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On the invertibility of mappings arising in 2D grid generation problems

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Dedicated to Jean Descloux on his 60th birthday

Summary. In adapting a grid for a Computational Fluid Dynamics problem one uses a mapping from the unit square onto itself that is the solution of an elliptic partial differential equation with rapidly varying coefficients. For a regular discretization this mapping has to be invertible. We will show that such result holds for general elliptic operators (in two dimensions). The Carleman-Hartman-Wintner Theorem will be fundamental in our proof. We will also explain why such a general result cannot be expected to hold for the (three-dimensional) cube.

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1. Introduction

The present paper deals with the invertibility of mappings that transform simply connected two-dimensional domains into the unit square. These mappings are used to generate so called structured grids in the physical domain to solve Computational Fluid Dynamics (CFD) problems. These grids are generated by mapping a uniform rectangular mesh from the unit square onto the physical domain. To enable a consistent discretization of the flow equations, it is necessary that the mesh in the physical domain is non-overlapping. Hence the mapping has to be invertible.

A typical example of 2D grid generation is illustrated by the diagram in Fig. 1. The boundary conforming mesh around a 2D airfoil (see Fig. 1.c) is obtained as the image of a uniform rectangular mesh in the unit square (Fig. 1.a) under a mapping T. The mapping T is constructed as a compound mapping $T = M \circ A$, where M provides a basic non-overlapping mesh in the physical domain, and

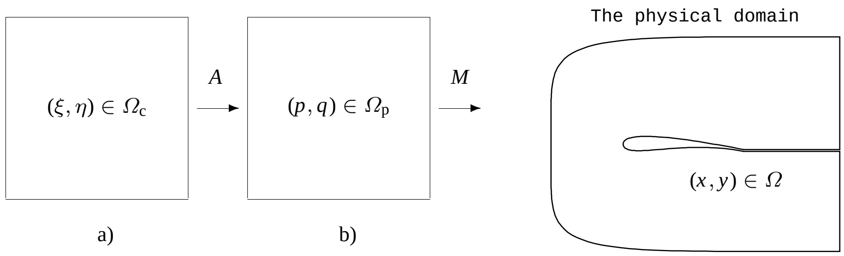

Fig. 1.

where A serves to adapt the mesh to improve the resolution of the geometry or the flow solution.

Since M provides a basic parameterization of the physical domain Ω, the unit square in Fig. 1.b is called the parametric domain (Ω_p). Similarly, since the compound mapping T provides the computational mesh in Ω on which the flow equations are solved, the unit square in Fig. 1.a is called the computational domain (Ω_c). The coordinates in Ω_c, Ω_p and Ω are denoted by $\boldsymbol{\xi} = (\xi, \eta)$, $\mathbf{p} = (p, q)$ and $\mathbf{x} = (x, y)$.

A way to construct the basic mapping M is to define the parametric coordinates p and q as solutions of the Laplace equation in Ω:

$$\Delta p = 0 \qquad \Delta q = 0, \tag{1.1}$$

with $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Mastin and Thompson [7] proved that if p and q are appropriately specified on the boundary $\partial\Omega$ of Ω, the resulting mapping M^{-1} from Ω to Ω_p has a non vanishing Jacobian $J_M = p_x q_y - p_y q_x$, which is a necessary condition for the mapping to be regular. The mesh spacing in Ω can be controlled to some extend by the specification of p and q on $\partial\Omega$. Winslow [13] replaced the Laplace equation (1.1) by isotropic diffusion equations

$$\nabla \cdot \frac{1}{w} \nabla p = 0 \qquad \nabla \cdot \frac{1}{w} \nabla q = 0, \tag{1.2}$$

with $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$. The weight function $w(x, y)$ enables more direct control over the mesh spacing.

An alternative way to enable mesh spacing control is to apply an additional mapping A, see Fig. 1. When the basic mapping M is defined by the Laplace system (1.1), Warsi [12] has shown that the compound mapping $T = M \circ A$ is given by

$$\begin{aligned} \Delta\xi(x, y) &= P\left(\xi_x, \xi_y, \eta_x, \eta_y; p_\xi, q_\xi, p_\eta, q_\eta, \xi_{pp}, \xi_{pq}, \xi_{qq}\right) \\ \Delta\eta(x, y) &= Q\left(\eta_x, \eta_y, \xi_x, \xi_y; p_\xi, q_\xi, p_\eta, q_\eta, \eta_{pp}, \eta_{pq}, \eta_{qq}\right), \end{aligned} \tag{1.3}$$

where the functions P and Q are nonlinear in $\xi_x, \xi_y, \eta_x, \eta_y$. In most applications however, the functions P and Q are specified directly rather than through specification of the adaptive mapping A [11].

Explicit use of an adaptive mapping A is incorporated in the algorithm of Hagmeijer [3], where it is assumed that a regular mapping M is given which provides sufficient resolution of the geometry in Ω. The additional mapping A is used to adapt the mesh in Ω with respect to a first approximation of the flow solution such that recalculation of the flow on the adapted mesh results in higher accuracy. The mapping A is defined by

$$\Lambda\nabla_{\mathbf{p}} \cdot W^{-1}\nabla_{\mathbf{p}}\, \xi = 0 \qquad \Lambda\nabla_{\mathbf{p}} \cdot W^{-1}\nabla_{\mathbf{p}}\, \eta = 0, \tag{1.4}$$

where Λ and W are diagonal matrices with strictly positive elements that are functions of p and q, and $\nabla_{\mathbf{p}} = \left(\frac{\partial}{\partial p}, \frac{\partial}{\partial q}\right)$. The boundary conditions for ξ, η on $\partial\Omega_{\mathrm{p}}$ are

$$\begin{array}{llll} \xi(0,q) = 0 & \xi(1,q) = 1 & \xi_q(p,0) = 0 & \xi_q(p,1) = 0 \\ \eta_{\mathrm{p}}(0,q) = 0 & \eta_{\mathrm{p}}(1,q) = 0 & \eta(p,0) = 0 & \eta(p,1) = 1. \end{array} \tag{1.5}$$

A variety of applications of the adaptive mapping defined by (1.4) and (1.5), see [3], [4] and [5], shows that, although heavily adapted meshes are produced, overlap never occurred. Hence it was suspected that the mapping defined by (1.4-1.5) is always invertible. This is the motivation for the present paper.

