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THE EXISTENCE OF THE PRINCIPAL EIGENVALUE FOR COOPERATIVE ELLIPTIC SYSTEMS IN A GENERAL DOMAIN

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1. Introduction

In the present paper we investigate the existence of the principal eigenfunction of the vector-valued elliptic eigenvalue problem

$$(\mathscr{L} - H)\Phi = \lambda B\Phi \quad \text{in} \quad \Omega, \qquad \Phi = 0 \quad \text{on} \quad \partial\Omega, \tag{1}$$

and its relationship with the maximum principle. We assume that the operator $\mathscr{L}$ is a diagonal matrix consisting of uniformly elliptic second-order partial differential operators and H and B are cooperative matrices with entries from $C\left(\bar{\Omega}\right)$. The domain $\Omega \subset \mathbf{R}^N$ is bounded. We do not assume that the boundary satisfies any regularity condition.

Systems of elliptic and parabolic differential equations appear in the investigations of models of population dynamics, combustion theory, and nerve conduction (e.g., see [1, 2]).

Elliptic spectral problems play an important role in the existence, uniqueness, and stability of solutions of such systems [3 - 6]. They are also closely related to the maximum principle and comparison theorems.

In most papers dealing with elliptic and parabolic systems (e.g., see [7 - 14]), the boundary of the domain Ω is assumed to be smooth. The absence of this condition results in difficulties even in the case of a single equation. There are at least two approaches for considering the case of a single equation in a domain with no condition of the domain regularity. In Perron regular domains [15], solutions continuous up to the boundary can be defined by means of the passage to the limit. In general domains, self-adjoint problems can be investigated in the weak sense with the help of minimizing the corresponding energy functional.

An eigenvalue problem with no assumption on the boundary regularity and with no use of the self-adjointness was analyzed in [16] (see also [17 - 19]). In that case it was necessary to determine in what sense the solution discontinuous at boundary points satisfies the boundary conditions. It turned out that such a solution can be obtained by an approximation procedure (the definition is given below).

In the general case, vector-valued elliptic systems are not self-adjoint even if the second-order operators involved are self-adjoint; therefore, the approach suggested in [16] for a scalar equation proves to be more natural for vector-valued problems.

Elliptic systems form a wide class of problems. Of these, we extract systems whose properties are similar to those of a single equation, namely, the so-called weakly-coupled and quasimonotone systems (these terms are explained below).

Let us outline the difficulties encountered if the boundary is not assumed to be regular. Just as in [16], we use the Krein–Rutman theorem. For using this theorem, it suffices to have a strictly positive compact resolving operator [3]. In the case of no boundary regularity, the proof of the compactness of the resolving operator becomes very cumbersome even for a single equation. However, quite a simple proof of this assertion was suggested in [20]. The method used there can be generalized to the case of systems, as shown in the

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present paper. For results on the regularity of the solution in domains with smooth boundaries and with finitely many corners, see [21, 22].

In the general case, the resolving operator of a system of equations is not strictly positive. Nevertheless, it follows from [23] that the assumptions of the Krein–Rutman theorem stipulating the strict positiveness and compactness of the resolving operator can be replaced by positiveness, irreducibility, and compactness. In what follows, we consider systems with resolving operators satisfying the latter conditions.

The paper [16] clarified the notion of validity of the boundary conditions for domains with nonsmooth boundaries. To this end, one must construct a special function u_0 by the following limit process involving a domain $\Omega \subset \mathbf{R}^N$ and an elliptic operator L.

Let $\{\Omega_j\}_{j=1}^{\infty}$ be a sequence of smooth domains approximating Ω from inside:

$$\Omega_j \subset \bar{\Omega}_j \subset \Omega_{j+1} \subset \cdots \subset \Omega \qquad \text{and} \qquad \bigcup_{j\in\mathbf{N}} \Omega_j = \Omega. \tag{2}$$

Next, let $\bar{c}$ be a number such that $L1+\bar{c} \geq 0$. Finally, let u_j stand for the solution of the problem $(L+\bar{c})\,u_j = 1$ in Ω_j, $u_j = 0$ on $\partial\Omega_j$. We define u_0 as $u_0(x) = \lim_{j\to\infty} u_j(x)$ for $x \in \Omega$; it follows from [16] that $u_j \rightharpoonup u_0$ in $W^{2,p}_{\mathrm{loc}}(\Omega)$ for any p and $u_j \to u_0$ in $C^1_{\mathrm{loc}}(\Omega)$.

Definition [16]. *Let u_0 be the above-defined function. A solution u of the elliptic equation $Lu = f$ (under appropriate assumptions on $L = -\sum a_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \sum b_i \frac{\partial}{\partial x_i} + c$ and f) satisfies the zero Dirichlet boundary conditions on $\partial\Omega$ in the BNV sense, namely, $u \overset{u_0}{=} 0$ on $\partial\Omega$, if $\lim_{j\to\infty} u(x_j) = 0$ for each sequence $x_j \to \partial\Omega$ such that $u_0(x_j) \to 0$.*

In the present paper we assume that the boundary condition occurring in (1) is satisfied in an appropriate BNV sense (see Definition 4 below).

It follows from the remark in [16, p. 73] that $u_0 \overset{v}{=} 0$ on $\partial\Omega$ for any $v \in W^{2,N}_{\mathrm{loc}}(\Omega)$ such that $v > 0$ and $Lv \geq 0$ in Ω. Therefore, the choice of u_0 is not a restrictive condition.

