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Sign change for Green function and eigenfunction of clamped plate type equations

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1. Introduction

By results of Boggio ([2], [3]) it is known that on balls in $\mathbb{R}^n$, with $n \geq 1$, the biharmonic Dirichlet problem is order preserving, that is, the solution u of

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \frac{\partial}{\partial n} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

for $\Omega = \{x \in \mathbb{R}^n; |x| < R\}$ satisfies $u \geq 0$ whenever $f \geq 0$. One also finds that the corresponding first eigenfunction is strictly positive. This biharmonic Dirichlet problem appears in the linearized clamped plate equation.

For the (second-order) Laplace problem such a positivity preserving property holds for every domain in $\mathbb{R}^n$. Although an early conjecture of Boggio and Hadamard ([14], [15]) claimed that such a result would hold for the biharmonic operator on arbitrary nice convex domains, numerous counterexamples have been constructed since then. On many domains both the biharmonic Dirichlet problem is not order preserving nor is the first eigenfunction of fixed sign. See the counterexamples to the Boggio-Hadamard Conjecture of [7], [8], [4], [6], [17] and [23]. Two-dimensional domains which are in an appropriate sense close to a ball however do have the sign preserving property. Such a result was obtained in [11].

The sign-preserving property for (1) is equivalent to having a positive Green function. By an application of Jentzsch's Theorem ([16]), or the Krein-Rutman Theorem ([19]), it follows that a strictly positive Green

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function implies that the first eigenvalue of

$$\begin{cases} \Delta^2\varphi = \lambda\varphi & \text{in } \Omega, \\ \varphi = \frac{\partial}{\partial n}\varphi = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

is simple and that the corresponding eigenfunction is positive. Hence in two dimensions the eigenfunction also remains positive if the domain is 'not far' from a ball. By a direct approach it has even been shown ([12]) that in higher dimensions the sign of the eigenfunction remains fixed under small perturbations of the domain.

A question in this area still left open was mentioned to us by James Serrin [24]. Suppose that one considers a smooth deformation of the domain, say $t \mapsto \Omega_t$ with $\Omega_0 = B$ and B a ball. For such a family of domains we rephrase this question as:

Can it happen that 'positivity preserving' and 'first eigenfunction positive' fail simultaneously?

Or in other words, if t_e is the largest number such that (2) on Ω_t has a simple first eigenvalue with positive eigenfunction for all $t \in [0, t_e)$ and if t_g is the largest number such that (1) on Ω_t is strongly positivity preserving for all $t \in [0, t_g)$, can it happen that $t_e = t_g$? By the argument using Jentzsch's result one finds $t_e \geq t_g$. We will show that for appropriately defined deformations the inequality is strict and hence the answer to the question above is negative: $t_e > t_g$.

In order to study this question we include a real parameter λ in (1) and consider for $\Omega \subset \mathbb{R}^n$, a bounded domain with $\partial\Omega \in C^{2m+1}$, the problem

$$\begin{cases} (-\Delta)^m u = \lambda u + f & \text{in } \Omega, \\ \left(\frac{\partial}{\partial n}\right)^k u = 0 & \text{on } \partial\Omega \text{ with } 0 \leq k \leq m-1, \end{cases} \tag{3}$$

and we allow $m \in \{2, 3, \ldots\}$. Positivity preserving properties of this system have been studied in [12]. For $\Omega = B_1 = \{x \in \mathbb{R}^n; |x| < 1\}$ there is $\lambda_{c,m,n} \in (-\infty, 0)$ such that (3) is positivity preserving if and only if $\lambda \in (\lambda_{c,m,n}, \lambda_{1,m,B_1})$.

Note that, since the realization in $L^2(\Omega)$, with $D = H^{2m,2}(\Omega) \cap H_0^{m,2}(\Omega)$, of the boundary value problem (3) is self-adjoint and has a compact inverse, the spectrum is discrete and consists of real eigenvalues. Since the coefficients involved are constant and the boundary is C^{2m+1}, the eigenfunctions are identical for the realization in $L^2(\Omega)$ as well as in $C(\bar{\Omega})$. Indeed, the eigenfunctions are in $C^{2m}(\bar{\Omega})$ by standard regularity results (see [1]). Note that for each domain Ω the first eigenvalue $\lambda_{1,m,\Omega}$ is well defined by the corresponding Rayleigh-quotient. Isoperimetric questions for the principal eigenvalue of the biharmonic Dirichlet problem have been studied by Talenti in [25].