The paper is organized as follows. In Sect. 2 we state the main result, which is proven in Sect. 3. In Sect. 4 some remarks will be made for 3D problems.

2. Main result

Let us denote the open unit square $(0,1)\times(0,1)$ in $\mathbb{R}^2$ by S and the sides by Γ_1 to Γ_4 in the following way

$$\begin{cases} \Gamma_1 = \{0\} \times (0,1), \\ \Gamma_2 = (0,1) \times \{1\}, \\ \Gamma_3 = \{1\} \times (0,1), \\ \Gamma_4 = (0,1) \times \{0\}. \end{cases}$$

Consider the problem

$$\text{(a)} \left[\begin{array}{ll} Lu = 0 & \text{in } S, \\ u = 0 & \text{on } \Gamma_1, \\ u = 1 & \text{on } \Gamma_3, \\ \frac{\partial}{\partial n}u = 0 & \text{on } \Gamma_2 \cup \Gamma_4, \end{array}\right. \quad \text{and} \quad \text{(b)} \left[\begin{array}{ll} Lv = 0 & \text{in } S, \\ v = 1 & \text{on } \Gamma_2, \\ v = 0 & \text{on } \Gamma_4, \\ \frac{\partial}{\partial n}v = 0 & \text{on } \Gamma_1 \cup \Gamma_3, \end{array}\right. \tag{2.1}$$

where we are looking for a solution $(u,v) \in W^{2,p}(S) \times W^{2,p}(S)$ with $p \in (\underset{\sim}{2}, \infty)$. For a domain in $\mathbb{R}^2$ with a Lipschitz boundary one has $W^{2,p}(S) \subset C^1(\bar{S})$ for $p > 2$. (See Theorem 7.26 of [2].)

The operator L in (2.1) is given by

$$L = a_1(x)\left(\frac{\partial}{\partial x_1}\right)^2 + a_2(x)\left(\frac{\partial}{\partial x_2}\right)^2 + b_1(x)\frac{\partial}{\partial x_1} + b_2(x)\frac{\partial}{\partial x_2}, \tag{2.2}$$

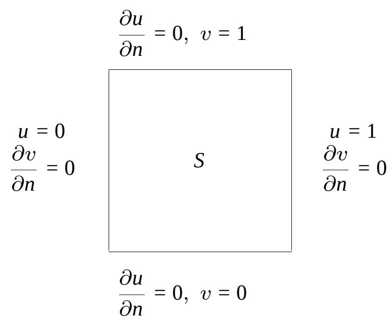

Fig. 2.

where the coefficients satisfy for some $c > 0$ and $\gamma \in (0, 1)$

$$a_i \in C^{0,1}(\overline{S}), \; a_i \geq c > 0 \text{ in } \overline{S}, \; i = 1, 2, \tag{2.3}$$

$$b_i \in C^{\gamma}(\overline{S}), \; i = 1, 2. \tag{2.4}$$

Remark 1. Observe that problem (1.4-1.5) is a special case of (2.1).

We have

Theorem 1. *Problem* (2.1) *possesses exactly one solution* $(u, v) \in C^2(\overline{S})$.

Moreover (u, v) *is a bijection from* $\overline{S}$ *(resp.* S*) into itself and*

$$\det \begin{pmatrix} u_{x_1} & u_{x_2} \\ v_{x_1} & v_{x_2} \end{pmatrix} > 0 \quad \text{on } \overline{S}. \tag{2.5}$$

Remark 2. The theorem implies that the mapping $A : \Omega_{\mathrm{c}} \to \Omega_{\mathrm{p}}$ (see Fig. 1) is regular.

3. Proof of the main result

We will start by studying the local behaviour of a solution to a two-dimensional elliptic problem near a stationary point. A powerful theorem of Carleman-Hartman-Wintner will yield the result that we need. We will use a generalized version of this theorem from Schulz ([10]). A tool in our proofs will be the Brouwer degree. For a mapping $\Phi \in C\left(\overline{\Omega}; \mathbb{R}^2\right)$, with $\Omega \subset \mathbb{R}^2$ open and bounded, the degree from Φ in Ω is well defined if $\Phi \neq 0$ on $\partial\Omega$. This degree is denoted by $\deg(\Phi, \Omega)$. For an introduction to the notion of degree one may see the first chapter of Deimling's book ([1]).

Let the operator $\tilde{L}$ on the domain Ω be as follows:

$$\tilde{L} = \tilde{a}_{11} \left(\frac{\partial}{\partial y_1} \right)^2 + \tilde{a}_{12} \frac{\partial}{\partial y_1} \frac{\partial}{\partial y_2} + \tilde{a}_{22} \left(\frac{\partial}{\partial y_2} \right)^2 + \tilde{b}_1 \frac{\partial}{\partial y_1} + \tilde{b}_2 \frac{\partial}{\partial y_2}, \tag{3.1}$$

with for some $c > 0$

$$\tilde{a}_{ij} \in C^{0,1}(\bar{\Omega}) \qquad \text{for } 1 \le i \le j \le 2, \tag{3.2}$$
$$\sum_{1\le i\le j\le 2} \tilde{a}_{ij}(x)\,\xi_i\xi_j \ge c\,|\xi|^2 \quad \text{for all } x \in \bar{\Omega}, \xi \in \mathbb{R}^2,$$