In Section 2 we introduce the relevant definitions and state the basic results. In Section 3 we give the assertions used in the proof of the maximum principle. In Section 4 we prove the main theorem for systems (1) with B being the unit matrix. Finally, using the results of the previous two sections, in Section 5 we justify the main theorem. In the last section, to make the exposition self-contained, we present a theorem of the Krein–Rutman–de Pagter type, which is needed in our investigation.

2. Definitions and the Main Result

The following assumptions are used throughout the investigation: the set Ω is a bounded open connected subset of $\mathbf{R}^N$; the operator $\mathscr{L}$ is a diagonal $k \times k$ matrix of elliptic operators L_μ $(1 \leq \mu \leq k)$:

$$L_\mu := -\sum_{i,j=1}^{N} a_{ij}^\mu(x)\frac{\partial^2}{\partial x_i\,\partial x_j} + \sum_{i=1}^{N} b_i^\mu(x)\frac{\partial}{\partial x_i} + c^\mu(x), \tag{3}$$

satisfying the conditions

$$\begin{gathered} c_0|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}^\mu(x)\xi_i\xi_j \leq C_0|\xi|^2, \qquad a_{ij}^\mu \in C(\Omega), \qquad b_i^\mu, c^\mu \in L^\infty, \\ \left(\sum_{i=1}^{k}\left(b_i^\mu(x)\right)^2\right)^{1/2} \leq b, \qquad |c^\mu(x)| \leq b, \end{gathered} \tag{4}$$

for some positive constants c_0, C_0, and b and for all $x \in \Omega$ and $\xi \in \mathbf{R}^k$; the entries of the $k \times k$ matrices H and B belong to the space $C\left(\bar{\Omega}\right)$. As was mentioned above, we do not assume any regularity of $\partial\Omega$.

Let $p \in (1,\infty)$, and let $L^p(\Omega)$ be the usual Lebesgue space. Note that $(L^p(\Omega))^k$ can be identified with $L^p(\omega)$, where

$$\omega = (\underbrace{\Omega, \Omega, \ldots, \Omega}_{k}). \tag{5}$$

Definition 1 (inequalities). *Let $D \subset \mathbf{R}^M$ be a bounded open set, and let $w \in L^p(D)$. Then*
1) $w > 0$ if $w \geq 0$ almost everywhere in D and w does not vanish identically;
2) $w \gg 0$ if $w|_{D^*} > 0$ on each open set $D^* \subset D$.

Definition 2 (matrices). *A $k \times k$ matrix A with entries $A_{ij} \in C\left(\bar{\Omega}\right)$ is said to be*
1) positive if $A_{ij}(x) \geq 0$ for all $i, j \in \{1, \dots, k\}$ and $x \in \bar{\Omega}$;
2) cooperative if $A_{ij}(x) \geq 0$ for all $i, j \in \{1, \dots, k\}$ with $i \neq j$ and for all $x \in \bar{\Omega}$.
A cooperative matrix A is said to be
3) completely mixed if $\tilde{A} + I$ is an irreducible matrix, where the entries of $\tilde{A}$ are given by the relation $\tilde{A}_{ij} = \|A_{ij}\|_\infty$.

In the literature, the cooperative property is known as essential positiveness or quasimonotonicity [11, 4].

Definition 3 (supersolutions). *A function $w \in \left(W^{2,p}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)\right)^k$ with $w > 0$ and $(\mathscr{L} - H)w \in \left(L^N(\Omega)\right)^k$ is referred to as*
1) a supersolution for the operator $\mathscr{L} - H$ if $(\mathscr{L} - H)w \geq 0$;
2) a strict supersolution for $\mathscr{L} - H$ if $(\mathscr{L} - H)w > 0$;
3) a strong supersolution for $\mathscr{L} - H$ if $(\mathscr{L} - H)w \gg 0$.

Next, let us define the validity of boundary conditions in the BNV sense. Let a sequence $\{\Omega_j\}$ consist of smooth domains approximating Ω from inside just as in (2), and let $M_\mu = L_\mu - c^\mu$. Let u_0^μ be the limit of functions u_j^μ that are solutions of the problem $M_\mu u^\mu = 1$ in Ω_j, $u^\mu = 0$ on $\partial\Omega_j$.

Definition 4 (Dirichlet boundary conditions). *Let u_0 be the above-constructed function. A function $u \in (C(\Omega))^k$ satisfies the Dirichlet boundary conditions in the BNV sense, i.e., $u \overset{u_0}{=} 0$, if for each $\mu \in \{1, 2, \dots, k\}$ and for each sequence $\{x^j\}_{j \in \mathbf{N}} \subset \Omega$, $x^j \to \partial\Omega$, the relation $\lim_{j\to\infty} u_0^\mu(x^j) = 0$ implies $\lim_{j\to\infty} u^\mu(x^j) = 0$.*

Let us state the eigenvalue problem to be investigated in the present paper. We say that $\Phi \in \left(W^{2,N}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)\right)^k$ is an eigenfunction of problem (1) corresponding to an eigenvalue λ if

$$(\mathscr{L} - H)\Phi = \lambda B\Phi \quad \text{in} \quad \Omega, \qquad \Phi \overset{u_0}{=} 0 \quad \text{on} \quad \partial\Omega. \tag{6}$$

Just as in [16], we set

$$\lambda_0 = \sup\left\{\lambda;\ \exists w \in \left(W^{2,N}_{\mathrm{loc}}(\Omega)\right)^k :\ (\mathscr{L} - H)w \geq \lambda B w \quad \text{and} \quad w \gg 0\right\}. \tag{7}$$

For smooth domains under appropriate conditions imposed on the operators in question and for $B = I$, it is known [12] that λ_0 is the first eigenvalue in the ordinary sense.

