qquad \textit{and} \qquad -\varepsilon^2 \Delta \overline{u}_{\varepsilon} \geq f\left(\overline{u}_{\varepsilon}\right) - v.$$

Figure 2: The order interval $\mathcal{U}_\varepsilon$



4.2. The order interval for the v component

As the upper bound for the v-component order interval we shall use the functions $\delta\overline{v}$, with $\overline{v}$ the solution to

$$\left\{\begin{array}{rcll} -\Delta v + \gamma v & = & \rho_+ & \text{in } \Omega; \\ v & = & 0 & \text{on } \partial\Omega. \end{array}\right. \tag{4.9}$$

Let e_1 and λ_1 be as in (2.2). To define the lower bound we first choose a constant $K > 0$ such that

$$Ke_1 \leq \underline{u}_{\varepsilon_1}. \tag{4.10}$$

Note that also $\underline{u}_\varepsilon \geq Ke_1$ holds true for all $\varepsilon < \varepsilon_1$. We define

$$\underline{v} = \frac{K}{\lambda_1 + \gamma} e_1. \tag{4.11}$$

Note that in Ω

$$-\Delta\left(\overline{v} - \underline{v}\right) + \gamma\left(\overline{v} - \underline{v}\right) = \rho_+ - Ke \geq \overline{u}_{\varepsilon_0} - \underline{u}_{\varepsilon_0} \geq 0,$$

and by the strong maximum principle we have for $x \in \Omega$ that $\overline{v}(x) > \underline{v}(x)$. We denote the order interval $[\underline{v}, \overline{v}]$ by $\mathcal{V}$ and let

$$\mathcal{V}_\delta = \delta\mathcal{V} = \left[\delta\underline{v}, \delta\,\overline{v}\right].$$

The following lemma is the counterpart of Lemma 4.2.

Lemma 4.3. *If $(\varepsilon, \delta, u, v)$ is a solution to $(P_{\varepsilon,\delta})$ with $u \in \mathcal{U}_\varepsilon$, then $v \in \delta\mathcal{V}$.*

Proof. It holds that $-\Delta\left(v - \delta\underline{v}\right) = \delta\left(u - Ke_1\right) \geq 0$ in Ω. Hence, by the maximum principle, $v \geq \delta\underline{v}$. Similarly we have that $v \leq \delta\,\overline{v}$ so that $v \in \delta\mathcal{V}$. In fact unless $\delta = 0$ there exists $\mu > 0$ such that, with e_1 as in (2.2),

$$\delta\underline{v} + \mu e_1 \leq v \leq \delta\,\overline{v} - \mu e_1, \tag{4.12}$$

with see for example [5, Lemma A1]. □

4.3. Proof of Theorem 3.1

We define the mapping $T_\varepsilon : \mathrm{R} \times X \to X$ (the space X is defined in (2.3)) by

$$T_\varepsilon (\delta, u, v) = \left(\left(-\varepsilon^2 \Delta\right)_0^{-1} (f(u) - v), \delta (-\Delta + \gamma)_0^{-1} u \right) - (u, v). \tag{4.13}$$

Here $h = (-\Delta)_0^{-1} w$ with $w \in C(\bar{\Omega})$ is interpreted as the unique function $h \in Y$ such that $-\Delta h = w$ in $\mathcal{D}'(\Omega)$-sense. It also holds that $S_\varepsilon : \mathrm{R} \times X \to X$ defined by $S_\varepsilon (\delta, u, v) = T_\varepsilon (\delta, u, v) + (u, v)$ maps bounded sets of $\mathrm{R} \times X$ into relatively compact subsets of X.

We are interested in points $(\delta, u, v) \in \mathrm{R}^+ \times X$ such that $T_\varepsilon (\delta, u, v) = 0$. Denote this solution set by $\mathcal{S}_\varepsilon$, that is

$$\mathcal{S}_\varepsilon = \{(\delta, u, v) \in \mathrm{R} \times X \,;\, T_\varepsilon (\delta, u, v) = 0\}.$$

Observe that $(0, u, v) \in \mathcal{S}_\varepsilon$ if and only if $v = 0$ and u is a solution to the scalar equation (1.2).

Remark 4.4. Recall that we denote by u_ε° the solution to (1.2) with maximum close to ρ_+. By a uniqueness result for u_ε°, see [5], we can assume that if u_ε is a solution to (1.2) and $u_\varepsilon \in \mathcal{U}_\varepsilon$, then $u_\varepsilon = u_\varepsilon^\circ$.

By these observations follows the existence for all $\varepsilon < \varepsilon_1$ of an element $(0, u_\varepsilon^\circ, 0) \in \mathcal{S}_\varepsilon$. Also, T_ε is Fréchet differentiable with respect to (u, v) in $(0, u_\varepsilon^\circ, 0)$ with derivative given by

$$d_{(u,v)} T_\varepsilon (0, u_\varepsilon^\circ, 0)(h_1, h_2) = \left(\left(-\varepsilon^2 \Delta\right)_0^{-1} (f'(u_\varepsilon^\circ) h_1 - h_2), 0 \right) - (h_1, h_2)$$

for $(h_1, h_2) \in X$. It holds that $d_{(u,v)} T_\varepsilon (0, u_\varepsilon^\circ, 0)$ is an isomorphism on X. Indeed, we have that $d_{(u,v)} T_\varepsilon (0, u_\varepsilon^\circ, 0)(h_1, h_2) = (0, 0)$ if and only if $h_2 = 0$ and

$$\begin{aligned} -\varepsilon^2 \Delta h_1 - f'(u_\varepsilon^\circ) h_1 &= 0 \quad \text{in } \Omega; \\ h_1 &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Since the eigenvalues of the operator $\left(-\varepsilon^2 \Delta - f'(u_\varepsilon^\circ)\right)$ are strictly bounded away from zero, see [5], we conclude that $h_1 = 0$. Since the mapping

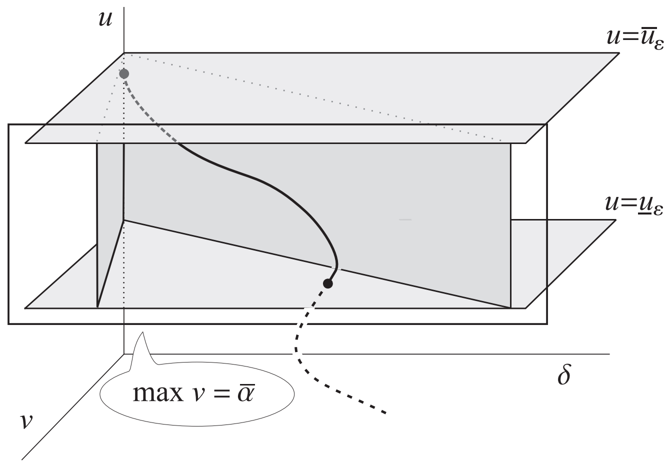
$$(h_1, h_2) \mapsto \left(\left(-\varepsilon^2 \Delta\right)_0^{-1} (f'(u_\varepsilon^\circ) h_1 - h_2), 0 \right)$$

is compact, $d_{(u,v)} T_\varepsilon (0, u_\varepsilon^\circ, 0)$ is surjective if and only if it is injective, and hence we conclude that $d_{(u,v)} T_\varepsilon (0, u_\varepsilon^\circ, 0)$ is an isomorphism.

By the implicit function theorem there exists for δsmall a smooth curve parameterized by δ of solutions to $T_\varepsilon (\delta, u, v) = (0, 0)$ passing through $(0, u_\varepsilon^\circ, 0)$. Moreover, in a small neighbourhood of $(0, u_\varepsilon^\circ, 0)$ this curve is the only solution to $T_\varepsilon (\delta, u, v) = (0, 0)$. We summarize this in the following proposition.

Proposition 4.5. *For all $\varepsilon < \varepsilon_1$ there exists a solution $(0, u_\varepsilon^\circ, 0)$ to the problem $T_\varepsilon (\delta, u, v) = 0$. Moreover, there exist $\kappa_\varepsilon, \eta_\varepsilon > 0$ and a smooth curve $(u_\varepsilon, v_\varepsilon)$: $(-\kappa_\varepsilon, \kappa_\varepsilon) \to X$ such that $T_\varepsilon (\delta, u_\varepsilon(\delta), v_\varepsilon(\delta)) = 0$ and if $T_\varepsilon (\delta, u, v) = 0$ with*

Figure 3: schematized bifurcation picture



$\|(u,v)\| < \eta_\varepsilon$ *then* $(u,v) = (u_\varepsilon(\delta), v_\varepsilon(\delta))$. *Finally, if* $T_\varepsilon(0,u,v) = 0$ *with* $u \in \mathcal{U}_\varepsilon$ *then* $(0,u,v) = (0,u_\varepsilon^\circ,0)$.

In the rest of this section we investigate this curve originating in $(0,u_\varepsilon^\circ,0)$, or rather the component in $\mathcal{S}_\varepsilon$ of $(0,u_\varepsilon^\circ,0)$. In the proof we use a version of the global implicit function of Rabinowitz [15], proven by Clément and Peletier [4]. It is shown that if for all $(0,u,v) \in \mathcal{C}_\varepsilon^+$ it holds that $v < \overline{\alpha}$ it follows then $\mathcal{C}_\varepsilon^+$ is bounded. This leads to a contradiction of this global implicit function theorem. Figure 3 gives an graphical description of the idea of the proof.