2. Varying the parameter in the resolvent

Let us denote by $\lambda_{1,m,\Omega} \in \mathbb{R}^+$ the first eigenvalue and by $G_{m,\lambda,\Omega}$ the Green kernel corresponding to (3). In other words, the solution of (3) for $\lambda < \lambda_{1,m,\Omega}$ can be written by means of this kernel:

$$u(x) = (\mathcal{G}_{m,\lambda,\Omega} f)(x) := \int_\Omega G_{m,\lambda,\Omega}(x,y) f(y)\, dy.$$

Since the boundary value problem is self-adjoint with respect to the standard inproduct $G_{m,\lambda,\Omega}(x,y) = G_{m,\lambda,\Omega}(y,x)$ for all $x \neq y \in \Omega$. By [1] it follows that $G_{m,\lambda,\Omega}(x,\cdot) \in C^{2m,\gamma}\left(\bar{\Omega}\backslash\{x\}\right)$ for all $x \in \Omega$, $\gamma \in (0,1)$.

Since all eigenvalues are real and satisfy $\lambda \geq \lambda_{1,m,\Omega}$ one finds that for all $\mu \in \mathbb{R}$ with $\lambda < \mu < \lambda_{1,m,\Omega}$ the solution of

$$\begin{cases} (-\Delta)^m u = \mu\, u + f & \text{in } \Omega, \\ \left(\frac{\partial}{\partial n}\right)^k u = 0 & \text{on } \partial\Omega \text{ with } 0 \leq k \leq m-1, \end{cases} \tag{4}$$

with $f \in C\left(\bar{\Omega}\right)$ is well defined by the following Neumann series:

$$u = \sum_{k=0}^{\infty} \Big(\mathcal{G}_{m,\lambda,\Omega}(\mu - \lambda)\Big)^k \mathcal{G}_{m,\lambda,\Omega} f. \tag{5}$$

Since $\mathcal{G}_{m,\lambda,\Omega}$ is a positive definite operator, the spectral radius of $\mathcal{G}_{m,\lambda,\Omega}$ satisfies $\nu(\mathcal{G}_{m,\lambda,\Omega}) = (\lambda_{1,m,\Omega} - \lambda)^{-1}$.

Lemma 1. *Suppose that Ω and $\lambda < \lambda_{1,m,\Omega}$ are such that*

$$G_{m,\lambda,\Omega}(x,y) \geq 0 \qquad \textit{for all } x \neq y \in \Omega.$$

Then, for $\mu \in \mathbb{R}$ with $\lambda < \mu < \lambda_{1,m,\Omega}$ one has

$$G_{m,\mu,\Omega}(x,y) > 0 \qquad \textit{for all } x \neq y \in \Omega.$$

A related result in an abstract setting can be found in [5].

Proof. Since the operators $\mathcal{G}_{m,\lambda,\Omega}$ are nonnegative, it follows that $\mathcal{G}_{m,\mu,\Omega} = \sum_{k=0}^{\infty} (\mu-\lambda)^k \mathcal{G}_{m,\lambda,\Omega}^{k+1} \geq (\mu - \lambda)\mathcal{G}_{m,\lambda,\Omega}^2$, or in other words:

$$G_{m,\mu,\Omega}(x,y) \geq (\mu - \lambda) \int_{z\in\Omega} G_{m,\lambda,\Omega}(x,z)\, G_{m,\lambda,\Omega}(z,y)\, dz. \tag{6}$$

Now fix $x \neq y \in \Omega$. Note that $G_{m,\lambda,\Omega}(x,\cdot)$ cannot be identically zero. Indeed, let $\varphi_{1,1}$ be the eigenfunction of (3) for $m = 1$ and use $(\varphi_{1,1})^m$ as a testfunction. Since $\varphi_{1,1} > 0$ in Ω and $(\varphi_{1,1})^m$ satisfies the boundary conditions we find that $G_{m,\lambda,\Omega}(x,\cdot) \equiv 0$ implies

$$0 < (\varphi_{1,1}(x))^m = \mathcal{G}_{m,\lambda,\Omega}\Big(\left((-\Delta)^m - \lambda\right)(\varphi_{1,1})^m\Big)(x) = 0,$$

a contradiction. Hence there are $\delta, \varepsilon > 0$ and $z_0 \in \Omega$ such that $B_\delta(z_0) \subset \Omega \setminus \{x\}$ and

$$G_{m,\lambda,\Omega}(x,z) > \varepsilon > 0 \text{ for } z \in B_\delta(z_0). \tag{7}$$

Next we show that $G_{m,\lambda,\Omega}(\cdot, y)$ cannot be identically zero on open sets. Suppose that $G_{m,\lambda,\Omega}(\cdot, y) = 0$ on some open set $U \subset \Omega$. Then the unique continuation principle for elliptic equations (see [20]) applied to $((-\Delta)^m - \lambda) G_{m,\lambda,\Omega}(\cdot, y) = 0$ on $\Omega \setminus \{y\}$, implies that $G_{m,\lambda,\Omega}(\cdot, y) \equiv 0$ on $\Omega \setminus \{y\}$, a contradiction. For $U = B_\delta(z_0)$ it follows that $G_{m,\lambda,\Omega}(\cdot, y) \not\equiv 0$.