Next, we note that if B satisfies the condition $\sum_{j=1}^k B_{ij}(x) > 0$ for all $1 \leq i \leq k$, then definition (7) is closely related with Barta type inequalities [25]. More precisely, we have

$$\lambda_0 \geq \inf_{1 \leq i \leq k,\, x \in \Omega} ((\mathscr{L} - H)w)_i(x)/(Bw)_i(x) \tag{8}$$

for all $w \in (C^2(\Omega))^k$ with $w \gg 0$. The main results of the present paper are stated in the following two theorems.

Theorem 1. *Let Ω, $\mathscr{L}$, H, and B satisfy the above-mentioned assumptions, and let λ_0 be the number given by (7). Suppose that*
a) there exists a positive strong supersolution for $\mathscr{L} - H$;
b) H is a cooperative matrix;
c) H is a completely mixed matrix;
d) B is a cooperative matrix;
e) $B_{ii}(x) > 0$ for some $i \in \{1, \dots, k\}$ and $x \in \Omega$.

Then the following assertions are valid:

i) λ_0 is a positive eigenvalue and corresponds to a strongly positive eigenfunction;

ii) λ_0 is the unique eigenvalue corresponding to a positive eigenfunction, and its algebraic multiplicity is equal to 1;

iii) there is no eigenvalue on $[0, \lambda_0)$.

Next, we consider the boundary value problem

$$(\mathscr{L} - H)u = \lambda Bu + f \quad \text{in} \quad \Omega, \qquad u \overset{u_0}{=} 0 \quad \text{on} \quad \partial\Omega. \tag{9}$$

Theorem 2. *Let the assumptions of Theorem 1 be satisfied. Let $f \in (\mathscr{L}^p(\Omega))^k$ with $f > 0$, and let λ_0 be the number given by (7). Then the following assertions hold:*

i) if $0 \le \lambda < \lambda_0$, then there exists a solution $u \in \left(W^{2,N}_{\text{loc}}(\Omega) \cap L^\infty(\Omega)\right)^k$ of problem (9), and $u \gg 0$.

If B is positive, then

ii) problem (9) with $\lambda = \lambda_0$ has no solution for any $f > 0$;

iii) problem (9) with $\lambda > \lambda_0$ has no positive solution for any $f > 0$.

Remark 1. If, in addition to the assumptions of Theorem 1, we suppose that B is a positive diagonal matrix ($B_{ij} \equiv 0$ for $i \neq j$ and $B_{ii} > 0$), then there is no eigenvalue in $(-\infty, \lambda_0)$, and assertion i) of Theorem 2 is valid for all $\lambda < \lambda_0$.

Remark 2. Since we impose no assumptions on the sign of c^μ, it follows that we can replace c^μ by $c^\mu - \gamma$ and $H_{\mu\mu}$ by $H_{\mu\mu} + \gamma$. Consequently, without loss of generality, we can assume that $H_{\mu\mu} \ge 0$ (H is positive) or even $H_{\mu\mu} \gg 0$.

Corollary 1. *Let the assumptions of Theorem 1 be satisfied, and let $\sum_{j=1}^k B_{ij}(x) > 0$ for all $1 \le i \le k$. Then*

$$\lambda_0 = \sup_{w \in (C^2(\Omega))^k, w \gg 0} \; \inf_{1 \le i \le k, x \in \Omega} ((\mathscr{L} - H)w)_i(x)/(Bw)_i(x). \tag{10}$$

This result was obtained in [25] for the Laplace operator. For results concerning more general second-order elliptic equations, see [10, 26]. The case of systems was investigated in [12].

Proof. We denote the expression occurring on the right-hand side in (10) by λ_B^*. It follows from (8) that $\lambda_0 \ge \lambda_B^*$. Using the first eigenfunction, whose existence is provided by Theorem 1, we obtain $\lambda_0 \le \lambda_B^*$. The proof of the corollary is complete.

3. The Maximum Principle, Subdomains, and Nonzero Boundary Conditions

In the case of elliptic equations, it is well known that if the resolving operator of the Dirichlet problem is positive on Ω, then the same is true for the Dirichlet problem in any subdomain $\Omega_s \subseteq \Omega$. A similar result is valid for cooperative systems.

Proposition 1. *Let Ω, $\mathscr{L}$, and H satisfy the above-stipulated assumptions. Suppose that conditions b) and c) of Theorem 1 are satisfied. If there exists a $u_* \in \left(W^{2,N}_{\text{loc}}(\Omega) \cap L^\infty(\Omega)\right)^k$ such that*

$$(\mathscr{L} - H)u_* \ge 0 \quad in \quad \Omega, \qquad u_* \overset{u_0,\Omega}{=} 0 \quad on \quad \partial\Omega, \tag{11}$$

and $u_* \ge 0$ in Ω, then the relation $u \ge 0$ is valid in Ω_s for each open set $\Omega_s \subseteq \Omega$ with smooth boundary $\partial\Omega_s$ and for $u \in \left(W^{2,N}(\Omega_s)\right)^k$ such that

$$(\mathscr{L} - H)u \ge 0 \quad in \quad \Omega_s, \qquad u \ge 0 \quad on \quad \partial\Omega_s. \tag{12}$$

Remark 3. The assumption that H is a completely mixed matrix on Ω does not imply that $H|_{\Omega_s}$ is also completely mixed on Ω_s. This explains why we do not require the validity of the condition $u \ge 0$ in (12). However, if H is a completely mixed matrix on Ω_s, then we can strengthen the last assertion as follows: either $u \equiv 0$, or $u \gg 0$ in Ω_s.

