# On a Dirichlet problem related to the invertibility of mappings arising in 2D grid generation

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## 1   Introduction

The present paper deals with the invertibility of mappings that transform simply connected two-dimensional domains into a convex domain. The mapping is defined by a system of second order elliptic equations. These mappings are used to generate so called structured grids in the physical domain to solve Computational Fluid Dynamics (CFD) problems. These grids are generated by mapping a uniform rectangular mesh from a rectangle onto the physical domain. To enable a consistent discretization of the flow equations, it is necessary that the mesh in the physical domain be non-overlapping. Hence it is necessary that the mapping be invertible.

A typical example of 2D grid generation is illustrated by the diagram in Figure 1. The boundary conforming mesh around a 2D airfoil is obtained as the image of a uniform mesh in rectangle $R$ under a mapping $T$.
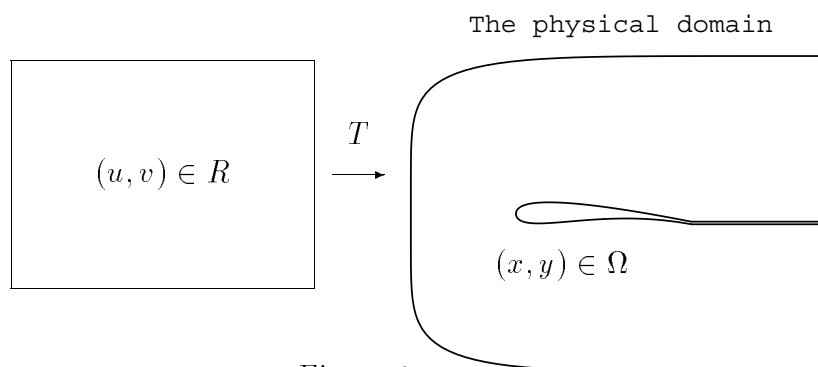


The physical domain

$(u, v) \in R$     $T \longrightarrow$     $(x, y) \in \Omega$

Figure 1.

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An elementary way to construct the basic mapping $T$ is to define the parametric coordinates $u$ and $v$ as solutions of the Laplace equation in $\Omega$:

$$\Delta u = 0 \qquad \Delta v = 0, \tag{1.1}$$

with $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ together with appropriate boundary conditions. The mapping $T$ is then defined by

$$T^{inv} = (u, v).$$

Mastin and Thompson considered such a problem in [14]. Winslow [21] replaced the Laplace equation (1.1) by isotropic diffusion equations

$$\nabla \cdot \frac{1}{w} \nabla u = 0 \qquad \nabla \cdot \frac{1}{w} \nabla v = 0, \tag{1.2}$$

with $\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$. The weight function $w(x, y)$ enables more direct control over the mesh spacing. .

The present paper deals with the mapping $T$ that is defined by a system of two elliptic partial differential equations with Dirichlet boundary conditions. The physical domain, a simply connected two-dimensional domain, is the image under $T$ of a convex domain. Existence, regularity and invertibility of the mapping $(u, v)$ is established in Corollary 3.

An alternative way to enable mesh spacing control is to apply an additional mapping $A$, see [20] and [8]. The regularity of such an additional mapping is studied in earlier work of the present authors ([2]). That paper is concerned with a mapping $T$ from the unit square onto itself that is defined by a similar elliptic system but with mixed Dirichlet and Neumann boundary conditions. Both problems are relevant for grid adaptation and generation problems.

The result in this paper depends strongly on a theorem of Carleman-Hartman-Wintner. This theorem is only true in two dimensional domains. In fact a straightforward generalization to more than two dimensional domains cannot be true. A counterexample to the proof of [15] for the three dimensional case can be found by using a special harmonic function due to Kellogg [12]. This function is shown in [2]. A direct counterexample can be found in [13].