Combining with (6) and (7), we have, with $\varepsilon^* = (\mu - \lambda)\varepsilon$,

$$G_{m,\mu,\Omega}(x,y) \geq \varepsilon^* \int_{z \in B_\delta(z_0)} G_{m,\lambda,\Omega}(z,y)\, dz > 0. \qquad \square$$

Lemma 2. *Suppose that Ω and $\lambda < \lambda_{1,m,\Omega}$ are such that*

$$G_{m,\lambda,\Omega}(x,y) \geq 0 \qquad \text{for all } x \neq y \in \Omega.$$

Then, for $\mu \in \mathbb{R}$ with $\lambda < \mu < \lambda_{1,m,\Omega}$, one has

$$\left(\frac{\partial}{\partial \boldsymbol{n}_x}\right)^m G_{m,\mu,\Omega}(x,y) > 0 \qquad \text{for all } x \in \partial\Omega,\ y \in \Omega, \tag{8}$$

where $\boldsymbol{n}_x$ denotes the inward normal.

Proof. Let $y \in \Omega$. By the previous lemma $G_{m,\mu,\Omega}(\cdot, y) > 0$ holds in Ω and since $\left(\frac{\partial}{\partial \boldsymbol{n}_x}\right)^k G_{m,\mu,\Omega}(\cdot, y) = 0$ on $\partial\Omega$ for $k < m$, it follows that

$$\left(\frac{\partial}{\partial \boldsymbol{n}_x}\right)^m G_{m,\mu,\Omega}(\cdot, y) \geq 0 \text{ on } \partial\Omega. \tag{9}$$

Next we fix $x \in \partial\Omega$. From the resolvent formula

$$\mathcal{G}_{m,\mu,\Omega} = \mathcal{G}_{m,\lambda,\Omega}\left(\mathcal{I} + (\mu - \lambda)\,\mathcal{G}_{m,\mu,\Omega}\right)$$

we find

$$G_{m,\mu,\Omega}(x,y) = G_{m,\lambda,\Omega}(x,y) + (\mu - \lambda) \int_\Omega G_{m,\lambda,\Omega}(x,z)\, G_{m,\mu,\Omega}(z,y)\, dz.$$

Using $G_{m,\mu,\Omega}(\cdot, y) \geq 0$ and (9), which also holds for $\mu = \lambda$, it follows that

$$\left(\frac{\partial}{\partial \boldsymbol{n}_x}\right)^m G_{m,\mu,\Omega}(x,y) = \left(\frac{\partial}{\partial \boldsymbol{n}_x}\right)^m G_{m,\lambda,\Omega}(x,y)\ +$$

$$(\mu - \lambda) \int_\Omega \left(\frac{\partial}{\partial \boldsymbol{n}_x}\right)^m G_{m,\lambda,\Omega}(x,z)\, G_{m,\mu,\Omega}(z,y)\, dz.$$

By the Green function estimates of Krasovskiĭ ([18]) the integral is well-defined for quite general $2m^{th}$-order elliptic boundary value problems on bounded domains in $\mathbb{R}^n$:

$$\left|\frac{\partial^{|\alpha|+|\beta|}}{\partial x^\alpha \partial y^\beta} G(x,y)\right| \leq c_{G,\alpha,\beta} \begin{cases} 1 & \text{if } 2m-n > |\alpha|+|\beta|, \\ \log\left(1+|x-y|^{-1}\right) & \text{if } 2m-n = |\alpha|+|\beta|, \\ |x-y|^{2m-n-|\alpha|-|\beta|} & \text{if } 2m-n < |\alpha|+|\beta|. \end{cases}$$

Since by the previous lemma $G_{m,\mu,\Omega}(z,y) > 0$ for all $z \in \Omega\backslash\{y\}$, we find that if $\left(\frac{\partial}{\partial n_x}\right)^m G_{m,\lambda,\Omega}(x,z) > 0$ for some $z \in \Omega$, then the estimate in (8) holds for all $y \in \Omega$. Note that $z \mapsto \left(\frac{\partial}{\partial n_x}\right)^m G_{m,\lambda,\Omega}(x,z) \in C\left(\bar{\Omega}\backslash\{x\}\right)$.