$$\tilde{b}_i \in L^\infty(\bar{\Omega}). \tag{3.3}$$

Proposition 2. *Let $\Omega \subset \mathbb{R}^2$ be open. Let $\tilde{L}$ be as in* (3.1) *with the coefficients satisfying* (3.2-3.3). *Suppose that $\phi \in W^{2,p}(\Omega)$, with $p > 2$, satisfies $\tilde{L}\phi = 0$ in Ω. Let $\hat{y} \in \Omega$ be such that $\nabla\phi(\hat{y}) = 0$. Then there exists $r > 0$ such that $\overline{B_r(\hat{y})} \in \Omega$ and either*

$$\nabla\phi \equiv 0 \qquad \text{on } \overline{B_r(\hat{y})}$$

or

$$\begin{cases} \nabla\phi \ne 0 & \text{for all } y \in \overline{B_r(\hat{y})} \setminus \{\hat{y}\}, \\ \deg\left(\nabla\phi, \overline{B_r(\hat{y})}\right) < 0. \end{cases}$$

Proof. From the uniform ellipticity of $\tilde{L}$ it follows that there exist $\lambda_1, \lambda_2 > 0$ and an orthogonal matrix Q, with $\det Q = 1$, such that

$$Q^{-1}\begin{pmatrix} \tilde{a}_{11}(\hat{y}) & \frac{1}{2}\tilde{a}_{12}(\hat{y}) \\ \frac{1}{2}\tilde{a}_{12}(\hat{y}) & \tilde{a}_{22}(\hat{y}) \end{pmatrix} Q = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

With the transformation $U : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$U(z_1, z_2) = \left(Q \begin{pmatrix} (\lambda_1)^{-\frac{1}{2}} & 0 \\ 0 & (\lambda_2)^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \end{pmatrix} \right)^{\mathrm{T}}$$

we find that $\varphi(z) := \phi(Uz) - \phi(\hat{y})$ satisfies a uniformly elliptic equation $\hat{L}\varphi = 0$ on $U^{\mathrm{inv}}\Omega$ where the operator $\hat{L}$ is as in (3.1) and satisfies $\hat{a}_{11}(0) = \hat{a}_{22}(0) = 1,\ \hat{a}_{12}(0) = 0$. Moreover

$$\begin{cases} \varphi(0) = 0, \\ \nabla\varphi(0) = 0. \end{cases}$$

Hence we are in a position to apply the version of the Carleman-Hartman-Wintner Theorem that is stated in Theorem 7.4.1 of [10]. We also use the result in Theorem 7.2.4 of [10]. Let $\Omega^* \subset \Omega$ denote the component of Ω that contains $\hat{y}$. Since $\varphi(z) = O(|z|)$ as $|z| \to 0$ it follows that either $\varphi(z) \equiv 0$ on $U^{\mathrm{inv}}\Omega^*$, or there exists $m \in \mathbb{N}^+$ with

$$\lim_{|z|\to 0} \frac{\varphi_{z_1} - \mathrm{i}\varphi_{z_2}}{(z_1 + \mathrm{i}z_2)^m} = \alpha \in \mathbb{C} \setminus \{0\}. \tag{3.4}$$

If $\varphi(z) \equiv 0$ on $U^{\mathrm{inv}}\Omega^*$ then $\phi(y) \equiv \phi(\hat{y})$ on Ω^*. Now suppose that $\varphi(z) \not\equiv 0$. Then there is $r^* > 0$ with $\overline{B_{r^*}(0)} \subset U^{\mathrm{inv}}\Omega$ and $\nabla\varphi(z) \ne 0$ for $z \in \overline{B_{r^*}(0)} \setminus \{0\}$, that is, 0 is an isolated zero for $\nabla\varphi$. Moreover, a homotopy argument shows that

$$\deg(\nabla\varphi, B_{r^*}(0)) = \deg\left(\left(\mathrm{Re}\left(\alpha(z_1 + \mathrm{i}z_2)^m\right), -\mathrm{Im}\left(\alpha(z_1 + \mathrm{i}z_2)^m\right)\right), B_{r^*}(0)\right).$$

Hence $\deg(\nabla\varphi, B_{r^*}(0)) = -m < 0$. Now take a ball $B_r(\hat{y})$, with $r > 0$, such that $B_r(\hat{y}) \subset UB_{r^*}(0)$. Since $\nabla\varphi \neq 0$ on $\overline{B_{r^*}(0)} \backslash \left(U^{\text{inv}} B_r(\hat{y})\right)$ we have

$$\deg(\nabla\varphi, B_{r^*}(0)) = \deg\left(\nabla\varphi, U^{\text{inv}}(B_r(\hat{y}))\right).$$

Since

$$\det\left(U'\right) = \det\left(Q\begin{pmatrix}(\lambda_1)^{-\frac{1}{2}} & 0\\ 0 & (\lambda_2)^{-\frac{1}{2}}\end{pmatrix}\right) > 0 \tag{3.5}$$

the matrix U' is nonsingular and orientation preserving. For a nonsingular linear mapping A the product formula for the degree, see Theorem 5.1 of [1], shows that

$$\deg(Ag(\cdot), D) = \deg(g(\cdot), D)\ \deg(A\ \cdot, K)\ ,$$

where K is the component of $g(D)$ containing 0, and

$$\deg(g(A\ \cdot), D) = \sum_{K_i} \deg(A\cdot\ -k_i, D)\ \deg(g(\cdot), K_i)$$

where K_i are the components of AD and $k_i \in K_i$. Theorem 1.1 of [1] shows that if $0 \in D$ and $\det A \neq 0$, then $\deg(A\ \cdot, D) = \operatorname{sgn}(\det A)$. From (3.5) it follows that we have

$$\deg\left(\nabla\varphi, U^{\text{inv}}(B_r(\hat{y}))\right) = \deg\left(\nabla\left(\phi(U\cdot) - \phi(\hat{y})\right), U^{\text{inv}}(B_r(\hat{y}))\right)$$

$$= \deg\left((\nabla\phi)(U\cdot)\ U', U^{\text{inv}}(B_r(\hat{y}))\right)$$

$$= \deg\left((\nabla\phi)(U\cdot), U^{\text{inv}}(B_r(\hat{y}))\right) = \deg(\nabla\phi(\cdot), B_r(\hat{y})).$$