## 2    Main result on smooth domains

Let the operator $L$ be given by

$$L = a_{11}(x) \left( \frac{\partial}{\partial x_1} \right)^2 + a_{12}(x) \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_1} + a_{22}(x) \left( \frac{\partial}{\partial x_2} \right)^2 + b_1(x) \frac{\partial}{\partial x_1} + b_2(x) \frac{\partial}{\partial x_2}, \tag{2.1}$$

where the coefficients satisfy for some $c > 0$ and $\gamma \in (0,1)$

$$a_{ij} \in C^{0,1}(\bar{\Omega}) \quad 1 \le i \le j \le 2,$$

$$\sum_{1 \le i \le j \le 2} a_{ij}\xi_i\xi_j \ge c\,|\xi|^2 \quad \text{on } \bar{\Omega}, \tag{2.2}$$

$$b_i \in C^\gamma(\bar{\Omega}), \quad 1 \le i \le 2. \tag{2.3}$$

The problem is as follows. For appropriate boundary values find $(u,v) \in C(\bar{\Omega}) \cap C^2(\Omega)$, satisfying

$$\text{(a)} \begin{bmatrix} Lu = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{bmatrix} \quad \text{and} \quad \text{(b)} \begin{bmatrix} Lv = 0 & \text{in } \Omega, \\ v = \psi & \text{on } \partial\Omega, \end{bmatrix} \tag{2.4}$$

such that $(u,v) : \bar{\Omega} \to I\!\!R^2$ is injective.

The physical problem in general involves non smooth domains and in most cases the mesh is defined by mapping a rectangle to the physical domain. The singularities for $(u,v)$ that occur because of corners come in two ways. For the physical domain having corners the elliptic p.d.e. has to be solved on a non smooth domain. Corners of the mesh domain, often a rectangle, that do not coincide with corners in the physical domain give rise to singularities of $\det(\nabla u, \nabla v)$. We will start with the case that both the physical and the mesh domain are smooth.

First we show the existence of an appropriate algebraic mapping between bounded, simply connected domains in $I\!\!R^2$ which have a Hölder smooth boundary.

**Proposition 1** *Let $\Omega$ and $\Sigma$ be Jordan domains in $I\!\!R^2$ with $\partial\Omega, \partial\Sigma \in C^{1,\gamma}$. Suppose $h$ is a $C^{1,\gamma}$ diffeomorphism from $\partial\Omega$ onto $\partial\Sigma$ that preserves the orientation. Then there is an extension $\Phi$ of $h$ such that*

$$\Phi \in C^{1,\gamma}\left(\bar{\Omega}; \bar{\Sigma}\right), \tag{2.5}$$

$$\Phi : \bar{\Omega} \to \bar{\Sigma} \text{ is a bijection} \tag{2.6}$$

*and*

$$\det \begin{pmatrix} \Phi_{1,x_1} & \Phi_{1,x_2} \\ \Phi_{2,x_1} & \Phi_{2,x_2} \end{pmatrix} > 0 \quad \text{on } \bar{\Omega}. \tag{2.7}$$

**Remark 1.1** Let us recall some definitions. We denote

$$\mathbf{T} = \left\{ x \in I\!\!R^2; \|x\| = 1 \right\}, \tag{2.8}$$

$$\mathbf{D} = \left\{ x \in I\!\!R^2; \|x\| < 1 \right\}. \tag{2.9}$$

i. For the definition of a Jordan domain see the appendix. For open bounded set $\Omega \subset I\!R^2$ to be a Jordan domain, it is sufficient that there exists a injective function $\omega \in C\left(\mathbf{T}\right)$ with $\partial\Omega = \omega\left(\mathbf{T}\right)$.

ii. The boundary $\partial\Omega$ satisfies $\partial\Omega \in C^{1,\gamma}$, if there is a parameterization $\omega \in C^{1,\gamma}\left(\mathbf{T}\right)$ of $\partial\Omega$ with $|\omega'| \neq 0$.

iii. The function $h$ is a $C^{1,\gamma}$ diffeomorphism from $\partial\Omega$ onto $\partial\Sigma$ with $\partial\Omega, \partial\Sigma \in C^{1,\gamma}$, if $\tilde{h} = \sigma^{inv} \circ h \circ \omega \in C^{1,\gamma}\left(\mathbf{T}; \mathbf{T}\right)$ and $\left|\tilde{h}'\right| \neq 0$. Here $\omega$ (resp. $\sigma$) is a $C^{1,\gamma}$ parameterization of $\partial\Omega$ (resp. $\partial\Sigma$) as in ii.