Now suppose that $\left(\frac{\partial}{\partial n_x}\right)^m G_{m,\mu,\Omega}(x,z) = 0$ for all $z \in \Omega$. Then for every $f \in C\left(\bar{\Omega}\right)$ the solution u of

$$\begin{cases} (-\Delta)^m u = \mu u + f & \text{in } \Omega, \\ \left(\frac{\partial}{\partial n}\right)^k u = 0 & \text{on } \partial\Omega \text{ with } 0 \leq k \leq m-1, \end{cases} \tag{10}$$

satisfies $\left(\frac{\partial}{\partial n}\right)^m u(x) = 0$. Let $d(z,\partial\Omega)$ denote the distance of z to $\partial\Omega$, that is

$$d(z,\partial\Omega) = \inf\{|z-z'|; z' \in \partial\Omega\}.$$

Note that $d(\cdot,\partial\Omega)$ is Lipschitz continuous for every Ω. However, since $\partial\Omega \in C^{2m+1}$, there exists a neighborhood $\Gamma_\delta = \left\{z \in \bar{\Omega}; d(z,\partial\Omega) < \delta\right\}$ such that $d(\cdot,\partial\Omega) \in C^{2m+1}(\Gamma_\delta)$, see [9, Lemma 14.16]. Let $w \in C^\infty\left(\bar{\Omega}\right)$ be such that $w = 1$ on $\Gamma_{\delta/2}$ and $w = 0$ on $\bar{\Omega}\backslash\Gamma_\delta$. Then u_*, defined by $u_*(z) = d(z,\partial\Omega)^m w(z)$, satisfies (10) for $f = ((-\Delta)^m - \mu) u_*$ and $\left(\frac{\partial}{\partial n}\right)^m u_*(x) > 0$, a contradiction. □

Corollary 1. *Suppose that Ω and $\lambda < \lambda_{1,m,\Omega}$ are such that*

$$G_{m,\lambda,\Omega}(x,y) \geq 0 \qquad \text{for all } x \neq y \in \Omega.$$

Then, after suitable normalization, the first eigenfunction $\varphi_{1,m,\Omega}$ is unique and satisfies for some $c_1 > 0$

$$\varphi_{1,m,\Omega}(x) \geq c_1\, d(x,\partial\Omega)^m \qquad \text{for all } x \in \Omega.$$

Remark 1. Note that the boundary conditions and the fact that $\varphi_{1,m,\Omega} \in C^{2m}\left(\bar{\Omega}\right)$ imply that there is $c_2 > 0$ such that

$$\varphi_{1,m,\Omega}(x) \leq c_2\, d(x,\partial\Omega)^m \qquad \text{for all } x \in \Omega.$$

Proof (of Corollary 1). Since $G_{m,\mu,\Omega}(x,y) > 0$ for $x \neq y \in \Omega$ and $\mu \in (\lambda, \lambda_{1,m,\Omega})$, one may use Jentzsch's Theorem (see [21, Theorem 6.6] or [16]) to find that $\mathcal{G}_{m,\mu,\Omega}$ has a unique (normalized) eigenfunction φ_1 with $\varphi_1 > 0$ a.e. in Ω. The stronger estimate is a consequence of $\varphi_{1,m,\Omega} = (\lambda_{1,m,\Omega} - \mu)\, \mathcal{G}_{m,\mu,\Omega}\varphi_{1,m,\Omega}$ and the estimates for $G_{m,\mu,\Omega}(x,y)$ in the previous two lemmata. □

Remark 2. For the biharmonic operator in one dimension, Schröder [22, Theorem 6.3] gives conditions where positivity breaks down. We take $\Omega = (-1,1)$ and consider

$$\begin{cases} u''''(x) = \lambda u(x) + f(x) & \text{for } -1 < x < 1 \\ u(-1) = u'(-1) = u(1) = u'(1) = 0. \end{cases} \tag{11}$$

Using his terminology we set $\mathcal{M} = \{(L_\lambda, B); \lambda \in I\}$ with $L_\lambda u = u'''' - \lambda u$ and $Bu = \Big(u(-1), u'(-1), u(1), u'(1)\Big)$. Let I be an interval containing 0. For $\lambda = 0$ the boundary value problem (11) is inverse positive. According to [22], inverse-positivity breaks down when:

1. $L_\lambda u = 0$, $Bu = 0$ has a nontrivial solution, or
2. $L_\lambda u = 0$, $\Big(u(-1), u'(-1), u''(-1), u(1)\Big) = 0$ has a nontrivial solution.