Hence $\deg(\nabla\phi, B_r(\hat{y})) = -m < 0$. $\square$

Next we will establish some results for problem (2.1.a):

$$\begin{cases} Lu = 0 & \text{in } S, \\ u = 0 & \text{on } \Gamma_1, \\ u = 1 & \text{on } \Gamma_3, \\ \frac{\partial}{\partial n}u = 0 & \text{on } \Gamma_2 \cup \Gamma_4, \end{cases} \tag{3.6}$$

where L in (2.2) satisfies (2.3) and

$$b_i \in L^\infty(\overline{S}). \tag{3.7}$$

Since problem (2.1.b) can be treated as (2.1.a) by exchanging the roles of x_1 and x_2, similar results hold for (2.1.b).

Theorem 3. *Assume that L satisfies* (2.3) *and* (3.7). *Then problem* (3.6) *possesses exactly one solution $u \in W^{2,p}(S)$, for all $p \in (2,\infty)$. Moreover, the following holds:*

$$0 < u(x) < 1 \qquad \textit{for } x \in S, \tag{3.8}$$

$$\frac{\partial}{\partial x_1} u(x) > 0 \qquad \textit{for } x \in \partial S, \tag{3.9}$$

and

$$\nabla u(x) \neq (0,0) \qquad \textit{for } x \in S. \tag{3.10}$$

Finally, if (2.4) *holds, then $u \in C^{2,\gamma}(S)$.*

Since we have mixed boundary conditions and a non smooth boundary, standard existence and regularity theory does not apply in a straightforward fashion. However, this difficulty can be removed by transforming (3.6) into a Dirichlet problem on an annulus. We start with this transformation.

Consider the mapping $T : \overline{S} \to \overline{A^+}$, where

$$A^{\pm} = \left\{ y \in \mathbb{R}^2; 1 < |y| < 2, \pm y_2 > 0 \right\},$$

that is defined by $y_1 = r\cos\varphi$, $y_2 = r\sin\varphi$ with $r = x_1 + 1$ and $\varphi = \pi x_2$:

$$T(x_1, x_2) = ((x_1+1)\cos(\pi x_2), (x_1+1)\cos(\pi x_2)). \tag{3.11}$$

One verifies that

$$\begin{cases} T \in C^{\infty}(\overline{S}), \\ T \text{ is a bijection from } \overline{S} \text{ onto } \overline{A^+}, \\ \det\left(T'(x)\right) \in [\pi, 2\pi] \quad \text{for } x \in \overline{S}. \end{cases} \tag{3.12}$$

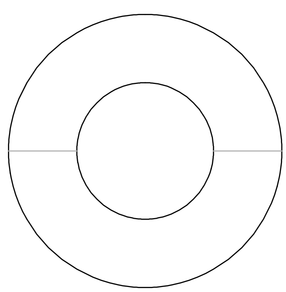

Fig. 3. A^+ and A^-

Problem (3.6) becomes:

$$\left[\begin{array}{ll} \tilde{L}w(y) = 0 & \text{for } y \in A^+, \\ w(y) = 0 & \text{for } y \in \partial A^+ \text{ with } |y| = 1, \\ w(y) = 1 & \text{for } y \in \partial A^+ \text{ with } |y| = 2, \\ \frac{\partial}{\partial n} w(y) = 0 & \text{for } y \in \partial A^+ \text{ with } y_2 = 0. \end{array}\right. \tag{3.13}$$

The operator $\tilde{L}$ is as in (3.1) where

$$\text{(3.14)}\quad \begin{cases} \tilde{a}_{11}(y_1,y_2) &= \left(\frac{y_1}{|y|}\right)^2 a_1 + y_2^2\ a_2\ , \\ \tilde{a}_{12}(y_1,y_2) &= 2y_1y_2\left(\frac{1}{|y|^2}\ a_1 - a_2\right), \\ \tilde{a}_{22}(y_1,y_2) &= \left(\frac{y_2}{|y|}\right)^2 a_1 + y_1^2\ a_2\ , \\ \tilde{b}_1(y_1,y_2) &= -y_1\ a_2 + \frac{y_1}{|y|}\ b_1 - y_2\ b_2\ , \\ \tilde{b}_2(y_1,y_2) &= -y_2\ a_2 + \frac{y_2}{|y|}\ b_1 - y_1\ b_2, \qquad \text{for } y \in \overline{A^+}, \end{cases}$$

with $a_i = a_i\left(T^{\text{inv}}(y)\right), b_i = b_i\left(T^{\text{inv}}(y)\right)$ for $i = 1,2$.

Next we extend the coefficients $\tilde{a}_{ij}$ and $\tilde{b}_i$ to the lower half of the annulus,

$$\text{(3.15)}\qquad A = \left\{y \in \mathbb{R}^2; 1 < |y| < 2\right\},$$

in the following way. For $y \in \overline{A}$ with $y_2 < 0$ we set

$$\text{(3.16)}\quad \begin{cases} \tilde{a}_{ii}(y_1,y_2) = \tilde{a}_{ii}(y_1,-y_2) \quad i = 1,2, \\ \tilde{a}_{12}(y_1,y_2) = -\tilde{a}_{12}(y_1,-y_2) \end{cases} \text{and} \begin{cases} \tilde{b}_1(y_1,y_2) = \tilde{b}_1(y_1,-y_2), \\ \tilde{b}_2(y_1,y_2) = -\tilde{b}_2(y_1,-y_2). \end{cases}$$

By using (2.3), (3.7) and

$$\text{(3.17)}\qquad \tilde{a}_{12} = 0 \qquad \text{for } y \in \partial A^+ \text{ with } y_2 = 0$$

we find that $\tilde{L}$ satisfies (3.2-3.3) for $\Omega = A$.