**Proof.** By Carathéodory's extension of the Riemann Mapping Theorem (see page 18 of [16]) there exists a mapping $f_\Omega \in C(\bar{\mathbf{D}}; \bar{\Omega})$ that is conformal from $\mathbf{D}$ onto $\Omega$, with $f_\Omega : \mathbf{T} \to \partial\Omega$ injective. Conformal includes $f_\Omega : \mathbf{D} \to \Omega$ being injective. By Theorems of Kellogg-Warschawski (see page 48 and 49 of [16]) we find $f_\Omega \in C^{1,\gamma}(\bar{\mathbf{D}}; \bar{\Omega})$ and $|f'_\Omega| \neq 0$ on $\bar{\mathbf{D}}$. Also a function $f_\Sigma$ exists with similar properties. Hence we may restrict ourselves to the case that $\Omega = \Sigma = \mathbf{D}$. Let $\tilde{h} \in C^{1,\gamma}\left(\mathbf{T}; \mathbf{T}\right)$ be as in Remark 1.1.iii. If we have an appropriate $\hat{\Phi} : \bar{\mathbf{D}} \to \bar{\mathbf{D}}$, with $\hat{\Phi} = \tilde{h}$ on $\mathbf{T}$, then $\Phi := f_\Sigma \circ \hat{\Phi} \circ f_\Omega^{inv}$ will be an extension of $h$. The claims (2.5-2.6) will be immediate and, indeed, (2.7) follows from

$$\det\left(\nabla\Phi\right) = |f'_\Sigma|^2 \ \det\left(\nabla\hat{\Phi}\right) \ |f'_\Omega|^{-2} \ .$$

It remains to show that there exists such a function $\hat{\Phi}$.

For $\tilde{h} \in C^{1,\gamma}\left(\mathbf{T}; \mathbf{T}\right)$ as above, there exists a $C^{1,\gamma}$ function $\alpha$, with the orientation preserving property implying $\alpha' > 0$, such that $\tilde{h}\left(\cos\varphi, \sin\varphi\right) = \left(\cos\alpha(\varphi), \sin\alpha(\varphi)\right)$. Setting $\vartheta\left(r\right) = \exp\left(1 - r^{-1}\right)$, we define an extension $\hat{\Phi} \in C^{1,\gamma}\left(\bar{\mathbf{D}}; \bar{\mathbf{D}}\right)$ by

$$\hat{\Phi}\left( \begin{array}{c} r\cos\varphi \\ r\sin\varphi \end{array} \right)^T = \left( \begin{array}{c} r\cos\left(\left(1 - \vartheta\left(r\right)\right)\ \varphi + \vartheta\left(r\right)\ \alpha(\varphi)\right) \\ r\sin\left(\left(1 - \vartheta\left(r\right)\right)\ \varphi + \vartheta\left(r\right)\ \alpha(\varphi)\right) \end{array} \right)^T.$$

The function $\hat{\Phi} : r\mathbf{T} \to r\mathbf{T}$ is a bijection for every $r \geq 0$ and a direct computation shows that

$$\det\left(\nabla\hat{\Phi}\right) = \left(1 - \vartheta\left(r\right)\right) + \vartheta\left(r\right)\ \alpha'(\varphi) > 0 \quad \text{on } \bar{\mathbf{D}}.$$

Notice that even if $\alpha' = 0$ somewhere we find $\det\left(\nabla\hat{\Phi}\right) > 0$ on $\mathbf{D}$. $\qquad\square$

The function $\Phi$ that we obtain above can be used for the assumptions in the next theorem.

**Theorem 2** *Let $\Omega$ be a simply connected domain in $I\!R^2$ with $\partial\Omega \in C^{1,\gamma}$ and let $\Sigma$ be a bounded, open and convex set in $I\!R^2$. Let $\Phi \in C^{1,\gamma}(\bar{\Omega}; I\!R^2)$ be such that*

$$\Phi : \bar{\Omega} \to \bar{\Sigma} \ \textit{ is a bijection,} \tag{2.10}$$

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*and*

$$\det \begin{pmatrix} \Phi_{1,x_1} & \Phi_{1,x_2} \\ \Phi_{2,x_1} & \Phi_{2,x_2} \end{pmatrix} > 0 \quad on \ \bar{\Omega}. \tag{2.11}$$