In our case, his third possibility equals the second by symmetry. The first possibility is reached for $\lambda = \lambda_1 > 0$, with λ_1 the first eigenvalue. Since

$$\varphi_1(x) = \cosh\left(\sqrt[4]{\lambda_1}\right)\cos\left(\sqrt[4]{\lambda_1}x\right) - \cos\left(\sqrt[4]{\lambda_1}\right)\cosh\left(\sqrt[4]{\lambda_1}x\right)$$

where λ_1 is such that $\varphi_1'(1) = \varphi_1'(-1) = 0$, we find that λ_1 is the smallest positive zero of

$$\tan\left(\sqrt[4]{\lambda_1}\right) = -\tanh\left(\sqrt[4]{\lambda_1}\right).$$

The second possibility is reached for $\lambda = \lambda_c < 0$ with

$$\varphi_c(x) = \sin(y)\cosh(y) - \cos(y)\sinh(y),$$

where $y = \sqrt[4]{\frac{-\lambda_c}{4}}\,(x+1)$. The number λ_c is determined by $\varphi_c(1) = 0$, that is, λ_c is the largest negative number such that

$$\tan\left(2\sqrt[4]{\frac{-\lambda_c}{4}}\right) = \tanh\left(2\sqrt[4]{\frac{-\lambda_c}{4}}\right).$$

A numerical calculation yields $\lambda_1 = 31.285243\ldots$ and $\lambda_c = -59.430266\ldots$. For $\lambda \in (\lambda_c, \lambda_1)$ one concludes that (11) is inverse positive.

In this fourth order example Lemma 2 is sharp in the sense that for $\lambda < \lambda_c$ and $|\lambda - \lambda_c|$ small there exist $y \in (-1,1)$ (close to 1) such that $\left(\frac{\partial}{\partial x}\right)^2 G_{2,\lambda,(-1,1)}(-1,y) < 0$. This can be shown by an explicit calculation of these Green functions.

3. Perturbing the domain

It is not obvious that for small perturbations of the domain the Green function or the eigenfunction remains positive. Such problems have been studied in [11] and [12]. In [11] it is shown that in 2-dimensional domains the Green function for $\lambda = 0$, and hence also the first eigenfunction, remains positive whenever the domain Ω is close to the disk in $\mathbb{R}^2$ in C^{2m+1}-sense. Whether or not the Green function remains positive for small perturbations of $\Omega \subset \mathbb{R}^n$ with $n > 2$ is still an open question. The eigenfunction however is shown to remain positive for small domain perturbations in any dimension ([12]).

For a precise statement we need to define a continuous family of C^k-domains. The families of domains that we will consider start with $\Omega_0 = B$, a ball, and will deform C^k-smoothly. We will be interested in the largest t_e, t_g such that the first eigenfunction, respectively the Green function on Ω_t, is strictly positive for all $t \in [0, t_e)$, respectively $t \in [0, t_g)$.

Definition 1. *Let k be a positive integer. We say that the collection $\{\Omega_t; t \in [0,1]\}$ is a continuous family of C^k-domains in $\mathbb{R}^n$ if there exists a family of functions $\left\{h_t \in C^k\left(\bar{\Omega}_0; \mathbb{R}^n\right); t \in [0,1]\right\}$ such that:*

1. *Ω_0 is bounded and $\partial\Omega_0 \in C^k$;*
2. *for every $t \in [0,1]$ the mapping $h_t : \bar{\Omega}_0 \to \bar{\Omega}_t$ is a C^k-diffeomorphism;*
3. *for every $t \in [0,1]$ we have $\lim_{s\to t} \|h_s - h_t\|_{C^k(\overline{\Omega}_0)} = 0$.*

3.1. Influence on a solution

Let $\{\Omega_t; t \in [0,1]\}$ be a continuous family of C^{2m+1}-domains in $\mathbb{R}^n$ and let $\{h_t; t \in [0,1]\}$ be corresponding mappings mentioned in Definition 1. For fixed $f \in C\left(\bar{\Omega}_0\right)$ we consider

$$\begin{cases} (-\Delta)^m u = f \circ \left(h_t^{-1}\right) & \text{in } \Omega_t, \\ \left(\frac{\partial}{\partial n}\right)^k u = 0 \quad \text{for } 0 \le k \le m-1 & \text{on } \partial\Omega_t. \end{cases} \tag{12}$$

We will be interested in the case that $\Omega_0 = B$, the unit ball in $\mathbb{R}^n$.