Proof. [Theorem 3.1] Recall that $\overline{\alpha}$ is a fixed constant such that $\alpha^* < \overline{\alpha} < \beta_+$. We define the following *'cylinder'* in $\mathrm{R}^+ \times X$:

$$K_\varepsilon = \left\{(\delta,u,v) \in \mathrm{R}_0^+ \times X\,;\,(u,v) \in \mathcal{U}_\varepsilon \times \delta\mathcal{V}\right\} = \bigcup_{\delta\geq 0} \{\delta\} \times \mathcal{U}_\varepsilon \times \delta\mathcal{V}$$

and let

$$\widetilde{K}_\varepsilon = K_\varepsilon \cap \left\{(\delta,u,v) \in \mathrm{R}_0^+ \times X\,;\,\max v \leq \overline{\alpha}\right\}$$

Observe that $\widetilde{K}_\varepsilon$, being the intersection of two closed sets, is closed and that $(0,u_\varepsilon^\circ,0)$ belongs to $\widetilde{K}_\varepsilon$.

Let $\varepsilon_{\overline{\alpha}} > 0$ be such that for all $\varepsilon < \varepsilon_{\overline{\alpha}}$ it holds that

$$\overline{v}(x) \leq \alpha^* \|\underline{v}\| / \overline{\alpha} \text{ for all } x \in \Omega\backslash\Omega_{(\varepsilon r_2/\varepsilon_0)}. \tag{4.14}$$

Such an $\varepsilon_{\overline{\alpha}}$ exists since $\overline{v}(x) = 0$ if $x \in \partial\Omega$. Let $\mathcal{T}_\varepsilon^+$ be the solutions to $T_\varepsilon(\delta,u,v) = 0$ in $\widetilde{K}_\varepsilon$, that is

$$\mathcal{T}_\varepsilon^+ = \mathcal{C}_\varepsilon^+ \cap \widetilde{K}_\varepsilon.$$

We claim that for all $\varepsilon < \varepsilon_{\overline{\alpha}}$ there exists an element $(\delta_\varepsilon, u_\varepsilon, v_\varepsilon) \in \mathcal{T}_\varepsilon^+$ with $\max v_\varepsilon = \overline{\alpha}$. We prove this claim by a contradiction argument.

Fix $\varepsilon < \varepsilon_{\overline{\alpha}}$ and suppose that all $(\delta, u, v) \in \mathcal{T}_\varepsilon^+$ satisfy $\max v < \overline{\alpha}$. Since all $(\delta, u, v) \in \widetilde{K}_\varepsilon$ satisfy $\delta \underline{v} \le v \le \overline{\alpha}$ we have for all $(\delta, u, v) \in \widetilde{K}_\varepsilon$ that $\delta \le \overline{\alpha} / \|\underline{v}\|$. Consequently we have for all $(\delta, u, v) \in \widetilde{K}_\varepsilon$ that $v \le \delta \overline{v} \le (\overline{\alpha} / \|\underline{v}\|)\, \overline{v}$. By the choice of $\varepsilon_{\overline{\alpha}}$, see (4.14), we have that if $(\delta, u, v) \in \widetilde{K}_\varepsilon$ then

$$v(x) \le \overline{\alpha} \text{ for all } x \in \Omega \backslash \Omega_{(\varepsilon r_2/\varepsilon_0)}.$$

Since $(0, u_\varepsilon^\circ, 0) \in \mathcal{T}_\varepsilon^+$ it holds that $\mathcal{T}_\varepsilon^+ \neq \emptyset$ and because $\widetilde{K}_\varepsilon$ is closed, $\mathcal{T}_\varepsilon^+$ is closed in the relative topology of $\mathcal{C}_\varepsilon^+$. But $\mathcal{T}_\varepsilon^+$ is also open. To see this, let $(\delta_0, u_0, v_0) \in \mathcal{T}_\varepsilon^+$. If $\delta_0 = 0$ we have that $(\delta_0, u_0, v_0) = (0, u_0, 0)$ with $u_0 \in \mathcal{U}_\varepsilon$. Hence by Proposition 4.5 and Remark 4.4, $u_0 = u_\varepsilon^\circ$. Moreover, if $\kappa > 0$ is small enough we know that

$$\left\{ (\delta, u, v) \in \mathrm{R}^+ \times X \,;\, \|(0, u_0, 0) - (\delta, u, v)\|_{\mathrm{R}\times X} < \kappa \right\} \cap \mathcal{S}_\varepsilon$$

consists of the smooth curve $(\delta, (u_\varepsilon(\delta), v_\varepsilon(\delta)))$, through $(0, u_0, 0)$. Moreover if δ is small and (δ, u, v) is on this curve, then $u \in \mathcal{U}_\varepsilon$ and by Lemma 4.3, $v \in \delta \mathcal{V}$. Hence for κ small enough

$$\left\{ (\delta, u, v) \in \mathrm{R}^+ \times X \,;\, \|(0, u_0, 0) - (\delta, u, v)\|_{\mathrm{R}\times X} < \kappa \right\} \cap \mathcal{C}_\varepsilon^+$$

consists precisely of the curve $\{(\delta, (u_\varepsilon(\delta), v_\varepsilon(\delta)) \,;\, 0 \le \delta < \kappa\}$, which is contained in $\widetilde{K}_\varepsilon$, and we have found a neighbourhood of (δ_0, u_0, v_0) in $\mathcal{C}_\varepsilon^+$ which is contained in $\mathcal{T}_\varepsilon^+$.

Now assume that $\delta_0 > 0$. We have that

$$\begin{cases} -\varepsilon^2 \Delta u_0 &=& f(u_0) - v_0 & \text{in } \Omega; \\ -\Delta v_0 &=& \delta_0 u_0 - \gamma v_0 & \text{in } \Omega; \\ u &=& v \,=\, 0 & \text{on } \partial\Omega, \end{cases}$$

with $(u_0, v_0) \in \mathcal{U}_\varepsilon \times \delta_0 \mathcal{V}$. In Ω we have that, see (4.10) and (4.11)

$$-\Delta(v_0 - \delta_0 \underline{v}) + \gamma(v_0 - \delta_0 \underline{v}) = \delta_0(u_0 - K e_1) \ge \delta_0(\underline{u}_\varepsilon - K e_1) \ge 0,$$

and $v_0 = \delta_0 \underline{v}$ on $\partial\Omega$. By the strong maximum principle there exists $\kappa_1 > 0$ such that $v_0 \ge \delta_0 \underline{v} + \kappa_1 e_1$. Similarly there exists $\kappa_2 > 0$ such that $v_0 \le \delta_0 \overline{v} - \kappa_2 e_1$. Concerning the function u_0, we have that

$$\left(-\varepsilon^2 \Delta + \omega\right)(u_0 - \overline{u}_\varepsilon) = f(u_0) + \omega u_0 - v_0 - \omega \rho_+ \le 0,$$

and there exists $\kappa_3 > 0$ such that $u_0 \le \overline{u}_\varepsilon - \kappa_3 e$. By our choice of $\varepsilon_{\overline{\alpha}}$ we obtain

$$v_0(x) \le \overline{\alpha} \quad \text{on } \Omega \backslash \Omega_{(\varepsilon r_2/\varepsilon_0)},$$

see (4.14). Since $v(x) \le \overline{\alpha} < \beta_+$ we find that

$$\left(-\varepsilon^2 \Delta + \omega\right)(u_0 - \underline{u}_\varepsilon) \ge (f(u_0) + \omega u_0 - v_0) - (f(\underline{u}_\varepsilon) + \omega \underline{u}_\varepsilon - v_0) \ge 0,$$

and there exists $\kappa_4 > 0$ such that $u_0 \ge \underline{u}_\varepsilon + \kappa_4 e$.

Let $\kappa' = \min\{\kappa_i \,;\, i = 1, \ldots, 4\}$ and $\kappa = \min\{\kappa', (\lambda_1 + \gamma)\kappa'/K\}$. Then it follows that the neighbourhood

$$N_\kappa = \left\{ (\delta, u, v) \in \mathrm{R}^+ \times X \,;\, \|(\delta_0 - \delta, u_0 - u, v_0 - v)\|_{\mathbb{R}\times X} < \kappa/2 \right\} \cap \mathcal{C}_\varepsilon^+$$

of (δ_0, u_0, v_0) in $\mathcal{C}_\varepsilon^+$ is contained in K_ε. We still have to show, by possibly decreasing κ, that $N_\kappa \subset \widetilde{K}_\varepsilon$. Since $\max v_0 < \overline{\alpha}$ this also holds for all v in an neighbourhood of v_0, $N_\kappa \subset \widetilde{K}_\varepsilon$. Since $\mathcal{C}_\varepsilon^+$ is connected, we deduce that $\mathcal{T}_\varepsilon^+ = \mathcal{C}_\varepsilon^+$.

Hence we see that $\mathcal{C}_\varepsilon^+ \subset \widetilde{K}_\varepsilon$. But this leads to a contradiction. Indeed, this implies that $\mathcal{C}_\varepsilon^+$ is bounded which means by [4, Theorem A] that there exists a point $(0,u,v) \in \mathcal{C}_\varepsilon^+$ with $(0,u,v) \neq (0, u_\varepsilon^\circ, 0)$. But $(0,u,v) \in \mathcal{C}_\varepsilon^+$, implies that $(u,v) \in \mathcal{U}_\varepsilon \times 0\mathcal{V}$ and by Proposition 4.5, see Remark 4.4 it holds that $u = u_\varepsilon^\circ$, a contradiction.