*Set $(\varphi, \psi) = \Phi_{|\partial\Omega}$. Then problem (2.4) possesses exactly one solution $u, v \in C^{1,\gamma}(\bar{\Omega}) \cap C^2(\Omega)$ with*

$$(u, v) : \bar{\Omega} \to \bar{\Sigma} \ \text{is a bijection} \tag{2.12}$$

*and*

$$\det \begin{pmatrix} u_{x_1} & u_{x_2} \\ v_{x_1} & v_{x_2} \end{pmatrix} > 0 \quad on \ \bar{\Omega}. \tag{2.13}$$

**Corollary 3** *Let $\Omega$ and $\Sigma$ be Jordan domains in $I\!\!R^2$ with $\partial\Omega, \partial\Sigma \in C^{1,\gamma}$. Suppose $h$ is a $C^{1,\gamma}$ diffeomorphism from $\partial\Omega$ onto $\partial\Sigma$ that preserves the orientation. Set $(\varphi, \psi) = h$.*
*Then problem (2.4) possesses exactly one solution $u, v \in C^{1,\gamma}(\bar{\Omega}) \cap C^2(\Omega)$, and $(u, v)$ satisfies (2.12) and (2.13).*

**Remark 3.1** From (2.11) it follows that the $C^{1,\gamma}$ smoothness of $\partial\Omega$ is transferred to $\partial\Sigma$. That is, also $\Sigma$ will have a $C^{1,\gamma}$ boundary. In fact (2.11) would also transfer corners of $\Omega$ to corners of $\Sigma$.

**Remark 3.2** If Hölder smoothness is replaced by Dini smoothness the results in this section remain true. One may also replace $C^{1,\gamma}$ with $C^{k,\gamma}$ for $k > 1$.

**Remark 3.3** A necessary condition on $(\varphi, \psi)$ to find a function $\Phi$ that satisfies (2.10) and (2.11) is the following. The boundary $\partial\Omega$ is the union of four counterclockwise ordered closed curves $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ such that

$$\begin{array}{llll} \varphi_\tau \geq 0, & \psi_\tau \geq 0 & \text{on } \Gamma_1 \\ \varphi_\tau \leq 0, & \psi_\tau \geq 0 & \text{on } \Gamma_2 \\ \varphi_\tau \leq 0, & \psi_\tau \leq 0 & \text{on } \Gamma_3 \\ \varphi_\tau \geq 0, & \psi_\tau \leq 0 & \text{on } \Gamma_4 \end{array} \tag{2.14}$$

and

$$\varphi_\tau^2 + \psi_\tau^2 > 0 \quad \text{on } \partial\Omega. \tag{2.15}$$

Here $\tau$ denotes the counterclockwise tangential direction. These conditions however do not imply that $\Sigma$ is convex. See Remark 5.3. Assuming convexity and regularity the conditions are sufficient for Proposition 1 and hence for Theorem 2.

**Proof of Theorem 2:** The proof will be done in several steps.

i. *Existence:* By Theorem 6.13 of [6] there exist unique solutions $u$ and $v$ in $C^0(\bar{\Omega}) \cap C^{2,\gamma}(\Omega)$. A theorem in [5] (see also page 111 of [6]) yields $u, v \in C^{1,\gamma}(\bar{\Omega})$. Let us denote

$$F(x) = (u(x), v(x)) \quad \text{for } x \in \bar{\Omega}. \tag{2.16}$$

ii. $\Sigma$ *contains* $F(\Omega)$: By assumption we have $F(\partial\Omega) = \Phi(\partial\Omega) = \partial\Sigma$. We will use two Theorems from the appendix for convex domains that use closed half spaces. Every closed half space in $I\!R^2$ can be written as

$$S = \left\{ y \in I\!R^2; w \cdot y \geq a \right\} \tag{2.17}$$

with some $w \in I\!R^2 \backslash \{0\}$ and $a \in I\!R$.

Let $S$, as in (2.17), be a closed half space containing $\bar{\Sigma}$. For $x \in \partial\Omega$ we have $F(x) = \Phi(x) \in \bar{\Sigma}$ and hence

$$w \cdot F(x) \geq a. \tag{2.18}$$

We also have

$$L(w \cdot F(x)) = w \cdot (Lu(x), Lv(x)) = 0 \quad \text{for } x \in \Omega.$$

Since $w \cdot F(\cdot) \not\equiv a$ on $\partial\Omega$ the strong maximum principle implies that