Lemma 3. *Let $\{\Omega_t; t \in [0,1]\}$ be a continuous family of C^{2m+1}-domains in $\mathbb{R}^n$ and let $f \in C\left(\bar{\Omega}_0\right)$. For each t let u_t denote the solution of (12). Then*

$$\lim_{s\to t} \left\|u_s \circ h_s \circ \left(h_t^{-1}\right) - u_t\right\|_{C^{2m}\left(\bar{\Omega}_t\right)} = 0.$$

Proof. For every $t \in [0,1]$ we will denote the unique solution of (12) by u_t. Let us define two auxiliary functions on Ω_t, namely $f_t = f \circ \left(h_t^{-1}\right)$ and $\tilde{u}_s = u_s \circ h_s \circ \left(h_t^{-1}\right)$. For simplicity we also write $g_s = h_s \circ \left(h_t^{-1}\right)$. We will compare u_t and $\tilde{u}_s$.

First notice that for m even

$$\int_{\Omega_s} \left(\Delta^{\frac{m}{2}} u_s\right)^2 dx = \int_{\Omega_s} u_s\, f_s\, dx;$$

this together with Cauchy-Schwarz and the Poincaré-inequality implies that

$$\|u_s\|_{H^{m,2}(\Omega_s)} \leq c_{R,m,n} \|f_s\|_{L^2(\Omega_s)}, \tag{13}$$

where the constant $c_{R,m,n}$ depends only on m, n and the radius R of the smallest ball that contains $\bigcup_{t\in[0,1]} \Omega_t$. For m odd, the same estimate (13) holds using $\int_{\Omega_s} \left|\nabla \left(\Delta^{\frac{m}{2}} u_s\right)\right|^2 dx = \int_{\Omega_t} u_s\, f_s\, dx$. Since $\|f_s\|_{L^2(\Omega_s)}$ is uniformly bounded for $s \in [0,1]$, so is $\|u_s\|_{H^{m,2}(\Omega_s)}$ and hence also $\|\tilde{u}_s\|_{H^{m,2}(\Omega_t)}$.

For $w \in C^2\left(\bar{\Omega}_s\right)$ we have $w \circ g_s \in C^2\left(\bar{\Omega}_t\right)$ and

$$(\Delta w) \circ g_s = A_s\left(w \circ g_s\right)$$

with

$$A_s = \sum_{k=1}^{n}\sum_{l=1}^{n} \left(\nabla \left(g_s^{-1}\right)_k \cdot \nabla \left(g_s^{-1}\right)_l\right) \circ g_s(\cdot)\ \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_l} + \sum_{k=1}^{n} \left(\Delta \left(g_s^{-1}\right)_k\right) \circ g_s(\cdot)\ \frac{\partial}{\partial x_k},$$

where $\left(g_s^{-1}\right)_k$ is the k^{th} component of g_s^{-1}. The function $\tilde{u}_s$ satisfies

$$\begin{cases} \left(-A_s\right)^m u = f_t & \text{in } \Omega_t, \\ \left(\frac{\partial}{\partial n}\right)^k u = 0 \quad \text{for } 0 \leq k \leq m-1 & \text{on } \partial\Omega_t. \end{cases} \tag{14}$$

Since $\left(-A_s\right)^m$ is uniformly elliptic on Ω_t uniformly in s, there are Λ and b such that $\left(-A_s\right)^m = \sum_{|\alpha|\leq 2m} a_\alpha^s(x) \left(\frac{\partial}{\partial x}\right)^\alpha$ satisfies

$$\Lambda^{-1}|\xi|^{2m} \leq \sum_{|\alpha|=2m} a_\alpha^s(x)\,\xi^\alpha \leq \Lambda\,|\xi|^{2m} \quad \text{and} \quad \|a_\alpha^s\|_{C^1} \leq b \text{ for } |\alpha| \leq 2m.$$