Hence for $\varepsilon < \varepsilon_{\overline{\alpha}}$ there exists an element $(\delta_\varepsilon, u_\varepsilon, v_\varepsilon) \in \mathcal{S}_\varepsilon$ with $(u_\varepsilon, v_\varepsilon) \in \mathcal{U}_\varepsilon \times \delta_\varepsilon \mathcal{V}$ and $\max v_\varepsilon = \overline{\alpha}$. Note that since $v_\varepsilon \in \delta_\varepsilon \mathcal{V}$, we have that

$$\delta_\varepsilon \|\underline{v}\| \leq \overline{\alpha} \leq \delta_\varepsilon \|\overline{v}\|,$$

that is, $\underline{\delta} = \overline{\alpha} / \|\overline{v}\|^{-1} \leq \delta_\varepsilon \leq \overline{\alpha} / \|\underline{v}\| = \overline{\delta}$. □

Remark 4.6. Note that from the proof of the theorem we have that $(u_\varepsilon, v_\varepsilon) \in \mathcal{U}_\varepsilon \times \delta_\varepsilon \mathcal{V}$.

5. Limiting behavior of $(\delta_\varepsilon, u_\varepsilon, v_\varepsilon)$

By Theorem 3.1 we know that given $\alpha^* < \overline{\alpha} < \beta_+$, there exists $\varepsilon_{\overline{\alpha}} > 0$ such that for all $\varepsilon < \varepsilon_{\overline{\alpha}}$ there exists a solution $(\delta_\varepsilon, u_\varepsilon, v_\varepsilon)$ to

$$(\mathrm{P}_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \Delta u &= f(u) - v \quad \text{in } \Omega; \\ -\Delta v &= \delta_\varepsilon u - \gamma v \quad \text{in } \Omega; \\ u &= v = 0 \quad \text{on } \partial\Omega. \end{cases}$$

with $\max v_\varepsilon = \overline{\alpha}$. Moreover we have that $\delta_\varepsilon \in [\underline{\delta}, \overline{\delta}]$ for all $\varepsilon < \varepsilon_{\overline{\alpha}}$. Since $u_\varepsilon \in \mathcal{U}_\varepsilon$ for all $\varepsilon < \varepsilon_{\overline{\alpha}}$, the set $\{u_\varepsilon\,;\, \varepsilon \leq \varepsilon_{\overline{\alpha}}\}$ is bounded in $L^\infty(\Omega)$. From this, and because

$$(5.1) \qquad \begin{cases} -\Delta v_\varepsilon &= \delta_\varepsilon u_\varepsilon - \gamma v_\varepsilon \quad \text{in } \Omega; \\ v_\varepsilon &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

it follows that the set $\{v_\varepsilon\,;\, \varepsilon < \varepsilon_{\overline{\alpha}}\}$ is bounded in $C^{1,\vartheta}(\bar{\Omega})$ with $0 < \vartheta < 1$. Let M be such that

$$(5.2) \qquad \|v_\varepsilon\|_{C^{1,\vartheta}(\bar{\Omega})} \leq M \quad \text{for all } \varepsilon < \varepsilon_{\overline{\alpha}}.$$

For fixed $\overline{\alpha}$ we now consider the behavior of the solutions $\{(\delta_\varepsilon, u_\varepsilon, v_\varepsilon)\,;\, \varepsilon \leq \varepsilon_{\overline{\alpha}}\}$ as $\varepsilon \to 0^+$. First we show that u_ε has a boundary layer of $O(\varepsilon)$ and approach $h_+(v_\varepsilon)$ in the interior of Ω.

Theorem 5.1. *Given $\mu > 0$ there exists $0 < \varepsilon(\mu) \leq \varepsilon_{\overline{\alpha}}$ and a constant c_μ such that for all $\varepsilon < \varepsilon(\mu)$ it holds that*

$$|u_\varepsilon(x) - h_+(v_\varepsilon(x))| < \mu \text{ for all } x \in \Omega_{(\varepsilon c_\mu)},$$

with $(\delta_\varepsilon, u_\varepsilon, v_\varepsilon)$ the solution as in Theorem 3.1.

Proof. First we show that there exists a constant $c_\mu > 0$ such that

$$u_\varepsilon(x) > h_+(v_\varepsilon(x)) - \mu \quad \text{for all } x \in \Omega_{(c_\mu\varepsilon)} \text{ and } \varepsilon < \varepsilon_\mu. \tag{5.3}$$

Since $u_\varepsilon \in \mathcal{U}_\varepsilon$ it follows that

$$u_\varepsilon(x) \geq \min\left\{c_0\varepsilon_0\varepsilon^{-1}\operatorname{dist}(x,\partial\Omega), \tau_+\right\}$$

with c_0 as in (4.3).

We assume that $\mu < h_+(\overline{\alpha}) - \tau_+$. By Condition 2.1 it holds that

$$m := \min\left\{-f'(t)\ ;\ t \in [h_+(\overline{\alpha}) - \mu, \rho_+]\right\} > 0. \tag{5.4}$$

and we define

$$\ell_\mu = \frac{m\mu}{2(\rho_+ - \mu - \tau_+)}.$$

Then we have for all $x, y \in \Omega$ and $s \in [\tau_+, h_+(v(y)) - \mu] \subset [\tau_+, \rho_+ - \mu]$ that

$$\begin{aligned}
\frac{f(s) - v(x)}{s - \tau_+} &\geq \frac{f(h_+(v(y)) - \mu) - v(x)}{\rho_+ - \mu - \tau_+} \\
&= \frac{f(h_+(v(y)) - \mu) - v(y) + v(y) - v(x)}{\rho_+ - \mu - \tau_+} \\
&\geq \frac{-f'(\theta_y)\mu - |v(y) - v(x)|}{\rho_+ - \mu - \tau_+}
\end{aligned}$$

with $\theta_y \in [h_+(v(y)) - \mu, h_+(v(y))] \subset [h_+(\beta^+) - \mu, \rho_+]$. Hence, using also (5.3), for all $x \in B(y, m\mu/2M)$ we have

$$\frac{f(s) - v(x)}{s - \tau_+} \geq \frac{m\mu - M|y - x|}{\rho_+ - \mu - \tau_+} \geq \ell_\mu,$$

i.e. for $s \in [\tau_+, h_+(v(y)) - \mu]$ and $x \in B(y, m\mu/2M)$

$$f(s) - v(x) \geq \ell_\mu(s - \tau_+).$$

Next we define $k_\mu = \sqrt{\lambda_B/\ell_\mu}$, where λ_B is the principal eigenvalue of

$$\begin{cases} -\Delta\varphi_B = \lambda_B\varphi_B & \text{in } B(0,1)\,; \\ \varphi_B = 0 & \text{on } \partial B(0,1)\,. \end{cases}$$

The corresponding eigenfunction is normalized such that $\max\varphi_B = 1$. Let $x_0 \in \Omega_{((c_0+k_\mu)\varepsilon)}$ and define the following family of functions on $B(x_0, k_\mu\varepsilon) \subset \Omega_{((c_0+k_\mu)\varepsilon)}$ by

$$\underline{w}_{\varepsilon,t}(x) = \tau_+ + t\varphi_B((x - x_0)/(k_\mu\varepsilon)) \text{ for } t \in [0, h_+(v_\varepsilon(x_0)) - \mu - \tau_+].$$

In $B(x_0, k_\mu\varepsilon)$ it holds that

$$-\varepsilon^2\Delta\underline{w}_{\varepsilon,t} = \ell_\mu\left(\underline{w}_{\varepsilon,t} - \tau_+\right).$$

Since $\underline{w}_{\varepsilon,t}(x) \in [\tau_+, h_+(v_\varepsilon(x_0)) - \mu]$ and $k_\mu\varepsilon \leq m\mu/2M$ we have on $B(x_0, k_\mu\varepsilon)$ that

$$-\varepsilon^2\Delta\underline{w}_{\varepsilon,t} \leq f\left(\underline{w}_{\varepsilon,t}\right) - v_\varepsilon$$

for all $t \in [0, h_+(v_\varepsilon(x_0)) - \mu - \tau_+]$. Since $\underline{w}_{\varepsilon,0} \leq u_\varepsilon$ on $B(x_0, k_\mu \varepsilon)$ it follows the sweeping principle that $\underline{w}_{\varepsilon,t} \leq u_\varepsilon$ for all $t \in [0, h_+(v_\varepsilon(x_0)) - \mu - \tau_+]$. In particular for $t = h_+(v_\varepsilon(x_0)) - \mu - \tau_+$ we have that

$$u_\varepsilon(x_0) \geq \underline{w}_{\varepsilon,t}(x_0) = \tau_+ + h_+(v_\varepsilon(x_0)) - \mu - \tau_+ = h_+(v_\varepsilon(x_0)) - \mu.$$

Since this can be done for every $x_0 \in \Omega_{((c_0+k_\mu)\varepsilon)}$, (5.3) holds with $c_\mu = c_0 + k_\mu$.