$$w \cdot F(x) > a \quad \text{for all } x \in \Omega. \tag{2.19}$$

Hence $F(\Omega) \subset S$. Since it holds for all appropriate $S$, Theorem B yields $F(\Omega) \subset \text{co}\left(\bar{\Sigma}\right) = \bar{\Sigma}$. Now suppose there is $x^* \in \Omega$ such that $F(x^*) \in \partial\Sigma$. We use Theorem A with $A = \bar{\Sigma}$ and $y = F(x^*)$ to get to a contradiction. By this theorem there is a closed half space $S$ such that $F(x^*) \in \partial S$ and $\bar{\Sigma} \subset S$. By (2.19) one finds $F(x^*) \in S \backslash \partial S$, a contradiction. Hence $F(\Omega) \subset \Sigma$ holds.

iii. $\Sigma$ *equals* $F(\Omega)$: Since $\Sigma$ is convex it is a Jordan domain. Then Theorem D.*i* (see the appendix) can be applied to show that $F(\Omega) = \Sigma$.

iv. *The Jacobian is positive on the boundary of* $\Omega$: We show that

$$\det(\mathcal{J}_F) > 0 \quad \text{on } \partial\Omega, \tag{2.20}$$

where we denote

$$\mathcal{J}_F(x) = \begin{pmatrix} u_{x_1}(x) & u_{x_2}(x) \\ v_{x_1}(x) & v_{x_2}(x) \end{pmatrix}.$$

By assumption we have $\det(\mathcal{J}_\Phi) > 0$ on $\Omega$ and hence on $\partial\Omega$. Although $F = \Phi$ on $\partial\Omega$ it does not straightforwardly imply that $\det(\mathcal{J}_F) > 0$ on $\partial\Omega$. Nevertheless, the result in (2.20) is true. Indeed, fix $y \in \partial\Omega$. From $F = \Phi$ on $\partial\Omega$ one deduces that

$$\frac{\partial}{\partial\tau} u = \frac{\partial}{\partial\tau}\Phi_1 \text{ and } \frac{\partial}{\partial\tau} v = \frac{\partial}{\partial\tau}\Phi_2,$$

where

- $\tau$ denotes the (counterclockwise) tangential direction of $\partial\Omega$ at $y$.

We will also use:

- $n$ for the interior normal direction of $\partial\Omega$ at $y$,
- $\zeta$ for the (counterclockwise) tangential direction of $\partial\Sigma$ at $\Phi(y)$,
- $\eta$ for the interior normal direction of $\partial\Sigma$ at $\Phi(y)$.

Notice that $\tau_1 = n_2$ and $\tau_2 = -n_1$. Since $\Sigma$ is convex, we have

$$\langle \eta, \Phi(x) - \Phi(y) \rangle > 0 \quad \text{for all } x \in \Omega, \tag{2.21}$$

and hence

$$\langle \eta, \mathcal{J}_\Phi(y)\, n \rangle = \frac{\partial}{\partial n} \langle \eta, \Phi(y) \rangle \geq 0. \tag{2.22}$$

Let $y(t)$ be a parameterization of $\partial\Omega$ near $y = y(0)$ with $y'(t) \neq 0$. We may assume that such a parameterization exists since $\partial\Omega$ is $C^{1,\gamma}$ near $y$. Then there is $c_1 \neq 0$ such that we have

$$\tau = c_1 \; y'(0).$$

Since $\Phi(y(t))$ parameterizes $\partial\Sigma$ near $\Phi(y)$ and $\frac{d}{dt}\left(\Phi(y(t))\right) = \mathcal{J}_\Phi(y(t)) \; y'(t)$ with $\det(\mathcal{J}_\Phi) \neq 0$ we find for some $c_2 \neq 0$ that

$$\zeta = c_2 \mathcal{J}_\Phi(y)\, \tau.$$

Hence it follows that

$$\langle \eta, \mathcal{J}_\Phi(y)\, \tau \rangle = c_2^{-1} \langle \eta, \zeta \rangle = 0, \tag{2.23}$$

and since $\det(\mathcal{J}_\Phi) \neq 0$ it follows then that

$$\langle \eta, \mathcal{J}_\Phi(y)\, n \rangle \neq 0,$$

and hence together with (2.22) that

$$\langle \eta, \mathcal{J}_\Phi(y)\, n \rangle > 0. \tag{2.24}$$

Now we will derive similar results for $F$. Since $\Sigma$ is convex we have that

$$\langle \eta, z - F(y) \rangle > 0 \quad \text{for all } z \in \Sigma.$$

Since $F(\Omega) \subset \Sigma$ we find

$$\langle \eta, F(x) - F(y) \rangle > 0 \quad \text{for all } x \in \Omega. \tag{2.25}$$