Proof of Proposition 1. Note that the Kato inequality [27], which can be applied to $u_\mu \in W^{2,1}_{\mathrm{loc}}(\Omega)$, and the cooperative property of H imply that

$$\begin{aligned}((\mathscr{L}-H)\min(u,0))_\mu &= \mathscr{L}_\mu\left(\chi_{[u_\mu<0]}u_\mu\right) - \sum_j H_{\mu j}\left(\chi_{[u_j<0]}u_j\right) \geq \chi_{[u_\mu<0]}\mathscr{L}_\mu u_\mu - \sum_j H_{\mu j}\chi_{[u_j<0]}u_j \\ &= \chi_{[u_\mu<0]}((\mathscr{L}-H)u)_\mu + \sum_j H_{\mu j}\left(\chi_{[u_\mu<0]} - \chi_{[u_j<0]}\right)u_j \\ &\geq \sum_j H_{\mu j}\left(\chi_{[u_\mu<0]} - \chi_{[u_j<0]}\right)u_j \\ &= \sum_{j\neq\mu} H_{\mu j}\chi_{[u_\mu<0]}\chi_{[u_j\geq 0]}u_j - \sum_{j\neq\mu} H_{\mu j}\chi_{[u_\mu\geq 0]}\chi_{[u_j<0]}u_j \geq 0\end{aligned}$$

in the sense of distributions. Consequently, $(\mathscr{L}-H)\min(u,0) \geq 0$ in Ω_s and $\min(u,0) = 0$ on $\partial\Omega_s$. Since u_s is a strong positive supersolution for $\mathscr{L}-H$ on Ω_s, it follows from [12] that $\min(u,0) \geq 0$ in Ω_s for each completely mixed subset of $\{1,\dots,k\}$. Therefore, $u \geq 0$ in Ω_s, which completes the proof.

Remark 4. Proposition 1 will be used in the proof of Theorems 1 and 2. Had these theorems already been proved, we could use them and prove Proposition 1 for $u \in \left(W^{2,N}_{\mathrm{loc}}(\Omega_s) \cap L^\infty(\Omega_s)\right)^k$ without requiring the smoothness of $\partial\Omega_s$. To this end, one should replace the boundary condition $u \geq 0$ on $\partial\Omega_s$ by the condition $\min(u,0) \overset{u_{0,\Omega}}{=} 0$ on $\partial\Omega_s$.

4. The Case $B = I$

Let us consider the special case of problem (1) with $B = I$ (the unit matrix). We assume that conditions a), b), and c) of Theorem 1 are satisfied and the inequality

$$\kappa \geq \sup_{\mu,x}\left(\sum_{j=1}^k H_{\mu j}(x) - c^\mu(x)\right) \tag{13}$$

is valid. We introduce the operator $\mathscr{L} - H + \kappa I$.

Lemma 1. *Let* $\mathbf{e} = (1,\dots,1)^{\mathrm{T}} \in \mathbf{R}^k$, $k \geq 1$, *and let* $\kappa > 0$ *satisfy inequality (13). If* u_0 *is the function constructed in Definition 4, then there exists a* $u_e = \left(u_e^1,\dots,u_e^k\right) \in \left(W^{2,N}_{\mathrm{loc}}(\Omega)\right)^k$ *such that* $(\mathscr{L}-H+\kappa I)u_e = \mathbf{e}$ *in* Ω, $u_e \overset{u_0}{=} 0$ *on* $\partial\Omega$, *and* $u_e \gg 0$ *in* Ω. *Moreover,* $u_0 \overset{u_e}{=} 0$ *on* $\partial\Omega$.

Remark 5. As a corollary of Lemma 1, we can claim that the assertions $u \overset{u_e}{=} 0$ on $\partial\Omega$ and $u \overset{u_0}{=} 0$ on $\partial\Omega$ are equivalent.

Proof of Lemma 1. Let $\{\Omega_i\}_{i\in\mathbf{N}}$ be a sequence of smooth domains approximating Ω from inside [just as in (2)], and let $u_{e,i} = \left(u_{e,i}^1,\dots,u_{e,i}^k\right)$ be the solution of the problem

$$(\mathscr{L}-H+\kappa I)u_{e,i} = \mathbf{e} \quad \text{in} \quad \Omega_i, \qquad u_{e,i} = 0 \quad \text{on} \quad \partial\Omega_i. \tag{14}$$

Since H is a completely mixed matrix in Ω, it follows that so is H in Ω_i for all sufficiently large i. This means that we can use Theorem 1.1 from [12]. Hence $u_{e,i} \gg 0$ for all sufficiently large i. Since Ω is bounded, we can assume that Ω lies in the half-space $\left\{x \in \mathbf{R}^N;\ x_1 > 0\right\}$. We set $d_\mu = \sup_x\left(c^\mu - \sum_{i=1}^k H_{\mu i} + \kappa\right)$ and consider $\sigma \in R$ such that $\sigma > \sup_{\mu,x}(2a_{11}^\mu)^{-1} + \left((b_1^\mu)^2\left((b_1^\mu)^2 + 4a_{11}^\mu(1+d_\mu)\right)^{1/2}\right)$ and $v(x) := \left(e^{\sigma d} - e^{\sigma x_1}\right)\mathbf{e}$, where d is the diameter of Ω.