Next we show that for ε small enough it holds that

$$u_\varepsilon(x) \leq h_+(v_\varepsilon(x)) + \mu \quad \text{in } \Omega.$$

Again we do this by sweeping. We choose a function $\psi \in C^2[0, \beta^+]$ such that

$$|\psi(s) - h_+(s)| \leq \mu/3 \quad \text{for } s \in [0, \beta^+] \tag{5.5}$$

and let $K = \max\{|\psi'(s)| + |\psi''(s)|\ ;\ s \in [0, \beta^+]\}$. Now we define

$$\overline{w}_{\varepsilon,t} = t(\psi(v_\varepsilon) + \mu/2) \quad \text{for } t \geq 1.$$

It holds with (5.2) that

$$\begin{aligned} -\varepsilon^2 \Delta \overline{w}_{\varepsilon,t} &= -\varepsilon^2 t \left(\psi''(v_\varepsilon)|\nabla v_\varepsilon|^2 + \psi'(v_\varepsilon)(\gamma v_\varepsilon - \delta_\varepsilon u_\varepsilon)\right) \\ &\geq -\varepsilon^2 t K \left(|\nabla v_\varepsilon|^2 + |\gamma v_\varepsilon - \delta_\varepsilon u_\varepsilon|\right) \geq -\varepsilon^2 t K_1, \end{aligned}$$

with K_1 independent of ε. On the other hand, by the mean value theorem, there exists a function $g(x) < -m$ such that by (5.5)

$$\begin{aligned} f(\overline{w}_{\varepsilon,t}) - v_\varepsilon &= f(\overline{w}_{\varepsilon,t}) - f(h_+(v_\varepsilon)) \\ &= g(x)(t\psi(v_\varepsilon) + t\mu/2 - h_+(v_\varepsilon)) \\ &\leq g(x)(t(\mu/6) + (t-1)h_+(v_\varepsilon)) \leq -mt\mu/6. \end{aligned}$$

Hence for all $\varepsilon \leq \sqrt{m\mu/6K_1}$ and all $t \geq 1$ we have that

$$-\varepsilon^2 \Delta \overline{w}_{\varepsilon,t} \geq f(\overline{w}_{\varepsilon,t}) - v_\varepsilon.$$

Since for t large enough $\overline{w}_{\varepsilon,t}(x) > u_\varepsilon(x)$ for all $x \in \bar{\Omega}$ and if $x \in \partial\Omega$ $\overline{w}_{\varepsilon,t}(x) \geq u_\varepsilon(x)$ for all $t \geq 1$ it follows from the sweeping principle that

$$u_\varepsilon(x) \leq \overline{w}_{\varepsilon,1}(x) = \psi(v_\varepsilon(x)) + \mu/2 \leq h_+(v_\varepsilon(x)) + \mu.$$

□

Now we consider the behavior of $(\delta_\varepsilon, u_\varepsilon, v_\varepsilon)$ as $\varepsilon \to 0^+$. To state our result we need the following lemma.

Lemma 5.2. *For every $0 < \alpha < \beta_+$ there exists a unique pair $(\widehat{\delta}_\alpha, \widehat{v}_\alpha) \in \mathrm{R}^+ \times C^2(\overline{\Omega})$ satisfying the following problem:*

$$\begin{aligned} -\Delta v + \gamma v &= \delta h_+(v) && \text{in } \Omega; \\ v &= 0 && \text{on } \partial\Omega; \\ \max v &= \alpha. \end{aligned} \tag{Q_α}$$

Proof. First we show that there exists a solution (δ, v) to (Q_a). For this we modify and extend $h_+(v)$ to a function $h(v)$ as follows

$$h(v) = \begin{cases} h_+(v) & \text{if } v \le \alpha; \\ h'_+(\alpha)(v-\alpha) + h_+(\alpha) & \text{if } v > \alpha. \end{cases}$$

A solution to the problem with this modified nonlinearity is also a solution to (Q_α).

Let Y be as in (2.3). We define $K : \mathbb{R}\times Y \to Y$ by

$$K(\delta, v) = v - \delta(-\Delta+\gamma)_0^{-1} p(v) \tag{5.6}$$

and denote by $\mathcal{T}$ the set $\{(\delta, v) \in \mathbb{R}\times Y\,;\, K(\delta, v) = 0\}$ of zeros of K. The meaning of $(-\Delta+\gamma)_0^{-1}$ in (5.6) is the same as that the operator defined in (4.13). We have that $(0,0) \in \mathcal{T}$ and this is the only element of the form $(0, v)$ in $\mathcal{T}$. The Fréchet derivative of K with respect to v is given by

$$d_v K(\delta, v) h = h - \delta(-\Delta+\gamma)_0^{-1} p(v) h$$

for $h \in Y$, and in particular

$$d_v K(0,0) h = h.$$

Similarly as in the proof of Theorem 3.1 we can apply the global implicit function theorem to conclude that the component, say $\mathcal{C}^+$, of $(0,0)$ in $\mathcal{T}\cap\mathbb{R}^+\times Y$ is unbounded. This implies the existence of an element $(\widehat{\delta}_\alpha, \widehat{v}_\alpha) \in \mathcal{C}^+$ with $\max \widehat{v} = \alpha$ as follows. Since $\mathcal{C}^+$ is connected it is sufficient to prove that there exists an element (δ, v) on the component such that $\max v > \alpha$. Suppose for all $(\delta, v) \in \mathcal{C}^+$ we have that $v \le \alpha$. Then for all $(\delta, v) \in \mathcal{C}^+$ we have

$$(-\Delta+\gamma) v = \delta h(v) \ge \delta h(\alpha).$$

Using the maximum principle this shows that

$$v \ge \frac{\delta h(\alpha)}{(\lambda_1+\gamma)} e_1$$

with e_1 as in (2.2). Hence

$$\delta \le \frac{\alpha}{h(\alpha)}(\lambda_1+\gamma),$$

in contradiction to the unboundedness of $\mathcal{C}^+$. The regularity of the solution follows from a usual bootstrap argument. This completes the existence part of the lemma.

To prove the uniqueness, suppose that $\widehat{\delta}_1 < \widehat{\delta}_2$ and that $\widehat{v}_1$ and $\widehat{v}_2$ are both solutions to (Q_α). We have that

$$\begin{aligned} (-\Delta+\gamma)(v_1 - v_2) &= \delta_1 h_+(v_1) + \delta_2 h_+(v_2) \\ &< \delta_2 h_+(v_1) + \delta_2 h_+(v_2) \\ &= \delta_2 m(x)(v_1 - v_2) \end{aligned}$$

with $0 \ge m(x) \in L^\infty(\Omega)$. Since also $v_1 \le v_2$ on $\partial\Omega$ we find by the strong maximum principle that $v_1(x) < v_2(x)$ for all $x \in \Omega$, a contradiction. □

The proof of Theorem 3.2 now follows almost immediately.

Proof. [Theorem 3.2] Since $\{(\delta_\varepsilon, v_\varepsilon); \varepsilon < \varepsilon_{\overline{\alpha}}\}$ is bounded in $\mathbb{R}\times C^{1,1}(\overline{\Omega})$, there exist by the Heine-Borel and Arzelà-Ascoli theorems $\delta \in \mathrm{R}$ and $v \in C^1(\overline{\Omega})$ and a sequence ε_n such that $(v_{\varepsilon_n}, \delta_{\varepsilon_n}) \to (v, \delta)$ as $\varepsilon_n \to 0^+$. By Theorem 5.1 we also have that $u_{\varepsilon_n} \to h_+(v)$ uniformly on compact subsets of Ω. For a test function $\varphi \in C_0^2(\Omega)$ we have that

$$\begin{aligned}\int_\Omega (\nabla v \cdot \nabla\varphi + \gamma v\varphi)\, dx &= \lim_{\varepsilon_n\to 0^+} \int_\Omega (\nabla v_{\varepsilon_n} \cdot \nabla\varphi + \gamma v_{\varepsilon_n}\varphi)\, dx \\ &= \lim_{\varepsilon_n\to 0^+} \delta_{\varepsilon_n} \int_\Omega u_{\varepsilon_n}\varphi\, dx \\ &= \delta \int_\Omega h_+(v)\,\varphi\, dx\end{aligned}$$

Hence we see (δ, v) is a $\mathrm{R}\times W_0^{1,2}(\Omega)$ solution to $(\mathrm{Q}_{\overline{\alpha}})$ and by uniqueness, $(\delta, v) = (\widehat{\delta}_{\overline{\alpha}}, \widehat{v}_{\overline{\alpha}})$. Since any subsequence of $(v_\varepsilon, \delta_\varepsilon)$ has a subsequence converging (by the same argument) to $(\widehat{\delta}_{\overline{\alpha}}, \widehat{v}_{\overline{\alpha}})$ we conclude that $\lim_{\varepsilon\to 0^+}\delta_\varepsilon = \widehat{\delta}_{\overline{\alpha}}$ and $\lim_{\varepsilon\to 0^+} v_\varepsilon = \widehat{v}_{\overline{\alpha}}$ in $C^1(\overline{\Omega})$. From Theorem 5.1 it follows that $\lim_{\varepsilon\to 0^+} u_\varepsilon = h_+(\widehat{v}_{\overline{\alpha}})$ uniformly on compact subsets of Ω. □

6. The minimizing solution

Next we show that under an additional assumption it holds that $\Phi_\varepsilon(u_\varepsilon) < 0$ for ε small enough. Then we show that u_ε is not the global minimizer of Φ_ε. This is accomplished by constructing a function $\widetilde{u}_\varepsilon$ for which it holds that $\Phi_\varepsilon(\widetilde{u}_\varepsilon) < \Phi_\varepsilon(u_\varepsilon)$.