From (2.25) and the fact that $L \langle \eta, F(\cdot) - F(y) \rangle = 0$ in $\Omega$, it follows by Hopf's boundary point lemma that

$$\langle \eta, \mathcal{J}_F(y) \ n \rangle = \frac{\partial}{\partial n} \langle \eta, F(y) \rangle > 0. \tag{2.26}$$

From the boundary conditions and the assumption that $\Omega$ is smooth near $y$ it follows that

$$\langle \eta, \mathcal{J}_F(y) \ \tau \rangle = \frac{\partial}{\partial \tau} \langle \eta, F(y) \rangle = 0. \tag{2.27}$$

Rewrite (2.26-2.27) as

$$\langle \eta, (u_n(y), v_n(y)) \rangle > 0 \tag{2.28}$$

and

$$\langle \eta, (u_\tau(y), v_\tau(y)) \rangle = 0 \tag{2.29}$$

Set $\xi_1 = \frac{\partial}{\partial n} \Phi_2(y)$ and $\xi_2 = -\frac{\partial}{\partial n} \Phi_1(y)$. We obtain by (2.24) that $\xi_1 \eta_2 - \xi_2 \eta_1 > 0$ and with (2.29) that

$$\det(\mathcal{J}_F) = (\xi_1 \eta_2 - \xi_2 \eta_1)^{-1} \det \left( \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix} \begin{pmatrix} u_\tau & u_n \\ v_\tau & v_n \end{pmatrix} \right) =$$

$$= (\xi_1 \eta_2 - \xi_2 \eta_1)^{-1} \det \begin{pmatrix} \xi_1 u_\tau + \xi_2 v_\tau & \xi_1 u_n + \xi_2 v_n \\ \eta_1 u_\tau + \eta_2 v_\tau & \eta_1 u_n + \eta_2 v_n \end{pmatrix} =$$

$$= (\xi_1 \eta_2 - \xi_2 \eta_1)^{-1} \det \begin{pmatrix} \xi_1 u_\tau + \xi_2 v_\tau & \xi_1 u_n + \xi_2 v_n \\ 0 & \eta_1 u_n + \eta_2 v_n \end{pmatrix} =$$

$$= (\xi_1 \eta_2 - \xi_2 \eta_1)^{-1} (\xi_1 u_\tau + \xi_2 v_\tau)(\eta_1 u_n + \eta_2 v_n)$$

Since

$$\xi_1 u_\tau + \xi_2 v_\tau = \xi_1 \Phi_{1,\tau} + \xi_2 \Phi_{2,\tau} = \det(\mathcal{J}_\Phi) > 0$$

we find, with (2.28), that

$$\det(\mathcal{J}_F) > 0 \quad \text{on } \partial\Omega. \tag{2.30}$$

v. *The global degrees related with the gradients are zero:* Let $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in I\!R^2 \backslash \{0\}$ and define

$$\phi(x) = \alpha u(x) + \beta v(x) \quad \text{for } x \in \partial\Omega.$$

We will show that

$$\deg(\nabla\phi, \Omega) = 0.$$

Because of (2.30) one finds $\nabla\phi \neq 0$ on $\partial\Omega$, and hence that the degree is well defined. Since this holds for all $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in I\!R^2 \backslash \{0\}$ we may define a homotopy

$h_1(t, x)$ with $h_1(0, x) = \nabla \phi(x)$ and $h_1(1, x) = \nabla u(x)$ and such that $h(t, x) \neq 0$ on $\partial \Omega$. Hence

$$\deg(\nabla \phi, \Omega) = \deg(\nabla u, \Omega).$$

As before notice that $\frac{\partial}{\partial \tau} \Phi_1 = \frac{\partial}{\partial \tau} u$ and moreover, if $\frac{\partial}{\partial \tau} \Phi_1 = 0$, then $\frac{\partial}{\partial n} \Phi_1$ and $\frac{\partial}{\partial n} u$ are both nonzero. They even have the same sign since both $F(\Omega) \subset \Sigma$ and $\Phi(\Omega) \subset \Sigma$ hold. It shows that

$$t \begin{pmatrix} \Phi_{1,\tau} \\ \Phi_{1,n} \end{pmatrix} + (1 - t) \begin{pmatrix} u_\tau \\ u_n \end{pmatrix} \neq 0 \quad \text{for all } t \in [0, 1] \text{ and } x \in \partial \Omega.$$