We have

$$\begin{aligned}((\mathscr{L}-H+\kappa I)v)_\mu &= \left(\sigma^2 a_{11}^\mu - b_1^\mu\sigma\right)e^{\sigma x_1} + \left(c^\mu - \sum_{j=1}^k H_{\mu j} + \kappa\right)\left(e^{\sigma d} - e^{\sigma x_1}\right) \\ &\geq \left(\sigma^2 a_{11}^\mu - b_1^\mu\sigma - d_\mu - 1\right)e^{\sigma x_1} + \left(c^\mu - \sum_{j=1}^k H_{\mu j} + \kappa\right)e^{\sigma d} + 1;\end{aligned}$$

consequently, $(\mathscr{L}-H+\kappa I)v \gg 1$ and $(\mathscr{L}-H+\kappa I)(u_{e,i} - v) \ll 0$. The maximum principle (see [12, Th. 1.1]) yields $0 < u_{e,i} < v$.

As a corollary, we can readily find that $u_{e,i} \to u_e$. Indeed, since $\{u_{e,i}(x)\}_{i\in\mathbf{N}}$, $x \in \Omega$, is a bounded increasing sequence, we find that it is convergent.

Choosing $c^* \in \mathbf{R}$ such that $M_\mu u_e^\mu = (-\kappa - c^\mu)\, u_e^\mu + \sum_{j=1}^k H_{\mu i} u_e^j + 1 \le c^* 1$, we obtain $0 \le u_e^\mu \le c^* u_0^\mu$ and $u_e \overset{u_0}{=} 0$ on $\partial\Omega$. By virtue of the results of [16, p. 73], this means that $u_0 \overset{u_e}{=} 0$ on $\partial\Omega$. The proof of the lemma is complete.

Lemma 2. *Let $f \in (L^\infty(\Omega))^k$. Then there exists a unique function $u \in \left(L^\infty(\Omega) \cap W^{2,N}_{\rm loc}(\Omega)\right)^k$ such that*

$$(\mathscr{L} - H + \kappa I)u = f \quad in \quad \Omega, \qquad u \overset{u_0}{=} 0 \quad on \quad \partial\Omega. \tag{15}$$

In addition, there exists a $C \in \mathbf{R}$ (independent of u and f) such that

$$\|u\|_{L_\infty} \le C\|f\|_{L_\infty}. \tag{16}$$

Proof. Let us consider the sequence of open domains Ω_i such that $\Omega_i \subset \bar{\Omega}_i \subset \Omega_{i+1}$ and $\Omega = \bigcup_{i=1}^\infty \Omega_i$. Let u_i, $i = 1, 2, \dots$, be a solution of the problem $(\mathscr{L} - H + \kappa I)u_i = f$ in Ω_i, $u_i = 0$ on $\partial\Omega_i$. Performing considerations with the help of the comparison theorems, we have

$$-z \le u_i \le z, \tag{17}$$

where $z := u_e\|f\|_{L^\infty}$. To complete the proof, we note that the boundary condition $u \overset{u_0}{=} 0$ on $\partial\Omega$ and relation (16) are satisfied by virtue of (17). The uniqueness follows from (16).

Proposition 2. *Let $f \in (L^n(\Omega))^k$. Then there exists a unique solution $u \in \left(W^{2,N}_{\rm loc}(\Omega) \cap L^\infty(\Omega)\right)^k$ of problem (15). Moreover, there exists a C (independent of u and f) such that*

$$\|u\|_{L^\infty} \le C\|f\|_{L_N}, \tag{18}$$

and the inequality $f > 0$ implies $u \gg 0$.

We use the ideas of the proof of Theorem 1.1 from [12].

Proof. Note that $(\mathscr{L} + \kappa I)$ is a diagonal matrix consisting of uniformly elliptic operators. By virtue of the choice of κ, the assumptions of Theorem 1.2 in [16] are satisfied. Consequently, under boundary conditions in the BNV sense (see Definition 4) the operator $(\mathscr{L} + \kappa I)^{-1}$ is defined in $L^N(\Omega)$.

Let $A := (\mathscr{L} + \kappa I)^{-1}H$ be the resolving operator of the problem $(\mathscr{L} + \kappa I)u = Hf$ in Ω, $u \overset{u_0}{=} 0$ on $\partial\Omega$, i.e., $A(f) = u$. Let us show that it is a compact irreducible operator in $\left(L^N(\Omega)\right)^k$.

Indeed, it follows from [20, Proposition 1.1] that $(\mathscr{L} + \kappa I)^{-1}$ is a compact operator. Therefore, since H is bounded, it follows that so is A.

Let us show that A is irreducible. Recall that A is said to be irreducible on $\left(L^N(\Omega)\right)^k$ if the set $\left\{f \in \left(L^N(\Omega)\right)^k;\ f_i(x) = 0 \text{ for all } x \in \mathrm{K}_i,\ 1 \le i \le k\right\}$ is not invariant under A for any measurable set $\mathrm{K} \subset \omega$ with $\mu(\mathrm{K}) > 0$ and $\mu(\omega \backslash \mathrm{K}) > 0$. Since the maximum principle is valid [16] in the case under consideration, it follows that each component of $(\mathscr{L} + \kappa I)^{-1}$ is irreducible. Using the fact that H is a completely mixed matrix (we assume that $H_{ii} \ge 0$), we find that A is irreducible and positive (see the proof of Lemma 1.3 in [12]).

Now, using the assertion of Theorem 5, we obtain $r(A) > 0$.

Let us now show that the operator $(I - A)^{-1}$ is well defined and satisfies $(I - A)^{-1} = \sum_{\nu=0}^\infty A^\nu$. Indeed, by Theorem 4, $r(A)$ (> 0) is an eigenvalue of A and of the adjoint operator A^*. By φ and ψ we denote the corresponding positive eigenfunctions.