Theorem 6.1. *Let* $\{(\delta_\varepsilon, u_\varepsilon, v_\varepsilon)\,;\, \varepsilon < \varepsilon_{\overline{\alpha}}\}$ *be as in Theorem* 3.1. *It holds that*

$$\lim_{\varepsilon\to 0^+}(\Phi_\varepsilon(u_\varepsilon)) = \frac{1}{2}\int_\Omega \left(h_+(\widehat{v}_{\overline{\alpha}})\,\widehat{v}_{\overline{\alpha}} - 2F(h_+(\widehat{v}_{\overline{\alpha}}))\right) dx. \tag{6.1}$$

In particular, a sufficient condition for $\Phi_\varepsilon(u_\varepsilon) < 0$ *for* ε *small enough is given by*

$$uf(u) < 2F(u) \quad \text{for } u \in [h_+(\overline{\alpha}), \rho_+]. \tag{6.2}$$

Proof. First note that by multiplying

$$-\varepsilon^2\Delta u_\varepsilon = f(u_\varepsilon) - v_\varepsilon$$

with u_ε and integrating over Ω we find that

$$\varepsilon^2\int_\Omega |\nabla u_\varepsilon|^2\, dx = \int_\Omega u_\varepsilon(f(u_\varepsilon) - v_\varepsilon)\, dx.$$

Hence

$$\begin{aligned}\Phi_\varepsilon(u_\varepsilon) &= \frac{1}{2}\int_\Omega u_\varepsilon(f(u_\varepsilon) - v_\varepsilon)\, dx + \frac{1}{2}\int_\Omega u_\varepsilon v_\varepsilon\, dx - \int_\Omega F(u_\varepsilon)\, dx \\ &= \frac{1}{2}\int_\Omega (u_\varepsilon f(u_\varepsilon) - 2F(u_\varepsilon))\, dx.\end{aligned}$$

Since $u_\varepsilon \to \widehat{u}_{\overline{\alpha}} = h_+ (\widehat{v}_{\overline{\alpha}})$, uniformly on compact subsets of Ω, (6.1) follows from the dominated convergence theorem.

Now, if (6.2) holds, then with $\widehat{u} = h_+ (\widehat{v}_{\overline{\alpha}})$ we have that

$$\frac{1}{2} h_+ (\widehat{v}_{\overline{\alpha}}) \widehat{v}_{\overline{\alpha}} - F (h_+ (\widehat{v}_{\overline{\alpha}})) = \frac{1}{2} \widehat{u} f (\widehat{u}) - F (\widehat{u}) < -m$$

for some $m > 0$. Hence $\lim_{\varepsilon \to 0^+} (\Phi_\varepsilon (u_\varepsilon)) < -m |\Omega|$ and the result follows. □

Remark 6.2. Condition (6.2) is clearly not necessary to have $\Phi_\varepsilon (u_\varepsilon) < 0$ for ε small. Since for $u \in I_+$

$$\frac{d}{du} (u f (u) - 2F (u)) = u f' (u) - f (u) < 0,$$

one has that (6.2) holds if $\overline{\alpha} h_+ (\overline{\alpha}) < 2F (h_+ (\overline{\alpha}))$. For the generic example $f (u) = u (u - 1) (a - u)$, (6.2) is satisfied if we choose

$$\overline{\alpha} \in \left(\frac{1}{3} (a + 1), \frac{2}{3} (a + 1) \right).$$

Recall that for this example $\alpha^* = \frac{1}{3} (a + 1)$ and since $a < \frac{1}{2}$ we have that $\frac{2}{3} (a + 1) < 1 = \rho_+$.

Next we give a heuristic argument why u_ε is not the global minimizer. As a *'first approximation'* of $\Phi_\varepsilon (u)$ we ignore the term $\frac{\varepsilon^2}{2} \int_\Omega |\nabla u|^2 \, dx$, which will be small unless u has a large gradient. Let us compare the values for two different functions, say u and $\widetilde{u}$, with $v = \delta G_\gamma u$, $\widetilde{v} = \delta G_\gamma \widetilde{u}$. We also assume that $u \approx h_+ (v)$ as is the case with the solutions constructed in the previous section. We compare:

$$\begin{aligned} &\frac{\delta}{2} \int_\Omega \widetilde{u} G_\gamma \widetilde{u} \, dx - \int_\Omega F (\widetilde{u}) \, dx - \left(\frac{\delta}{2} \int_\Omega u G_\gamma u \, dx - \int_\Omega F (u) \, dx \right) \\ &= \int_\Omega (\widetilde{u} - u) v \, dx - \int_\Omega (F (\widetilde{u}) - F (u)) \, dx. \\ &\quad + \frac{\delta}{2} \int_\Omega (\widetilde{u} - u) G_\gamma (\widetilde{u} - u) \, dx \end{aligned}$$

Let us first look at

$$\int_\Omega (\widetilde{u} - u) v \, dx - \int_\Omega (F (\widetilde{u}) - F (u)) \, dx = \int_\Omega (\widetilde{u} v - F (\widetilde{u})) - (uv - F (u)) \, dx.$$

The (real) function $P (u) = uv - F (u)$ with $v \in I_0$ has critical points $f (u) = v$, i.e. $u = h_\sigma (v)$ for $\sigma \in \{-, 0, +\}$. Also, $u = h_\pm (v)$ are local minima since $P'' (h_\pm (v)) = -f' (h_\pm (v)) > 0$. Moreover we have for $v > \alpha^*$ that $\int_{h_-(v)}^{h_+(v)} (f (u) - v) \, du < 0$, while $\int_{h_-(v)}^{h_+(v)} (f (u) - v) \, du > 0$ for $v < \alpha^*$. Since

$$P (h_+ (v)) = \int_{h_-(v)}^{h_+(v)} (v - f (u)) \, du + P (h_- (v)),$$

we see that

$$P\left(h_{+}\left(v\right)\right) < P\left(h_{-}\left(v\right)\right) \text{ if } v < \alpha^{*} \tag{6.3}$$

and

$$P\left(h_{-}\left(v\right)\right) < P\left(h_{+}\left(v\right)\right) \text{ if } v > \alpha^{*}. \tag{6.4}$$

Hence one would expect that a function $\widetilde{u}$ taking values close to $h_{-}\left(v\right)$ where $v > \alpha^{*}$ instead of close to $h_{+}\left(v\right)$ as u does, would have a lower energy. On the other hand, there exist a constant μ such that

$$\int_{\Omega} \left(\widetilde{u} - u\right) G_{\gamma} \left(\widetilde{u} - u\right) \, dx \geq \mu \left\|\widetilde{u} - u\right\|_{L^2(\Omega)}^2 ,$$

and hence $\int_{\Omega} \left(\widetilde{u} - u\right) G_{\gamma} \left(\widetilde{u} - u\right) \, dx$ increases if $\left\|\widetilde{u} - u\right\|_{L^2(\Omega)}^2$ increases. Jumping between $h_{+}\left(v\right)$ and $h_{-}\left(v\right)$ or from the zero boundary value to $h_{\pm}\left(v\right)$ introduces gradient terms which cannot be ignored.

We now show that if the jump is made on small subset $\Omega' \subset \Omega$ we can construct a suitable modification $\widetilde{u}_{\varepsilon}$ of u_{ε} which will indeed have a lower energy.

First we define for $\varepsilon < \varepsilon_{\overline{\alpha}}$ the following modification of u_{ε}:

$$\widetilde{u}_{\varepsilon}\left(x\right) = \min\left\{u_{\varepsilon}\left(x\right), \max\left\{h_{-}\left(v_{\varepsilon}\left(x\right)\right), \frac{\left|x - x_{\varepsilon}\right| - \nu}{\varepsilon}\right\}\right\}, \tag{6.5}$$

with $x_{\varepsilon} \in \Omega$ such that $v_{\varepsilon}\left(x_{\varepsilon}\right) = \overline{\alpha}$ and where $\nu > 0$ shall be chosen appropriately and independently of ε for ε small enough. The following theorem is a more detailed form of Theorem 3.3.

Theorem 6.3. *There exists ε_{*} such that for all $\varepsilon < \varepsilon_{*}$ the function u_{ε} is not the global minimizer of the energy functional Φ_{ε}. In particular we have that there exist $\nu > 0$, independent of ε and $\varepsilon_{*} > 0$ such that for all $\varepsilon < \varepsilon_{*}$ it holds for $\widetilde{u}_{\varepsilon}$ defined in* (6.5) *that*

$$\Phi_{\varepsilon}\left(\widetilde{u}_{\varepsilon}\right) < \Phi_{\varepsilon}\left(u_{\varepsilon}\right).$$

If we also assume that (6.2) *holds, the global minimizer is nontrivial and we have the existence of at least two nontrivial solutions.*

Remark 6.4. Generically a minimizer is expected to be stable in the sense mentioned before. Indeed, one directly shows that for a solution $U(t)$ of (1.1), with $\tau = 0$ and $V(t) = \delta\, G_{\gamma}\left(U(t)\right)$, one finds that

$$\begin{aligned} \frac{\partial}{\partial t}\Phi_{\varepsilon}\left(U\right) &= \int_{\Omega} \left(-\varepsilon^2 \Delta U + \delta\, G_{\gamma} U - f(U)\right) U_t \, dx \\ &= -\int_{\Omega} \left(-\varepsilon^2 \Delta U + \delta\, G_{\gamma} U - f(U)\right)^2 dx \;\leq\; 0. \end{aligned}$$

Hence, if $\Phi_{\varepsilon}\left(U\right)$ is locally strongly convex near the minimizer u^{*}, the function U that starts with u_0 in a corresponding neighborhood of u^{*}, with $V(t) = \delta\, G_{\gamma}\left(U(t)\right)$ for all

$t \geq 0$, is expected to converge to u^* for $t \to \infty$. Since we do not know the qualitative properties of the global minimizer well enough we are not able to verify the above.