Hence we may use the homotopy $h_2(t, x) = t \nabla \Phi_1 + (1 - t) \nabla u$ and we find that

$$\deg(\nabla u, \Omega) = \deg(\nabla \Phi_1, \Omega).$$

Since $\nabla \Phi_1 \neq 0$ on $\bar{\Omega}$ we have $\deg(\nabla \Phi_1, \Omega) = 0$.

vi. *The Jacobian is positive inside $\Omega$:* It remains to prove that

$$\det(\mathcal{J}_F) > 0 \quad \text{on } \bar{\Omega}. \tag{2.31}$$

Indeed, if (2.31) holds then $F$ on $\bar{\Omega}$ is locally injective and Theorem D.*ii* yields that $F : \bar{\Omega} \to \bar{\Sigma}$ is a bijection.

We will show (2.31) by a contradiction argument. Suppose that $\det(\mathcal{J}_F(\bar{x})) = 0$ for some $\bar{x} \in \Omega$. Then there are $(\alpha, \beta) \neq (0, 0)$ such that $\alpha \nabla u(\bar{x}) + \beta \nabla v(\bar{x}) = 0$. Set

$$\phi(x) = \alpha u(x) + \beta v(x) \quad \text{for } x \in \Omega.$$

As a consequence of Theorem C, we find that the zeros of $\nabla \phi$ are isolated, and that the local degree at a zero of $\nabla \phi$ is negative. That is, if $\nabla \phi(a) = 0$ there is $\varepsilon > 0$ such that $\nabla \phi \neq 0$ on $\partial B_\varepsilon(a) \subset \Omega$ and $\deg(\nabla \phi, B_\varepsilon(a)) < 0$. The additivity property of the degree shows that $\deg(\nabla \phi, \Omega) < 0$. Since we already showed that this degree is zero we have a contradiction. □

# 3 Domains with corners

We restrict ourselves to domains $\Omega$ and $\Sigma$ with Lipschitz boundary consisting of finitely many sufficiently differentiable curves and finitely many corners. Problems involving corners of these domains can be roughly distinguished into four different types. For the sake of simplicity we will leave out the case that on the boundary of $\Omega$ or $\Sigma$ two curves meet in a $C^1$ way (angle equals $\pi$). Notice that the convexity of $\Sigma$ implies that its corners are convex.

I. The case that the corners (all being convex) of $\Omega$ are mapped to the corners of $\Sigma$. Then it is still possible to have $\Phi \in C^{1,\gamma}(\bar{\Omega}; I\!R^2)$ mapping $\Omega$ onto $\Sigma$ with $\det(\nabla\Phi) > 0$ on $\bar{\Omega}$.

II. The boundary near a convex corner $x^*$ of $\Omega$ is mapped to a smooth part of $\partial\Sigma$. Then there will be two possibilities. Either one has $\Phi \in C^1(\bar{\Omega}; I\!R^2)$ and $\det(\nabla\Phi(x)) \to 0$ when $x \to x^*$, or $\det(\nabla\Phi(x))$ is bounded on $\Omega$ near $x^*$ but $\Phi \notin C^1(\bar{\Omega}; I\!R^2)$ .

III. The boundary near a concave corner $x^*$ of $\Omega$ is mapped to a smooth part of $\partial\Sigma$. Then there will not be a $\Phi \in C^1(\bar{\Omega}; I\!R^2)$ that maps $\Omega$ to $\Sigma$. However it is possible to have $\Phi \in C^0(\bar{\Omega}; I\!R^2) \cap C^1(\Omega; I\!R^2)$ with $\det(\nabla\Phi)$ bounded on $\Omega$ near $x^*$.

IV. A smooth part of $\partial\Omega$ is mapped on a neighborhood of a corner on $\partial\Sigma$. Similar features appear as in III.

In the next proposition and theorem we consider the first case. The proposition shows an algebraically defined regular mapping from $\Omega$ onto a rectangle.