Let us consider the function u_e defined in Lemma 1. Recall that $u_e \gg 0$. We have $(\mathscr{L} + \kappa I)u_e = Hu_e + \mathrm{e} \gg Hu_e$, whence $u_e \gg (\mathscr{L} + \kappa I)^{-1}Hu_e$. Consequently, $\langle\psi, u_e\rangle > \langle\psi, Au_e\rangle = \langle A^*\psi, u_e\rangle = r(A)\,\langle\psi, u_e\rangle$ and $\langle\psi, u_e\rangle > 0$. This means that $r(A) < 1$, whence we obtain the desired assertion.

To complete the proof of the proposition, we note that, by virtue of the above-stipulated requirements, for any $f \in \left(L^N(\Omega)\right)^k$ there exists a u such that $u = (I - A)^{-1}(\mathscr{L} + \kappa I)^{-1}f$. This is equivalent to the relation $u - (\mathscr{L} + \kappa I)^{-1}Hu = (\mathscr{L} + \kappa I)^{-1}f$, i.e., u is a solution of problem (15).

It remains to prove relation (18). It follows from Lemma 2 and the inequality $r(A) < 1$ that $\|u\|_\infty \le (M/(1-r(A)))\,\|(\mathscr{L}+\kappa I)^{-1}f\|_\infty$. On the other hand, the generalized modification of the Aleksandrov–Bakelman-Pucci theorem [15, Th. 9.1; 16] yields $\|(\mathscr{L}+\kappa I)^{-1}f\|_\infty \le C\|f\|_{L^N}$.

Finally, since H is a completely mixed matrix, we have $(I-A)^{-1}(\mathscr{L}+\kappa I)^{-1}f \ge A^k(\mathscr{L}+\kappa I)^{-1}f \gg \mathbf{0}$ for $f > 0$. This completes the proof.

Corollary 2. *The problem*

$$(\mathscr{L}-H)\varphi = \lambda_1\varphi \quad in \quad \Omega, \qquad \varphi \overset{\text{u}\Omega}{=} 0 \quad on \quad \partial\Omega,$$

has a positive eigenvalue λ_1 such that the corresponding eigenfunction satisfies $\varphi \in \left(W^{2,N}_{\text{loc}}(\Omega)\cap L^\infty(\Omega)\right)^k$ and $\varphi \gg 0$.

Proof. Let $S_\kappa := (\mathscr{L}-H+\kappa I)^{-1}$, defined in $\left(L^N(\Omega)\right)^k$, be the inverse operator corresponding to boundary conditions defined in the BNV sense (see Definition 4); it is positive and irreducible; by virtue of the Sobolev embedding theorems and inequalities (18), it is also compact. Theorem 3 (see below) implies the existence of the principal eigenvalue μ of the operator S_κ corresponding to an eigenfunction $\varphi > 0$.

Consequently, $\lambda_1 = 1/\mu - \kappa$ is the principal eigenvalue of the operator $(\mathscr{L}-H)$ corresponding to an eigenfunction φ such that $\varphi \gg 0$. Let us show that $\lambda_1 > 0$.

By assumption, there exists a strongly positive supersolution w of the problem $(\mathscr{L}-H)$, that is, $(\mathscr{L}-H+\kappa I)w > \kappa w > 0$. By Proposition 2, the function $\tilde{w} = S_\kappa(\mathscr{L}-H+\kappa I)w$ satisfies the conditions $\tilde{w} \in \left(W^{2,N}_{\text{loc}}(\Omega)\cap L^\infty(\Omega)\right)^k$ and $\tilde{w} \gg 0$. Let us consider the sequence $\{\Omega_i\}$ of smooth domains contained in Ω and satisfying condition (2). Let $S_{\kappa,i} := (\mathscr{L}-H+\kappa I)^{-1}$ be the resolvent of the operator of problem (15) considered in Ω_i. It follows from Proposition 1 that $w \ge S_{\kappa,i}(\mathscr{L}-H+\kappa I)w$ on Ω_i. Since $S_{\kappa,i}(\mathscr{L}-H+\kappa I)w = S_{\kappa,i}(\mathscr{L}-H+\kappa I)\tilde{w} \to \tilde{w}$ as $i \to \infty$, we find that $\tilde{w} \le w$ on Ω and $\tilde{w} = S_\kappa(\mathscr{L}-H+\kappa I)w > \kappa S_\kappa w \ge \kappa S_\kappa \tilde{w}$ on Ω. Let μ stand for the principal eigenvalue of S_κ. Since μ is also the principal eigenvalue of the adjoint operator S^*_κ corresponding to the eigenfunction ψ, we have $0 < \kappa\langle\psi, S_\kappa\tilde{w}\rangle < \langle\psi,\tilde{w}\rangle = (1/\mu)\langle S^*_\kappa\psi,\tilde{w}\rangle = (1/\mu)\langle\psi, S_\kappa\tilde{w}\rangle$, where $\langle\cdot,\cdot\rangle$ stands for the pairing between the space $\left(L^N(\Omega)\right)^k$ and the dual space. Consequently, $\lambda_1 = 1/\mu - \kappa > 0$. The proof of the corollary is complete.