Proof. We fix $\alpha \in (\alpha^*, \overline{\alpha})$ and let

$$m = \inf_{\alpha \leq s \leq \overline{\alpha}} \left\{ \left(sh_+ (s) - F(h_+ (s)) \right) - \left(sh_- (s) - F(h_- (s)) \right) \right\}.$$

Then $m > 0$ by our assumptions on f. Since $v_\varepsilon \leq \bar{\delta}\bar{v}$ it holds that $\{x_\varepsilon \, ; \, \varepsilon < \varepsilon_{\overline{\alpha}}\}$ is uniformly bounded away from $\partial\Omega$. Also by the equicontinuity of $\{v_\varepsilon \, ; \, \varepsilon < \varepsilon_{\overline{\alpha}}\}$ there exists $\nu_0 > 0$ such that

$$v_\varepsilon (x) \geq \alpha \quad \text{for all } \varepsilon < \varepsilon_{\overline{\alpha}} \text{ and } x \text{ such that } |x - x_\varepsilon| < \nu_0.$$

For $\widetilde{u}_\varepsilon$ as in (6.5) we assume that $\nu \leq \nu_0$. Observe $\widetilde{u}_\varepsilon \leq u_\varepsilon$ and that $\widetilde{u}_\varepsilon$ is a local modification of u_ε around the point x_ε. Indeed, let us define

$$\begin{aligned} P_\varepsilon^\nu &= \{x \in \Omega \, ; \, |x - x_\varepsilon| < \nu + \varepsilon\rho_+\}, \\ Q_\varepsilon^\nu &= \{x \in \Omega \, ; \, |x - x_\varepsilon| < \nu + \varepsilon h_- (\overline{\alpha})\}, \\ R_\varepsilon^\nu &= P_\varepsilon^\nu \backslash Q_\varepsilon^\nu. \end{aligned}$$

Because $h_- (\overline{\alpha}) \leq 0$ we have the inclusion

$$Q_\varepsilon^\nu \subset \{x \in \Omega \, ; \, |x - x_\varepsilon| < \nu\} \subset P_\varepsilon^\nu.$$

If $x \in \Omega \backslash P_\varepsilon^\nu$, then

$$h_- (v_\varepsilon (x)) \leq h_- (0) < h_+ (0) = \rho_+ \leq \frac{|x - x_\varepsilon| - \nu}{\varepsilon}.$$

Since $u_\varepsilon (x) \leq \rho_+$ it holds that $\widetilde{u}_\varepsilon (x) = u_\varepsilon (x)$ for $x \in \Omega \backslash P_\varepsilon^\nu$.

If $x \in Q_\varepsilon^\nu$, then $h_- (v_\varepsilon (x)) \geq h_- (\overline{\alpha})$ and $\widetilde{u}_\varepsilon (x) = \min \{u_\varepsilon (x), h_- (v_\varepsilon (x))\}$. But since $u_\varepsilon (x) \sim h_+ (v_\varepsilon (x))$ we have that $\widetilde{u}_\varepsilon (x) = h_- (v_\varepsilon (x))$. We find

$$(\widetilde{u}_\varepsilon v_\varepsilon - F(\widetilde{u}_\varepsilon)) - (h_+ (v_\varepsilon) v_\varepsilon - F(h_+ (v_\varepsilon))) \leq -m \quad \text{on } Q_\varepsilon^\nu. \tag{6.6}$$

We also introduce the following neighborhood of R_ε^ν :

$$\tilde{R}_\varepsilon^\nu = \{x \in \Omega \, ; \, \nu + \varepsilon h_- (\overline{\alpha}) - \varepsilon \leq |x - x_\varepsilon| \leq \nu + \varepsilon\rho_+ + \varepsilon\}.$$

With these preliminaries in place we prove that, by decreasing ν if necessary, it holds for ε small enough that

$$\Phi_\varepsilon (\widetilde{u}_\varepsilon) < \Phi_\varepsilon (u_\varepsilon).$$

We write the difference in energy in the following form:

$$\Phi_\varepsilon (\widetilde{u}_\varepsilon) - \Phi_\varepsilon (u_\varepsilon) = I_1 + I_2 + I_3, \tag{6.7}$$

with

$$
\begin{aligned}
I_1 &= \frac{\varepsilon^2}{2}\int_\Omega \left(|\nabla \widetilde{u}_\varepsilon|^2 - |\nabla u_\varepsilon|^2\right)\, dx;\\
I_2 &= \frac{\delta_\varepsilon}{2}\int_\Omega (\widetilde{u}_\varepsilon - u_\varepsilon)\, G_\gamma\,(\widetilde{u}_\varepsilon - u_\varepsilon)\, dx;\\
I_3 &= \delta_\varepsilon \int_\Omega (\widetilde{u}_\varepsilon - u_\varepsilon)\, G_\gamma u_\varepsilon\, dx - \int_\Omega \left(F(\widetilde{u}_\varepsilon) - F(u_\varepsilon)\right) dx\\
&= \int_\Omega \left((\widetilde{u}_\varepsilon v_\varepsilon - F(\widetilde{u}_\varepsilon)) - (u_\varepsilon v_\varepsilon - F(u_\varepsilon))\right) dx.
\end{aligned}
$$

In estimating the terms I_1, I_2 and I_3 we shall denote by C different constants all of which are independent of ε and ν.

We start with I_3 which is the main negative term in (6.7). Since $|u_\varepsilon - h_-(v_\varepsilon)|$ is uniformly small on Q_ε^ν by choosing ε small enough and using (6.6), we have that

$$(\widetilde{u}_\varepsilon v_\varepsilon - F(\widetilde{u}_\varepsilon)) - (u_\varepsilon v_\varepsilon - F(u_\varepsilon)) < -m/2 \text{ on } Q_\varepsilon^\nu$$

for ε small enough. The integrand of I_3 is zero on $\Omega \backslash P_\varepsilon^\nu$ and is uniformly bounded on R_ε^ν, so we have

$$I_3 \le C\,|R_\varepsilon^\nu| - \frac{m}{2}\,|Q_\varepsilon^\nu|\,. \tag{6.8}$$

Next we consider I_2. Denote by $\mathcal{G}_\gamma(x,y)$ the Green's function of $(-\Delta+\gamma)$ on Ω with homogeneous Dirichlet boundary conditions. We can write v_ε as

$$v_\varepsilon(x) = \delta_\varepsilon \int_\Omega \mathcal{G}_\gamma(x,y)\, u_\varepsilon(y)\, dy.$$

The maximum principle and the estimate for the fundamental solution for $N > 2$ (see e.g. [9, page 17]) imply that for $\gamma \ge 0$

$$\mathcal{G}_\gamma(x,y) \le \mathcal{G}_0(x,y) \le -\Gamma(x,y) = \frac{1}{N(N-2)\,\omega_N}\,|x-y|^{2-N}\,, \tag{6.9}$$

where ω_N is the volume of the unit ball. For $N = 2$

$$\mathcal{G}_\gamma(x,y) \le \mathcal{G}_0(x,y) \le \frac{1}{2\pi}\log\frac{\operatorname{diam}(\Omega)}{|x-y|}, \tag{6.10}$$

where $\operatorname{diam}(\Omega) = \sup\{|x-y|\,; x,y \in \Omega\}$. Whenever $\gamma \in (-\lambda_1, 0)$ we may use

$$\mathcal{G}_\gamma(x,y) \le c_\gamma \mathcal{G}_0(x,y)$$

(e.g. combine [3, Theorem 4.1 and 7.22]). For $N>2$ we have that

$$\begin{aligned} G_{\gamma}\left(u_{\varepsilon}-\widetilde{u}_{\varepsilon}\right)(x) &= \int_{P_{\varepsilon}^{\nu}} \mathcal{G}_{\gamma}(x,y)\left(u_{\varepsilon}-\widetilde{u}_{\varepsilon}\right)(y)\,dy \\ &\leq C\int_{P_{\varepsilon}^{\nu}} \mathcal{G}_{\gamma}(x,y)\,dy \\ &\leq C\int_{\{y\,;\,|x-y|\leq 2\varepsilon\rho_{+}+2\nu\}} |x-y|^{2-N}\,dy \\ &= C\int_{r=0}^{2\varepsilon\rho_{+}+2\nu} r\,dr = 2C\left(\varepsilon\rho_{+}+\nu\right)^{2}. \end{aligned}$$

Hence,

$$\begin{aligned} I_{2} &= \frac{\delta_{\varepsilon}}{2}\int_{P_{\varepsilon}^{\nu}} \left(\widetilde{u}_{\varepsilon}-u_{\varepsilon}\right) G_{\gamma}\left(\widetilde{u}_{\varepsilon}-u_{\varepsilon}\right) dx \\ &= \frac{\delta_{\varepsilon}}{2}\int_{P_{\varepsilon}^{\nu}} \left(u_{\varepsilon}-\widetilde{u}_{\varepsilon}\right) G_{\gamma}\left(u_{\varepsilon}-\widetilde{u}_{\varepsilon}\right) dx \\ &\leq \overline{\delta}C\left(\nu+\varepsilon\rho_{+}\right)^{2}\int_{P_{\varepsilon}^{\nu}} \left(u_{\varepsilon}-\widetilde{u}_{\varepsilon}\right) dx \\ &\leq C\left(\nu+\varepsilon\rho_{+}\right)^{2}\left|P_{\varepsilon}^{\nu}\right|. \qquad (6.11) \end{aligned}$$

Similarly for $N=2$

$$I_{2}\leq C\left(\nu+\varepsilon\rho_{+}\right)^{2}\left|P_{\varepsilon}^{\nu}\right|\log\left(\frac{\operatorname{diam}(\Omega)}{\nu+\varepsilon\rho_{+}}.\right) \qquad (6.12)$$