Note that (3.16) gives the restrictions on the regularity of the coefficients $\tilde{a}_{ij}$ and $\tilde{b}_i$. Indeed $\tilde{b}_2 \in C^\gamma(\overline{A^+})$ does not imply $\tilde{b}_2 \in C^\gamma(\overline{A})$ and $\tilde{a}_{ij} \in C^1(\overline{A^+})$ does not imply $\tilde{a}_{ij} \in C^1(\overline{A})$.

The problem on the annulus becomes

$$\text{(3.18)}\qquad \left[\begin{array}{ll} \tilde{L}w(y) = 0 & \text{for } y \in A, \\ w(y) = 0 & \text{for } |y| = 1, \\ w(y) = 1 & \text{for } |y| = 2. \end{array}\right.$$

Lemma 4. *Let $\tilde{L}$ as in* (3.1) *satisfy* (3.2-3.3) *for $\Omega = A$. Then the following holds.*

1. *There exists a unique solution $w \in W^{2,p}(A) \cap C(\overline{A})$ for all $p > 1$.*
2. *The solution w satisfies*

$$\text{(3.19)}\qquad \begin{cases} 0 < w(y) < 1 & \textit{for } y \in A, \\ \frac{\partial}{\partial n} w(y) > 0 & \textit{for } |y| = 2, \\ \frac{\partial}{\partial n} w(y) < 0 & \textit{for } |y| = 1, \end{cases}$$

where n denotes the outward normal.

Proof. By Theorem 9.15 and Corollary 9.18 of [2] one finds that (3.18) has a unique solution $w \in W^{2,p}(A) \cap C(\overline{A})$ for all $p > 1$. Using the strong maximum principle for the solution w on the annulus, we find (see Lemma 3.4 of [2]) the estimates in (3.19). $\square$

In the next lemma we show the relation between problems (3.6) and (3.18).

Lemma 5. *Suppose that the coefficients of L satisfy* (2.3) *and* (3.7). *Let $\tilde{L}$ be as above.*

1. *If $u \in W^{2,p}(S)$, for $p > 2$, satisfies* (3.6), *then w, defined by*

$$(3.20)\quad w(r\cos\varphi, r\sin\varphi) = u\left(r-1, \frac{1}{\pi}|\varphi|\right) \quad 1 \le r \le 2,\ -\pi < \varphi \le \pi,$$

is a $W^{2,p}(A)$-solution of (3.18).

2. *If $w \in W^{2,p}(A)$, for $p > 2$, satisfies* (3.18), *then u, defined by*

$$(3.21)\quad u(x_1,x_2) = w((x_1+1)\cos(\pi x_2), (x_1+1)\sin(\pi x_2)) \quad x \in \overline{S},$$

is a $W^{2,p}(S)$-solution of (3.6).

Proof. 1). The only difficulty appears where $y_2 = 0$. Since $u \in W^{2,p}(S)$ it follows that $w_{|_{A^+}} \in W^{2,p}(A^+)$ and $w_{|_{A-}} \in W^{2,p}(A^-)$. Since $w_{|_{A^+}} \in C^1(\overline{A^+})$, $w_{|_{A-}} \in C^1(\overline{A^-})$, and by symmetry $\frac{\partial}{\partial y_1} w(y_1,y_2) = \frac{\partial}{\partial y_1} w(y_1,-y_2)$ and $\frac{\partial}{\partial y_2} w(y_1,+0) = 0 = \frac{\partial}{\partial y_2} w(y_1,-0)$ we find $w \in C^1(\overline{A})$. Finally, since $\tilde{a}_{12}(y) \to 0$ for $y_2 \to 0$ and since $\left(\frac{\partial}{\partial y_i}\right)^2 w(y_1,y_2) = \left(\frac{\partial}{\partial y_i}\right)^2 w(y_1,-y_2)$ for $i = 1,2$, we find that w satisfies (3.18) in L^p-sense.

2.) From $w \in W^{2,p}(A)$ it follows that $w \in C^1(\overline{A})$. Since (3.18) has a unique solution in $W^{2,p}(A)$ and $\hat{w}$, defined by $\hat{w}(y_1,y_2) = w(y_1,-y_2)$ is also a solution, we find $w = \hat{w}$ and $\frac{\partial}{\partial y_2} w(y_1,0) = 0$ for $1 < |y_1| < 2$. Hence one finds that $\frac{\partial}{\partial x_2} u(x_1,x_2) = 0$ for $x_2 \in \{0,1\}$. $\square$

We will show some results for the map $\nabla w : \overline{A} \to \mathbb{R}^2$ by using a degree argument. Since $w \in C^1(\overline{A})$ we have $\nabla w \in C(\overline{A}; \mathbb{R}^2)$. From (3.19) it follows that $\nabla w \ne 0$ on ∂A. Therefore the (Brouwer) degree from ∇w in A is well defined.

Lemma 6. *Let $\tilde{L}$ as in* (3.1) *satisfy* (3.2-3.3) *for $\Omega = A$. Then the function $w \in W^{2,p}(A)$, with $p > 2$, that solves* (3.18) *satisfies* $\deg(\nabla w, A) = 0$.