Corollary 3. *Let $\lambda < \lambda_1$ and $f \in \left(L^N(\Omega)\right)^k$. Then there exists a unique function $u \in \left(W^{2,N}_{\text{loc}}(\Omega)\cap L^\infty(\Omega)\right)^k$ such that $(\mathscr{L}-H)u = \lambda u + f$ in Ω and $u \overset{\text{u}\Omega}{=} 0$ on $\partial\Omega$. In addition, if $f > 0$, then $u \gg 0$.*

Proof. If $\lambda = -\kappa$, then the assertion of the corollary follows from Proposition 2. Only inequality (13) restricts κ; therefore, the assertion is valid for all $\lambda \le \kappa$. If $\lambda \in (-\kappa, \lambda_1)$, then we can follow the lines of the proof of Proposition 2. Indeed, note that $\nu\left((\mathscr{L}-H+\kappa)^{-1}(\kappa+\lambda)\right) = (\lambda_1+\kappa)^{-1}(\kappa+\lambda) < 1$ and $(\mathscr{L}-H+\kappa)^{-1}$ is strictly positive; consequently, the function $u = \sum_{k=0}^{\infty}\left((\mathscr{L}-H+\kappa)^{-1}(\kappa+\lambda)\right)^k(\mathscr{L}-H+\kappa)^{-1}f$ is defined and provides the desired solution of our problem. The proof of the corollary is complete.

5. The Hess Lemma. The Cooperative Property of B

Let us investigate the existence of a positive eigenvalue λ_B with a positive eigenfunction Φ for the problem

$$(\mathscr{L}-H)\Phi = \lambda_B B\Phi \quad \text{in} \quad \Omega, \qquad \Phi \overset{\text{u}\Omega}{=} 0 \quad \text{on} \quad \partial\Omega. \tag{19}$$

Proposition 3. *Let assumptions a) - e) of Theorem 1 be valid. Then there exist $\lambda_B > 0$ and $\Phi \in \left(W^{2,N}_{\text{loc}}(\Omega)\cap L^\infty(\Omega)\right)^k$ with $\Phi \gg 0$ such that relation (19) holds.*

*Proof**. Without loss of generality, we can assume that $B_{ii} > -1$ for all $i \in \{1,\ldots,k\}$. Consider the operator $K_\alpha : \left(L^N(\Omega)\right)^k \to \left(L^N(\Omega)\right)^k$ defined by the formula $K_\alpha = (\mathscr{L}-H+\alpha I)^{-1}(B+I)$, where $\alpha \ge 0$ and $(\mathscr{L}-H+\alpha I)^{-1}$, and assume that the Dirichlet conditions are posed in the way described in Definition 4. It follows from Corollary 3 that K_α is a compact positive irreducible operator for any $\alpha \ge 0$.

To continue the proof, we need the following lemma, which states a useful property of K_α.

* In the proof we use the ideas of [5, 28].

Lemma 3. *There exist $\alpha > 0$ and $w \in \left(W^{2,N}_{\rm loc}(\Omega) \cap L^\infty(\Omega)\right)^k$ with $w \gg 0$ such that $\alpha K_\alpha w \geq w$.*

Proof. Let $i \in \{1, \dots, k\}$ and $\sigma > 0$, and let $\delta > 0$ and $x_0 \in \Omega$ be chosen so that $B_{\delta,x_0} \subset \Omega$ and $B_{ii}(x) \geq \sigma$ for $x \in B_{\delta,x_0}$. We set $B_{\delta,x_0} = \left\{x \in \mathbf{R}^N;\ |x - x_0| < \delta\right\}$.

Let us consider the eigenvalue problem

$$(L_i - H_{ii})\, v = \lambda v \quad \text{in} \quad B_{\delta,x_0}, \qquad v = 0 \quad \text{on} \quad \partial B_{\delta,x_0}. \tag{20}$$

Straightforward calculations show that $((\mathscr{L} - H)^{-1}\mathbf{e})_i$ is a positive strict supersolution of problem (20). Then under the Dirichlet boundary conditions on B_{δ,x_0}, the operator $T_i = \left((L_i - H_{ii})|_{B_{\delta,x_0}}\right)^{-1}$ is a positive compact irreducible operator (by virtue of the strong maximum principle). Consequently, the operator T_i has the first positive eigenvalue $\tilde\lambda$, which corresponds to some eigenfunction $\tilde\varphi$. We continue $\tilde\varphi$ by zero outside B_{δ,x_0}. Let us consider $\tilde\Phi = (0, \dots, \tilde\varphi, \dots, 0)^T$, where $\tilde\varphi$ stands in the ith position, and the set $w = \alpha K_\alpha \tilde\Phi$ with $\alpha = \tilde\lambda/\sigma$.

Let us show that $\tilde\Phi \leq \alpha K_\alpha \tilde\Phi$. Indeed, since $\tilde\Phi > 0$, we have $w \gg 0$. Consequently, $(B + I)\tilde\Phi > (\sigma + 1)\tilde\Phi$ and

$$(\mathscr{L} - H + \alpha I)\left(w - \tilde\Phi\right) = \alpha(B + I)\tilde\Phi - \left(\tilde\lambda + \alpha\right)\tilde\Phi > 0 \quad \text{on} \quad B_{\delta,x_0}. \tag{21}$$

Note that, although the strong maximum principle can fail for the system on B_{δ,x_0} (this system is not necessarily completely mixed on a subdomain of Ω), the componentwise strong maximum principle is still valid (see Proposition 1). Consequently, (21), together with the inequality $w - \tilde\Phi \gg 0$ on $\partial B_{\delta,x_0}$, implies $w - \tilde\Phi \gg 0$ on B_{δ,x_0}. In addition, from the relation $w \gg 0 = \tilde\Phi$ on $\Omega \backslash B_{\delta,x_0}$ we obtain $\alpha K_\alpha \tilde\Phi \gg \tilde\Phi$ on Ω. Since αK_α is a strongly positive operator, we find that $\alpha K_\alpha w = (\alpha K_\alpha)^2\,\tilde\Phi \gg \alpha K_\alpha \tilde\Phi = w \gg \tilde\Phi > 0$. This completes the proof of the lemma.