By [1, Theorem 15.2], respectively [1, Theorem 7.3], we find that for all $s \in [0,1]$ there exist constants $C_{\Omega_t,\Lambda,b,n,2m,p}$ and $C_{\Omega_t,\Lambda,b,n,2m,\gamma}$, which do not depend on s, such that

$$\|\tilde{u}_s\|_{H^{2m,p}(\Omega_t)} \leq C_{\Omega_t,\Lambda,b,n,2m,p}\left(\|f_t\|_{L^p(\Omega_t)} + \|\tilde{u}_s\|_{L^p(\Omega_t)}\right) \text{ and}$$

$$\|\tilde{u}_s\|_{C^{2m+\gamma}(\bar{\Omega}_t)} \leq C_{\Omega_t,\Lambda,b,n,2m,\gamma}\left(\|f_t\|_{C^\gamma(\bar{\Omega}_t)} + \|\tilde{u}_s\|_{C^0(\bar{\Omega}_t)}\right).$$

Since we have a uniform bound for $\|\tilde{u}_s\|_{H^{m,2}(\Omega_t)}$ we may use Sobolev embeddings (on the fixed domain Ω_t) and a bootstrapping argument to find a bound for $\|\tilde{u}_s\|_{C^{2m+\gamma}(\bar{\Omega})}$ which is uniform in $s \in [0,1]$. By the property that $h_s \to h_t$ in C^{2m+1}, and hence $g_s \to g_t$ in C^{2m+1} as $s \to t$, it follows that $a^s_\alpha \to a^t_\alpha$ in C^1 for $s \to t$. Since $\|\tilde{u}_s\|_{C^{2m+\gamma}(\bar{\Omega})}$ is uniformly bounded we find that

$$\left\| \left((-\Delta)^m - (-A_s)^m \right) \tilde{u}_s \right\|_{C^\gamma} \to 0 \text{ for } s \to t.$$

From $(-\Delta)^m \tilde{u}_s = f_t + \left((-\Delta)^m - (-A_s)^m \right) \tilde{u}_s$ it follows that

$$\lim_{s \to t} \|\tilde{u}_s - u_t\|_{C^{2m}(\bar{\Omega}_t)} = 0. \qquad \square$$

3.2. Dependence on the first eigenfunction

The proof in [12] of the result that the eigenfunction remains positive for Ω close to the ball uses the estimate

$$c_1\, d(x, \partial B)^m \le \varphi_{1,m,\Omega}(x) \le c_2\, d(x, \partial B)^m \text{ for } x \in B. \tag{15}$$

Since the proof only uses this estimate, and not the fact that B is a ball, we may refine this proof in order to show the following result.

Theorem 1. *Let $\{\Omega_t; t \in [0,1]\}$ be a continuous family of C^{2m+1}-domains in $\mathbb{R}^n$ with $t_0 \in [0,1]$ and $\Omega_{t_0} = \Omega$. Suppose that $\lambda_{1,m,\Omega}$ is the first eigenvalue of (3), that $\lambda_{1,m,\Omega}$ is simple, and that*

$$0 < \underline{c} = \inf_{x \in \Omega} \frac{\varphi_{1,m,\Omega}(x)}{d(x, \partial\Omega)^m} \le \sup_{x \in \Omega} \frac{\varphi_{1,m,\Omega}(x)}{d(x, \partial\Omega)^m} = \bar{c} < \infty. \tag{16}$$

Then for all $\varepsilon > 0$ there is $\delta > 0$ such that for $t \in [0,1]$ with $|t - t_0| < \delta$:

1. *the first eigenvalue λ_{1,m,Ω_t} is simple and $|\lambda_{1,m,\Omega} - \lambda_{1,m,\Omega_t}| \le \varepsilon$;*
2. *the corresponding eigenfunction φ_{1,m,Ω_t}, after suitable normalization, satisfies*

$$\underline{c} - \varepsilon = \inf_{x \in \Omega_t} \frac{\varphi_{1,m,\Omega_t}(x)}{d(x, \partial\Omega_t)^m} \le \sup_{x \in \Omega_t} \frac{\varphi_{1,m,\Omega_t}(x)}{d(x, \partial\Omega_t)^m} = \bar{c} + \varepsilon \tag{17}$$

Proof. See [12, Lemma 5.1 and Theorem 5.2]. For h_t as in Definition 1, one finds that there exists $\delta > 0$ such that for $|t - t_0| < \delta$

$$\left\| \varphi_{1,m,\Omega}(\cdot) - \varphi_{1,m,\Omega_t}\left(h_t \circ h_{t_0}^{-1}(\cdot) \right) \right\|_{C^{2m}(\bar{\Omega}_{t_0})} \le \varepsilon.$$