Proof. By using Tietze's Theorem there exists an extension of ∇w, denoted by F, satisfying $F \in C(\overline{D}_2; \mathbb{R}^2)$, where

$$D_r = \left\{y \in \mathbb{R}^2; |y| < r\right\}.$$

Since A and D_1 are disjoint open sets of D_2 such that $0 \notin F\left(\overline{D}_2 \setminus (D_1 \cup A)\right)$ (notice that $\overline{D}_2 \setminus (D_1 \cup A) = \partial A$), we have, by the additivity of the degree (see property d2, p. 17 of [1]), that

$$\deg(F,A) = \deg(F,D_2) - \deg(F,D_1)\,.$$

It follows from (3.19) that for $t \in [0,1]$ we have

$$0 \notin ((1-t)F + tI)\left(\bar{D}_2 \setminus (D_1 \cup A)\right).$$

By the homotopy invariance of the degree (see property d3, p. 17 of [1]) we obtain

$$\deg(F,D_2) = \deg(F,D_1) = 1$$

and hence $\deg(\nabla w, A) = \deg(F,A) = 0$. □

Lemma 7. *Let* $\tilde{L}$ *as in* (3.1) *satisfy* (3.2-3.3) *for* $\Omega = A$. *Then the function* $w \in W^{2,p}(A)$, *with* $p > 2$, *that solves* (3.18) *satisfies* $\nabla w \neq 0$ *in* A.

Proof. Suppose that $\nabla w(\hat{y}) = 0$ for some $\hat{y} \in A$. By Proposition 2 there exists $B_r(\hat{y}) \subset A$ such that $w = w(\hat{y})$ on $B_r(\hat{y})$ or $\hat{y}$ is the only zero of ∇w in $B_r(\hat{y})$. Since A is connected the first possibility implies that $w \equiv w(\hat{y})$, which contradicts the boundary conditions for w. Therefore $\hat{y}$ is an isolated zero of ∇w and its local degree (index) is negative. It follows that there are at most finitely many zeros of ∇w and the total degree of ∇w on A is negative by the additivity property, contradicting Lemma 6. □

Proof of Theorem 3. Existence and uniqueness. Lemma 4 and Lemma 5 imply that there exists exactly one solution $u \in W^{2,p}(S)$ for $p > 2$ of (3.6).

The inequalities. With (3.12) the estimates in (3.19) take care of (3.8) and

$$\frac{\partial}{\partial x_1} u(x) > 0 \qquad \text{for } x \in \{0,1\} \times [0,1]\,. \tag{3.22}$$

By Lemma 7 we find that $\nabla w \neq 0$ in A. Together with the continuity of ∇w and (3.12) it implies that

$$\begin{cases} \frac{\partial}{\partial x_1} u(x) > 0 & \text{for } x \in (0,1) \times \{0,1\} \\ \nabla u \neq 0 & \text{in } S\,. \end{cases}$$

Hölder type regularity. If we assume that L satisfies (2.4) instead of (3.7) the solution satisfies $u \in C^{2,\gamma}(\bar{S})$. This is shown as follows. Indeed, since the solution w of (3.18) is in $W^{2,p}(A)$, for all $p \in (1,\infty)$, we find by Theorem 7.26 of [2] that $w \in C^{1,\gamma}(\bar{A})$. The function w satisfies

$$\left(\tilde{a}_{11}\left(\frac{\partial}{\partial y_1}\right)^2 + \tilde{a}_{12}\frac{\partial}{\partial y_1}\frac{\partial}{\partial y_2} + \tilde{a}_{22}\left(\frac{\partial}{\partial y_2}\right)^2 + \tilde{b}_1\frac{\partial}{\partial y_1}\right) w = -\tilde{b}_2 \frac{\partial}{\partial y_2} w, \tag{3.23}$$

where the right hand side is in $C^{\gamma}(\bar{A})$. Note that $\tilde{b}_2 \frac{\partial}{\partial y_2} w \in C^{\gamma}(\bar{A})$ holds since $\frac{\partial}{\partial y_2} w = 0$ for $y_2 = 0$. Since the boundary ∂A is smooth, w is constant on ∂A, $\tilde{a}_{11}, \tilde{a}_{12}, \tilde{a}_{22}, \tilde{b}_1 \in C^{\gamma}(\bar{A})$ and the right hand side of (3.23) is in $C^{\gamma}(\bar{A})$ it follows from Schauder type estimates (Theorem 9.19 of [2]) that $w \in C^{2,\gamma}(\overline{A})$. The properties of the transformation in (3.11) imply that $u \in C^{2,\gamma}(\overline{S})$.

This finishes the proof of Theorem 3. $\square$

Proof of Theorem 1. Theorem 3 shows that there exists a unique solution u, v in $W^{2,p}(\overline{S})$ and even that $u, v \in C^2(\overline{S})$.

We start by showing that (2.5) holds. Let us denote

$$D(x) = \det \begin{pmatrix} u_{x_1}(x) & u_{x_2}(x) \\ v_{x_1}(x) & v_{x_2}(x) \end{pmatrix}.$$

From $u, v \in C^1(\overline{S})$ it follows that $D \in C(\overline{S})$. Since $D(0) = u_{x_1}(0)\, v_{x_2}(0) > 0$ it will be sufficient to show that $D \neq 0$. By the estimate for u in (3.9) and a similar one for v we have $D > 0$ on ∂S. We will argue by contradiction to show that $D > 0$ in S. Suppose that $D(\hat{x}) = 0$ for some $\hat{x} \in S$. Then there is $(\alpha, \beta) \neq 0$ with $\alpha \nabla u(\hat{x}) + \beta \nabla v(\hat{x}) = 0$. We obtain from the boundary conditions of u and v that

$$\alpha \nabla u(x) + \beta \nabla v(x) = \left(\alpha u_{x_1}(x), \beta v_{x_2}(x)\right) \quad \text{for } x \in \partial S.$$

From (3.9) it follows that $u_{x_1}(x) > 0$ for $x \in \partial S$ and similarly $v_{x_2}(x) > 0$ for $x \in \partial S$. It shows that

$$(1-t)\left(\alpha u_{x_1}(x), \beta v_{x_2}(x)\right) + t(\alpha, \beta) \neq 0 \quad \text{for } x \in \partial S.$$

Hence by homotopy invariance we find that

$$\deg(\nabla(\alpha u + \beta v), S) = \deg((\alpha, \beta), S) = 0.$$

We also have that $L(\alpha u + \beta v) = 0$. Then Proposition 2 implies that the zeros of $\nabla(\alpha u + \beta v)$ are isolated and that the local degree at such a zero is negative. Additivity of the degree shows that $\nabla(\alpha u + \beta v) \neq 0$ on $\overline{S}$, a contradiction. This completes the proof of (2.5).