Let us continue the proof of Proposition 3. Since K_α is a compact irreducible operator, it follows from Theorem 3 that the operator K_α has the first eigenvalue $1/\alpha_1 > 0$, which corresponds to some eigenfunction Φ_1; thus $\Phi_1 = \alpha_1 K_\alpha \Phi_1$. By Lemma 3, there exists a $w > 0$ such that $\alpha K_\alpha w \geq w$. Hence $1/\alpha_1 = r(K_\alpha) \geq 1/\alpha$, where $r(K_\alpha)$ is the spectral radius of K_α.

By varying α, we construct a sequence $(\alpha_n, \Phi_n)_{n\geq 1}$ with $\alpha_0 = \alpha$ such that $0 < \alpha_n \leq \alpha_{n-1}$, $\Phi_n = \alpha_n K_{\alpha_{n-1}} \Phi_n > 0$, and $\|\Phi_n\| = 1$ for $n \geq 1$. From the sequence $(\alpha_n, \Phi_n)_{n\geq 1}$, we can extract a subsequence such that $\alpha_n \to \lambda > 0$ with $\Phi = \lambda K_\lambda \Phi$; we denote this subsequence by (α_n, Φ_n). Consequently, $\Phi \overset{\rm u0}{=} 0$ on $\partial\Omega$, and $(\mathscr{L} - H + \lambda I)\Phi = \lambda(I + B)\Phi \implies (\mathscr{L} - H)\Phi = \lambda B \Phi$, i.e., $\lambda = \lambda_B$, which completes the proof.

Remark 6. Using the existence of a positive supersolution at the first stage, we have found a value λ such that $\Phi = \lambda K_\alpha \Phi$ with $\lambda \leq \alpha$, where the equality does not necessarily take place.

Proof of Theorems 1 and 2. We readily obtain $\lambda_B = \lambda_0$. Using the function Φ defined in Proposition 3, we find that the assumptions of Theorem 1 with $(\mathscr{L} - H)$ replaced by $(\mathscr{L} - H - \lambda B)$ are valid for all $\lambda \in [0, \lambda_B)$. Using the results of Section 3 with $B = I$, we complete the proof.

6. Krein–Rutman–de Pagter Results

A real vector space with a partial order $(E, \geq)$ is called a *vector lattice* if for any $f, g \in E$ one has $f \vee f \in E$, where $f \vee g$ is the least upper bound of $\{f, g\}$. A lattice equipped with a norm $(E, \geq, \|\cdot\|)$ is called a *Banach lattice* if $(E, \|\cdot\|)$ and $(E, \geq)$ are a Banach lattice and a vector lattice, respectively, such that the inequality $|f| \leq |g|$ yields $\|f\| \leq \|g\|$. Here $|f| = f \vee (-f)$. A set $A \subseteq E$ is called an *ideal lattice* if the conditions $|f| \leq |g|$ and $g \in A$ imply that $f \in A$. A positive operator $S \in L(E)$ is said to be irreducible if $\{0\}$ and E are the only closed lattice ideals invariant under S.

Theorem 3. *Let E be a Banach lattice with $\dim(E) > 1$, and let $T \in L(E)$ be a positive irreducible compact operator. Then the spectral radius r of the operator T satisfies the condition $r > 0$, and there exists a φ, $0 < \varphi \in E$, such that $T\varphi = r\varphi$. Moreover, r is the unique eigenvalue corresponding to a positive eigenfunction, and this eigenvalue is algebraically simple.*

This theorem is a combination of the well-known Krein-Rutman theorem from [29] and the important de Pagter result [23] allowing one to replace the positiveness of the spectral radius of the operator T by the irreducibility. The latter condition is easier to verify. In $E = L^p(\omega)$ (with the Lebesgue measure and with $1 \le p < \infty$), where ω is an open set in $\mathbf{R}^n$, any closed ideal has the form $\{f \in L^p(\omega);\ f = 0$ almost everywhere in $K\}$ [30, p. 158]. This means that the irreducibility of a positive operator S on E is equivalent to the fact that $M_{\chi_{\omega\setminus K}} \circ S \circ M_{\chi_K} \neq 0$ for each measurable set $K \subset \omega$ with $\mu(K), \mu(\omega\setminus K) > 0$. Here the operator M_{χ_K} stands for the multiplication by the characteristic function χ_K of K. For (compact) nuclear operators Theorem 3 is known as the Entsch theorem. In the present paper, we consider the case in which $E = (L^p(\Omega))^k$ for $p \in (1, \infty)$. Note that E can be identified with $L^p(\omega)$, where ω is defined in (5).

Theorem 4 (the Krein–Rutman theorem). *Let $T \in L(E)$ be a compact positive operator with a strictly positive spectral radius r. Then there exists a $\varphi \in E$, $\varphi > 0$, such that $T\varphi = r\varphi$.*

Theorem 5 (the de Pagter theorem). *Let E be a Banach lattice with* $\dim(E) > 1$, *and let $T \in L(E)$ be a compact positive irreducible operator. Then this operator has a positive spectral radius r.*

It remains to prove the uniqueness in Theorem 3. Since T is a positive compact operator, it follows that its adjoint $T' \in L(E')$ is also positive and compact and has the same spectral radius $r > 0$. By the Krein–Rutman theorem, it has a positive eigenfunction $\varphi \in E'$ with $T'\varphi = r\varphi$. It follows from Theorem V.5.2 in [30] that φ is the unique eigenfunction of T; moreover, r is an algebraically simple eigenvalue of T.

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