Finally we consider the gradient terms. We set $I_{1}=I_{1a}+I_{1b}$ with

$$\begin{aligned} I_{1a} &= \frac{\varepsilon^{2}}{2}\int_{R_{\varepsilon}^{\nu}} \left(\left|\nabla\widetilde{u}_{\varepsilon}\right|^{2}-\left|\nabla u_{\varepsilon}\right|^{2}\right) dx, \\ I_{1b} &= \frac{\varepsilon^{2}}{2}\int_{Q_{\varepsilon}^{\nu}} \left(\left|\nabla\widetilde{u}_{\varepsilon}\right|^{2}-\left|\nabla u_{\varepsilon}\right|^{2}\right) dx. \end{aligned}$$

Since $|\nabla v_{\varepsilon}|$ and $h_{-}'(v_{\varepsilon})$ are uniformly bounded,

$$\begin{aligned} I_{1b} &= \frac{\varepsilon^{2}}{2}\int_{Q_{\varepsilon}^{\nu}} \left(\left|h_{-}'(v_{\varepsilon})\nabla v_{\varepsilon}\right|^{2}-\left|\nabla u_{\varepsilon}\right|^{2}\right) dx \\ &\leq \frac{\varepsilon^{2}}{2}\int_{Q_{\varepsilon}^{\nu}} \left(h_{-}'(v_{\varepsilon})\left|\nabla v_{\varepsilon}\right|\right)^{2} dx \leq C\varepsilon^{2}\left|Q_{\varepsilon}^{\nu}\right|. \qquad (6.13) \end{aligned}$$

On $\tilde{R}_{\varepsilon}^{\nu}$ it holds that $-\Delta u_{\varepsilon}=\varepsilon^{-2}\left(f(u_{\varepsilon})-v_{\varepsilon}\right)$ and we find by using [9] that

$$\begin{aligned} &\sup_{x\in\tilde{R}_{\varepsilon}^{\nu}}\left[\operatorname{dist}\left(x,\partial\tilde{R}_{\varepsilon}^{\nu}\right)\left|\nabla u_{\varepsilon}(x)\right|\right] \\ &\leq \widetilde{C}\left(\sup_{x\in\tilde{R}_{\varepsilon}^{\nu}}|u_{\varepsilon}|+\sup_{x\in\tilde{R}_{\varepsilon}^{\nu}}\left\{\operatorname{dist}\left(x,\partial\tilde{R}_{\varepsilon}^{\nu}\right)^{2}\varepsilon^{-2}\left|f(u_{\varepsilon}(x))-v_{\varepsilon}(x)\right|\right\}\right), \end{aligned}$$

and the right-hand side of the inequality is bounded, independent of ε. Since for $x \in R_\varepsilon^\nu$ it holds that dist $\left(x, \partial \tilde{R}_\varepsilon^\nu\right) \geq \varepsilon$ we have that

$$\varepsilon \left|\nabla u_\varepsilon\right| \leq C \quad \text{on } R_\varepsilon^\nu.$$

Where $\widetilde{u}_\varepsilon(x) = \frac{|x-x_\varepsilon|-\nu}{\varepsilon}$ we have that $\varepsilon \left|\nabla \widetilde{u}_\varepsilon\right| \leq C$ and where $\widetilde{u}_\varepsilon(x) = h_-(v_\varepsilon(x))$ we have that $\varepsilon \left|\nabla \widetilde{u}_\varepsilon\right| = \varepsilon \left|h_-'(v_\varepsilon(x))\right| \left|\nabla v_\varepsilon\right| \leq \varepsilon C$. This results in

$$I_{1a} \leq C \left|R_\varepsilon^\nu\right|. \tag{6.14}$$

Putting everything together, (6.8), (6.11), (6.14) and (6.13), it follows that for $N \geq 3$

$$\begin{aligned} \Phi_\varepsilon(\widetilde{u}_\varepsilon) - \Phi_\varepsilon(u_\varepsilon) &\leq C\left|R_\varepsilon^\nu\right| + C\varepsilon^2\left|Q_\varepsilon^\nu\right| + C\left|R_\varepsilon^\nu\right| - m\left|Q_\varepsilon^\nu\right| + C(\nu+\varepsilon\rho_+)^2\left|P_\varepsilon^\nu\right| \\ &\leq C\left|Q_\varepsilon\right|\left(\frac{\left|R_\varepsilon^\nu\right|}{\left|Q_\varepsilon^\nu\right|} + \varepsilon^2 - \frac{m}{2C} + (\nu+\varepsilon\rho_+)^2\frac{\left|P_\varepsilon^\nu\right|}{\left|Q_\varepsilon^\nu\right|}\right). \end{aligned}$$

Notice that for $N = 2$ the additional logarithm adds only a lower order term.

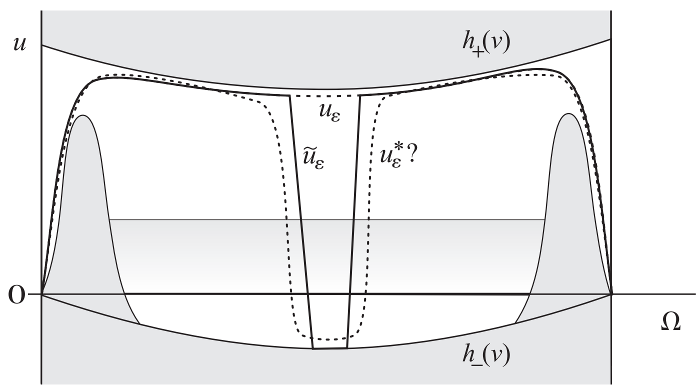
Now, recalling that C is independent of ν we fix $\nu = \min\left(\nu_0, \frac{1}{2}\sqrt{\frac{m}{2C}}\right)$ in case $N > 2$ (for $N = 2$ with an obvious modification). Then, for this fixed ν we have as $\varepsilon \downarrow 0$ that

$$\frac{\left|R_\varepsilon^\nu\right|}{\left|Q_\varepsilon^\nu\right|} \to 0 \quad \text{and} \quad \frac{\left|P_\varepsilon^\nu\right|}{\left|Q_\varepsilon^\nu\right|} \to 1,$$

and hence

$$\Phi_\varepsilon(\widetilde{u}_\varepsilon) - \Phi_\varepsilon(u_\varepsilon) < 0. \qquad \square$$

Figure 4: draft of the function with lower energy $\widetilde{u}_\varepsilon$



By this theorem we have that the minimizer, say u_ε^* of the functional Φ_ε is different from solution u_ε. We expect that u_ε^* has the same type of transition layer as the function $\widetilde{u}_\varepsilon$. Moreover if, for ε small, the system has the trivial solution (when $f(0) = 0$), the stable boundary layer solution and another one being the global minimizer, there exist at least a fourth solution to the problem. Indeed, there exists a solution of mountain pass type between. Generically a fifth solution may exists as a mountain pass between boundary layer solution and the minimizer.

7. Some numerical results

Although we may conclude from the previous arguments that for a certain range of parameters next to the boundary layer type solution another stable solutions exist, we have not obtained any analytical evidence how this solution looks. In order to have some guess we have used numerical methods. Numerical results for related (mostly time-dependent) problems are found in [8], [14], [13] and [19].

A complication is the fact that the system is neither cooperative nor competitive which means that one cannot expect some ordering to be preserved. That is, there is no guarantee that the numerical iteration will converge and, if it converges, that is does so to the solution one is interested in. To counter this problem we used the following (double) iterative scheme and we started with an appropriate function.

7.1. Scheme

The iteration scheme that was used consisted of the following steps:

1. Fix an initial function u_0 and set $n = 0$;
2. For given u_n solve for v_n by $v_n = (-\Delta + \gamma)_0^{-1} u_n$;
3. For given v_n solve for u_{n+1} satisfying $-\varepsilon^2 \Delta u_{n+1} = f(u_{n+1}) - v_n$ with Dirichlet boundary condition by
   (a) Set $U_0 = u_n$ and $k = 0$;
   (b) For given U_k solve for U_{k+1} by
   $$U_{k+1} = \left(-\varepsilon^2 \Delta + 1\right)_0^{-1} \left(f(U_k) + U_k - v_n\right);$$
   (c) If U_{k+1} is 'good enough' set $u_{n+1} = U_{k+1}$; else go back to b with $k := k+1$;
4. If u_{n+1} is 'good enough' then set $\tilde{u} = u_{n+1}$ and $\tilde{v} = v_n$; else $n := n+1$ and go back to 2.