Since

$$\left(\frac{\partial}{\partial \boldsymbol{n}_x} \right)^k \varphi_{1,m,\Omega} = \left(\frac{\partial}{\partial \boldsymbol{n}_x} \right)^k \left(\varphi_{1,m,\Omega_t} \circ h_t \circ h_{t_0}^{-1} \right) = 0$$

for all $k \in \{0, \dots, m-1\}$, it follows that

$$\left|\varphi_{1,m,\Omega}(x) - \varphi_{1,m,\Omega_t}\left(h_t \circ h_{t_0}^{-1}(x)\right)\right| \le \varepsilon\, d(x, \partial\Omega)^m,$$

implying (17). □

3.3. Conclusion for the sign change

Theorem 2. *Let $\{\Omega_t \subset \mathbb{R}^n; t \in [0,1]\}$ be a continuous family of C^{2m+1}-domains with $\Omega_0 = B$. Set*

$$t_g = \sup\{t \in [0,1]\,; G_{m,0,\Omega_s}(x,y) > 0 \textit{ for all } x \ne y \in \Omega \textit{ and } s \le t\}.$$

Then there is $\varepsilon > 0$ such that for all $t < t_g + \varepsilon$ the first eigenvalue is simple and the corresponding eigenfunction satisfies

$$\varphi_{1,m,\Omega_t} > 0 \quad \textit{in } \Omega_t. \tag{18}$$

Proof. Suppose that $G_{m,0,\Omega_{t_g}}(x,y)$ changes sign. Since $G_{m,0,\Omega_{t_g}}$ is continuous on $\Omega^2 \backslash \{(x,x)\,; x \in \Omega\}$ there is $f \in C\left(\bar{\Omega}_{t_g}\right)$ with $f \ge 0$ and the corresponding solution u being negative somewhere. Let u_t be the solution of (12) with this f on the right hand side. Then we find by Lemma 3 that $u_t \circ h_t \circ h_{t_g}^{-1} \to u_{t_g} = u$ uniformly. Since $u_t \ge 0$ for all $t < t_g$ and $u_{t_g} \not\ge 0$ we obtain a contradiction. Hence we have $G_{m,0,\Omega_{t_g}}(x,y) \ge 0$ for all $x, y \in \Omega_{t_g}^2$. Then we find by Corollary 1 that $\varphi_{1,m,\Omega_{t_g}} \ge c\, d(x)^m$ for some $c > 0$. By Theorem 1 it follows that there is $\varepsilon > 0$ such that (18) holds for all $t \in [t_g, t_g + \varepsilon)$ and hence for all $t \in [0, t_g + \varepsilon)$. □

We have proven that $G_{m,0,\Omega_{t_g}} \ge 0$ and $\varphi_{1,m,\Omega_{t_g+\varepsilon}} > 0$ for small positive ε. In order to answer the question whether or nor the Green function remains positive under small perturbations of the domain, that is $t_g > 0$, one would have to show that $G_{m,0,\Omega_t} \succ 0$ implies $G_{m,0,\Omega_{t+\varepsilon}} \succ 0$ for small positive ε. As mentioned before this question is still open in dimensions $n \ge 3$. The symbol $\succ$ denotes a strict ordering in an appropriate lattice. In [10] one finds $\succ$ defined as follows for the Green function of (3) with $2m > n$: $G \succ 0$ if and only if there is $c > 0$ such that for all $x, y \in \Omega$

$$G(x,y) \ge c\left(d(x,\partial\Omega)\, d(y,\partial\Omega)\right)^{m-\frac{1}{2}n} \min\left(1, \frac{d(x,\partial\Omega)\, d(y,\partial\Omega)}{|x-y|^2}\right)^{\frac{1}{2}n}.$$

Note that from our proofs one finds that $G_{m,\lambda,\Omega_t} \succ 0$ implies $G_{m,\mu,\Omega_t} \succ 0$ for $\mu \in (\lambda, \lambda_{1,m,\Omega_t})$. Assuming that the Green function remains positive in $\succ$-sense under domain perturbations for all $\lambda \in (\lambda_{c,t}, \lambda_{1,t})$, a schematic graph of t and λ for which the Green function is positive should have the appearance that is shown in Fig. 1. In the grey area $G_{m,\lambda,\Omega_t} > 0$ holds true.

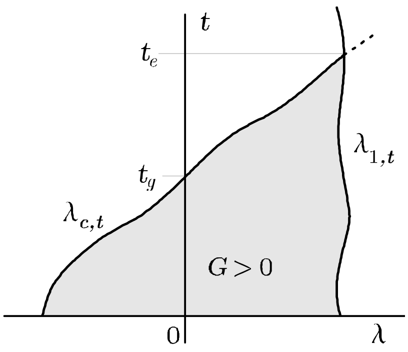

Fig. 1.

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