We will again use a degree argument to show that $(u, v) : \overline{S} \to \overline{S}$ (resp. $S \to S$) is a bijection. Here we will use the function $F : \overline{S} \to \mathbb{R}^2$, defined by $F(x) = (u(x), v(x))$. By the estimates in Theorem 3 we have that $F \in C^1(\overline{S}; \overline{S})$. In fact the boundary conditions and the inequality in (3.9) show that $F_{|\partial S} : \partial S \to \partial S$ is a bijection. It also shows that $F(S) \subset S$. Now we fix $(\alpha, \beta) \in S$ and consider $\deg(F - (\alpha, \beta), S)$.

The properties of u, v show that $F(x) - (\alpha, \beta) = (u(x) - \alpha, v(x) - \beta)$ is always directed outward of S at $x \in \partial S$. By a homotopy argument we have

$$\deg(F(\cdot) - (\alpha, \beta), S) = \deg(I \;\cdot - (\alpha, \beta), S) = 1.$$

Hence there exists $\hat{x} \in S$ with $F(\hat{x}) = (\alpha, \beta)$, that is, F is onto. We finish by showing that F is one to one. Since F is in $C^1(\overline{S})$ it follows that

$$F(x) = F(\hat{x}) + (x - \hat{x}) \begin{pmatrix} u_{x_1}(\hat{x}) & v_{x_1}(\hat{x}) \\ u_{x_2}(\hat{x}) & v_{x_2}(\hat{x}) \end{pmatrix} + o\left(|x - \hat{x}|\right).$$

Then there is a ball $B_r(\hat{x})$ such that $F(x) \neq F(\hat{x})$ for $x \in B_r(\hat{x}) \setminus \hat{x}$, and the local degree is well defined. We have

$$\begin{aligned}
\deg(F(\cdot) - (\alpha,\beta), B_r(\hat{x})) &= \deg(F(\cdot) - F(\hat{x}), B_r(\hat{x})) \\
&= \deg\left((\cdot - \hat{x})\begin{pmatrix} u_{x_1}(\hat{x}) & v_{x_1}(\hat{x}) \\ u_{x_2}(\hat{x}) & v_{x_2}(\hat{x}) \end{pmatrix}, B_r(\hat{x})\right) \\
&= \deg\left((\cdot)\begin{pmatrix} u_{x_1}(\hat{x}) & v_{x_1}(\hat{x}) \\ u_{x_2}(\hat{x}) & v_{x_2}(\hat{x}) \end{pmatrix}, B_r(0)\right) \\
&= \operatorname{sgn}(D(\hat{x})) = 1.
\end{aligned}$$

In the last equality we used Theorem 1.1 of [1], which shows $\deg(Q, \Omega) = \operatorname{sgn}(\det Q)$ for linear maps Q with $\det Q \neq 0$ and $\Omega \ni 0$. By the additivity property of the degree there exists exactly one $\hat{x} \in S$ with $F(\hat{x}) = (\alpha, \beta)$. □

Remark. The basic theorem that is used in the proofs above is the result of Carleman-Hartman-Wintner. One may give a somewhat different proof of (2.5) that does not use a degree argument. We still need the C.-H.-W. result. The alternative proof uses that C.-H.-W. implies that a stationary point of a non trivial C^1 solution w of $\tilde{L}w = 0$ is a saddle point. That means, if w has a stationary point at $\hat{y} \in A$, then $A^+_{w(\hat{y})}$ and $A^-_{w(\hat{y})}$, defined by

$$A^{\pm}_{w(\hat{y})} = \left\{ y \in \bar{A}; \pm(w(y) - w(\hat{y})) > 0 \right\} \tag{3.24}$$

consists locally of at least two components (for all small r the sets $A^+_{w(\hat{y})} \cap B_r(\hat{y})$ and $A^-_{w(\hat{y})} \cap B_r(\hat{y})$ have both at least two components). The Jordan curve Theorem implies that either $A^+_{w(\hat{y})}$ or $A^-_{w(\hat{y})}$ has at least two components. Let us say $A^+_{w(\hat{y})}$ has two components. Since $\{|y| = 2\}$ lies in one component of $A^+_{w(\hat{y})}$ the other component $\mathscr{C}$ of $A^+_{w(\hat{y})}$ has empty intersection with $\partial\Omega$. Hence $w = w(\hat{y})$ on $\partial\mathscr{C}$. The maximum principle [9] implies that $w \equiv w(\hat{y})$ in $\mathscr{C}$, a contradiction. Together with the strong maximum principle [9] it shows that $\nabla u \neq 0$ (and similarly $\nabla v \neq 0$). In a similar fashion $\alpha\nabla u + \beta\nabla v \neq 0$ on $\bar{S}$ for $(\alpha, \beta) \neq 0$. One concludes by showing that $(u(x_a), v(x_a)) = (u(x_b), v(x_b))$ for some $x_a \neq x_b$ implies $\alpha\nabla u + \beta\nabla v = 0$ somewhere in S.