7.2. Details

In the present case we used $\gamma = 0$ and $\varepsilon = .03$,

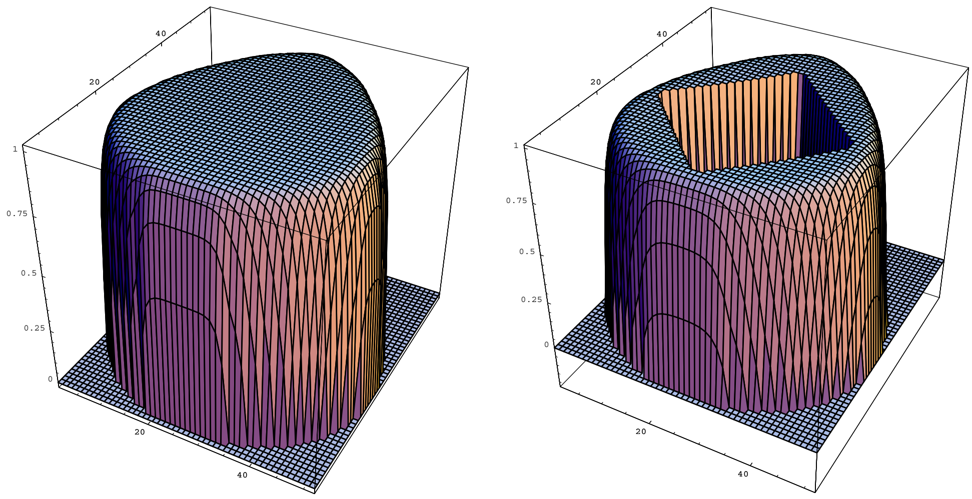
$$f(u) = u(1-u)\left(\tfrac{1}{4} - u\right),$$

and a two-dimensional egg-shaped domain

$$\Omega = \left\{ x \in \mathrm{R}^2; x_1^2 + (1 - \frac{2}{5} x_2)(x_2 + \frac{6}{25})^2 < 1 \right\}$$

on which we put a uniform rectangular mesh of size $h = \frac{1}{25}$. The Δ-operator was discretized by finite differences. Throughout the numerical experiments we stayed with these data and have only considered the behaviour depending on δ. The actual δ for which the calculation has been done are found in Figure 6. By we denote iterations that started with the $u_0 = u^*$ close to the solution for $\delta = 0$; by we denote iterations that started with the $u_0 = u^{**}$. This function was obtained by changing u^* with a square-shaped internal jump to 0. By exception, due to the slow convergence, the iteration for $\delta = .31$ has been started with the solution for $\delta = .32$. By 'good enough' in c we took $k = 9$; 'good enough' in 4 was decided if no significant difference was observed in the last two iterations. For $\delta = .3$ starting with u^{**} it meant $n = 45$ (bringing the total number of solved systems during that iteration to 450).

Figure 5: The two initial configurations.



(a) u^* (b) u^{**}

The programme was run on a PC using Mathematica 4.0. Step 2 used the Mathematica command `LinearSolve`. For Step b, due to its high repetition, it turned out to be more efficient to invert (Mathematica's `Inverse`) and store the corresponding matrix.

7.3. Results

The numerical results indicate the following:

1. For $\delta \in [0, .47]$ and $u_0 = u^*$ the iteration process converges to a boundary layer type solution u.

2. For $\delta \in [.475, .8]$ and $u_0 = u^*$ the iteration process converges to a solution having both boundary and internal layer.

3. For $\delta \in [0, .3]$ and $u_0 = u^{**}$ the iteration process converges to a boundary layer type solution u.

4. For $\delta \in [.31, 1.6]$ and $u_0 = u^{**}$ the iteration process converges to a solution having both boundary and internal layer.

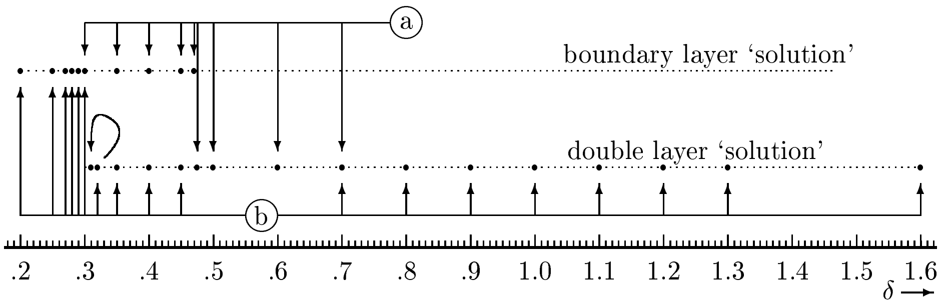
5. For $\delta \in [.31, .6]$ the internal layer is close to circular.

6. For increasing δ the internal layer moves outward until collapsing with the boundary layer.

The iteration itself showed a rapid deformation of the initial square hole to a circular layer but that circular layer only slowly converged to the stationary internal layer of the (approximated) solution.

The actual cases that have been calculated can be found in Figure 6. The higher dots correspond with boundary layer type solutions; the lower dots with solutions that have both a boundary and an internal layer. We would like to recall that we do not have any analytical guarantee that these functions are indeed approximations of actual solutions. The present method did not converge for $\delta \in \{1.8, 1.9, 2\}$. For $\delta = 1.75$ the iteration converged to $u = v = 0$.

Figure 6: table of actual δ with the initial function

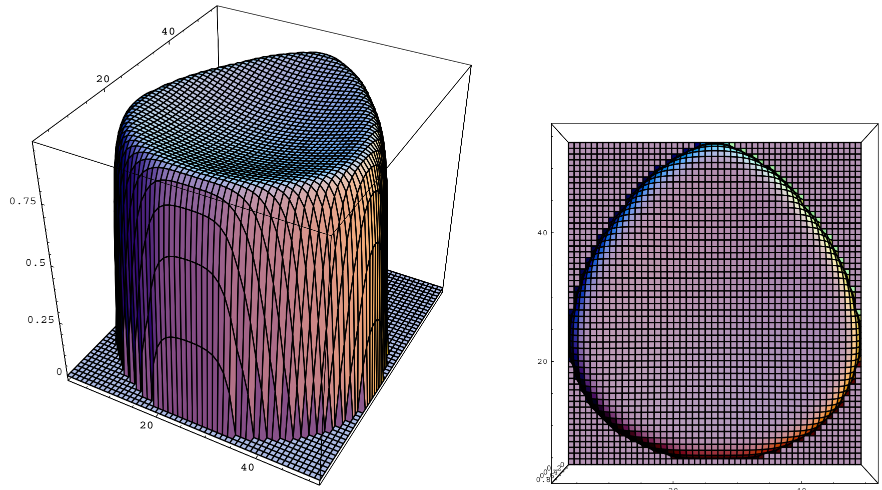


We fact expect that these solutions form an S-shaped bifurcation curve for the parameter δ. In fact, if we set out $\left(\delta, \min\left\{u\left(x\right); x \in \Omega_{(.2)}\right\}\right)$ and thus leaving out the boundary layer, the curve would have the shape of a reflected S. As the third solution in between is expected to be unstable it cannot be determined by the iteration scheme used here. All of this is motivation for making Conjecture 3.4.

7.4. Numerical illustrations

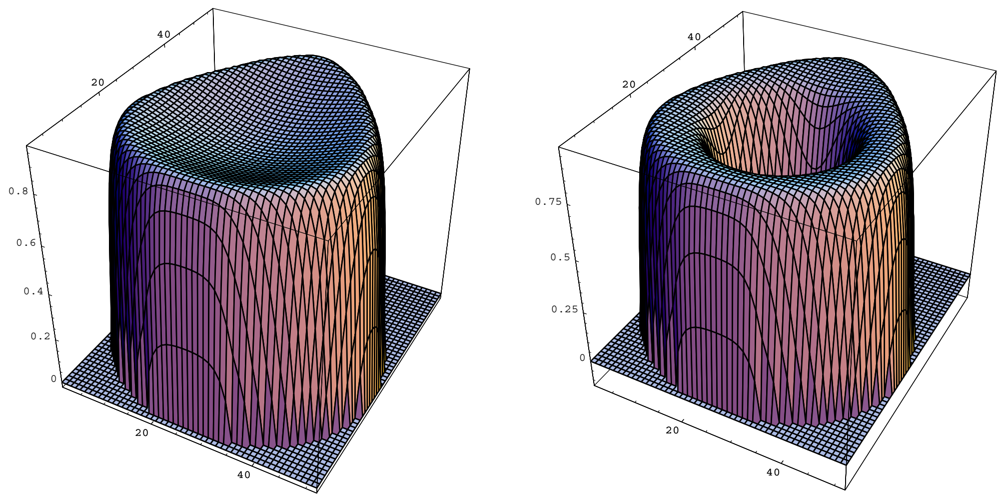
Here we present some of the numerical approximations which have been calculated, and which give some numerical support for the conjecture made in the previous section.

Figure 7: $\delta = 0.3$, starting either with u^* or u^{**}.



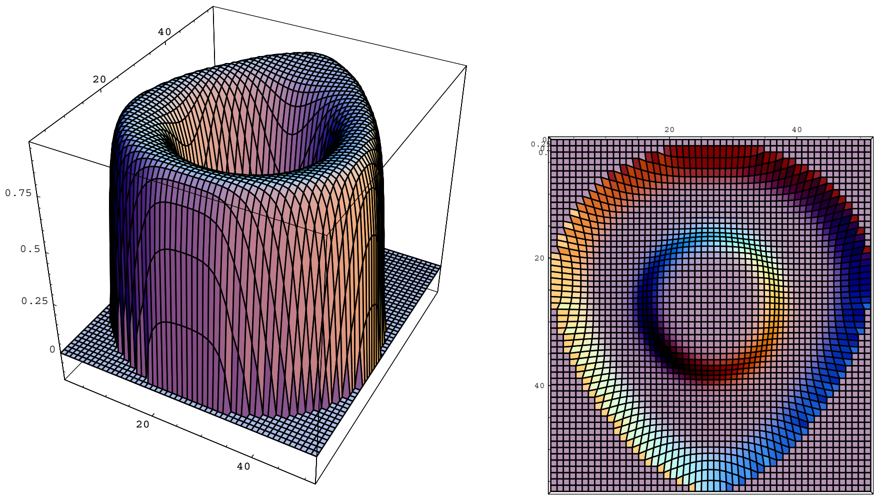
(a) View from the side (b) View from above

Figure 8: $\delta = 0.35$



(a) Starting with u^* (b) Starting with u^{**}

Figure 9: $\delta = 0.475$, starting either with u^* or u^{**}.



(a) View from the side

(b) View from below

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