4. In three dimensions

A similar way of adapting the grid in three dimensions leads to a problem on a cube. Let this cube be denoted by: $K = \{(x_1, x_2, x_3); 0 < x_i < 1\}$. The elliptic problem will be the following. Find $u = (u_1, u_2, u_3) \in C^2\left(\bar{K}; \bar{K}\right)$ such that

$$\left[\begin{array}{lll}
Lu_i = 0 & \text{in } K, & \\
u_i = 0 & \text{on } \partial K \cap \{x_i = 0\}, & \\
u_i = 1 & \text{on } \partial K \cap \{x_i = 1\}, & \\
\frac{\partial}{\partial n} u_i = 0 & \text{on } \partial K \cap \{0 < x_i < 1\}, & \\
 & & \text{with } i \in \{1, 2, 3\}.
\end{array}\right. \tag{4.1}$$

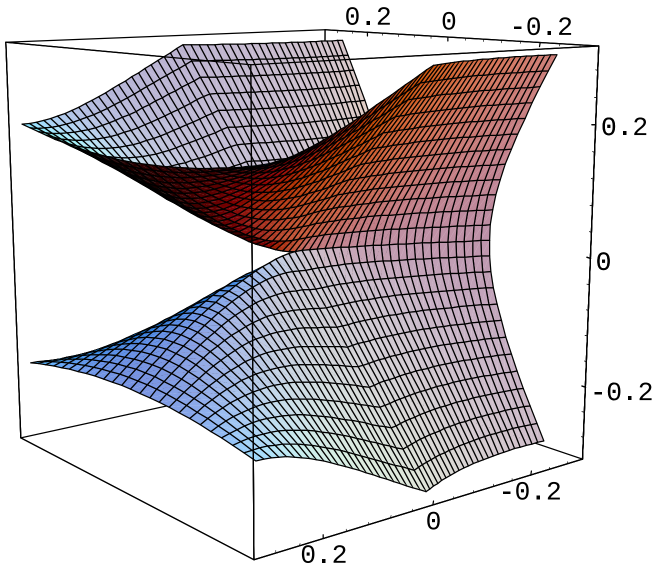

Fig. 4. Kellogg's example, the front

When $L = \Delta$ the identity is the solution, that is $u(x) = x$, which is clearly an invertible mapping. Using a perturbation argument one may expect that for elliptic operators near the Laplacian the solution will still be invertible. However the situation is less clear for general second order elliptic operators L. We will explain the differences between the two dimensional and higher dimensional case in the following.

Let the function w be a non constant solution of a uniformly elliptic equation

$$\left(\sum_{i,j=1}^{n} a_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} \right) w = 0 \qquad \text{in } \Omega,$$

where Ω is a regular domain in $\mathbb{R}^n$. Our proof (for $n = 2$) uses basically three ingredients. The Carleman-Hartman-Wintner Theorem shows that a singularity ($\nabla w(y) = 0$) implies that the level sets $\Omega^+_{w(y)}$ and $\Omega^-_{w(y)}$ (see (3.24)) both consist locally near y of at least two disconnected sets. The Jordan Curve Theorem shows that $\overline{\Omega^+_{w(y)}} \cup \left(\mathbb{R}^2 \backslash \Omega\right)$ has at least two components. Thirdly, the maximum principle shows that every component intersects the boundary. Put together, $\{x \in \Omega; w(x) = w(y)\}$ consists of at least two (intersecting) curves that run up to the boundary, that is, if $\nabla w(y) = 0$ then $\{x \in \partial\Omega; w(x) = w(y)\}$ contains at least four components. The degree argument that we used is the appropriate mathematical tool here.

One might try to repeat such a proof for higher dimensions. The maximum principle still holds. But both other ingredients are no longer true. A singular point ($\nabla w(y) = 0$) does not necessarily give (locally) two separate sheets in the set $\{x \in \Omega; w(x) = w(y)\}$ (There is no straightforward higher dimensional

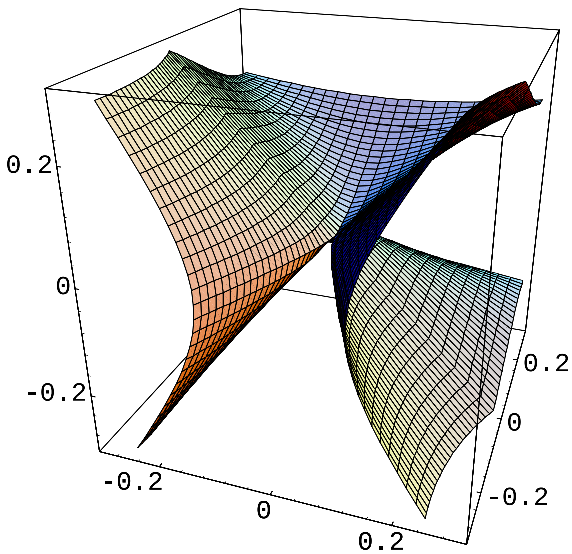

Fig. 5. Kellogg's example, the front

equivalent of C.-H.-W.). Even if there are two sheets, with their intersecting curve containing y, it is not clear that one can use Jordan's Theorem on one of these sheets. The obstructions are related with the fact that the local degree at a singularity in higher dimensions no longer has a fixed sign.

A stationary point that doesn't show at the boundary can be found by the example on p. 276 of Kellogg's book ([6]). The function $w(x, y, z) = z^2 - x^2 - y\left(y^2 - 3x^2\right)$ is harmonic and has zero gradient at 0. However, the intersection of the zero level set $\{(x, y, z); w(x, y, z) = 0\}$ and the boundary of the cube $[-.3, .3]^3$ consists of a single curve. Even at the singular point the level set is one sheet. In Figs. 4 and 5 this level set inside the cube is shown. Compare with Mastin and Thompson in [8]. Their arguments do not seem to be sufficient for the Theorem in 3 dimensions that is stated.

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