**Proposition 4** *Let $\Omega$ be a Jordan domain such that $\partial\Omega$ consists of four $C^{k,\gamma}$-curves, with $k \geq 1$, that are joined by strictly convex corners (the angles $\alpha_i$ are in $(0, \pi)$ ). Say $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ counterclockwise oriented with $\Gamma_i = \gamma_i([0,1])$ and the corners will be at $\gamma_i(1) = \gamma_{i+1}(0)$ $(i \mod 4)$. Moreover assume $\varphi, \psi : \partial\Omega \to I\!R^2$ satisfy $\varphi|_{\Gamma_i}, \psi|_{\Gamma_i} \in C^{k,\gamma}$,*

$$\begin{array}{llll} \varphi_\tau > 0 & and & \psi_\tau = 0 & on\ \Gamma_1, \\ \varphi_\tau = 0 & and & \psi_\tau > 0 & on\ \Gamma_2, \\ \varphi_\tau < 0 & and & \psi_\tau = 0 & on\ \Gamma_3, \\ \varphi_\tau = 0 & and & \psi_\tau < 0 & on\ \Gamma_4, \end{array}$$

*and that $(\varphi, \psi)(\partial\Omega) = \partial R$ for some open rectangle $R$.*
*Then there is an extension $\Phi$ of $(\phi, \psi)$ such that*

$$\Phi \in C^{k,\gamma}\left(\bar{\Omega}; \bar{R}\right), \tag{3.1}$$

$$\Phi : \bar{\Omega} \to \bar{R}\ is\ a\ bijection \tag{3.2}$$

*and*

$$\det\begin{pmatrix} \Phi_{1,x_1} & \Phi_{1,x_2} \\ \Phi_{2,x_1} & \Phi_{2,x_2} \end{pmatrix} > 0 \quad on\ \bar{\Omega}. \tag{3.3}$$

**Remark  4.1** If $\Gamma_i = \gamma_i([0,1])$ we mean by $\psi_\tau > 0$ on $\Gamma_i$ that

$$\begin{cases} \psi_{\tau+} > 0 & on\ \gamma_i([0,1)), \\ \psi_{\tau-} > 0 & on\ \gamma_i((0,1]), \end{cases}$$

where $\tau^+$ $(\tau^-)$ is the upper (lower) counterclockwise tangential direction.

**Proof.** We start with a series of rather technical transformations and fix a corner at $(0,0)$ with $\gamma'_{i+1}(0) = (1,0)$. By $\vartheta = \vartheta(x^2 + y^2)$ we denote a positive $C^\infty$-function with $\vartheta(r) = 1$ for $r < \varepsilon$ and $\vartheta(r) = 0$ for $r > 2\varepsilon$ where $\varepsilon$ is small enough. It will be used to construct transformations that act locally involving just one corner. We restrict ourselves to the corner at $(0,0) = \gamma_1(1) = \gamma_2(0)$.

i. By transformations of the type $(x,y) \mapsto (x + c\vartheta\, y, y)$ one may enlarge or reduce any strictly convex $(0 < \alpha < \pi)$ corner at $0$ in a regular way to a corner with angle $\frac{1}{2}\pi$. Hence we may suppose that at the corner $\gamma'_1(1) = (0,-1)$ and $\gamma'_2(0) = (1,0)$.
Higher order derivatives of the second component $\left(\gamma_2^{(m)}(0)\right)_2$, $2 \leq m \leq k$, can be set to $0$ by $(x,y) \mapsto (x + c\vartheta\, y^2 P_1(y), y)$, where $P_1$ is a polynomial. In a similar way one takes care of $\left(\gamma_1^{(m)}(1)\right)_1$.

ii. By a transformation $(x,y) \mapsto ((1 + c_1\,\vartheta)x, y)$ we may assume that $\psi_{\tau+} = 1$ at $(0,0)$, and $(x,y) \mapsto ((1 + \vartheta\, xP_2(x))x, y)$ is used to obtain $\left(\frac{\partial}{\partial\tau+}\right)^m \psi = 0$ at $(0,0)$. Similarly one takes care of $\varphi$.

iii. The transformation $(x,y) \mapsto (1 - \vartheta)(x,y) + \vartheta\,(x^2 - y^2, 2xy)$ stretches the corner such that in the new parameterization we find $\gamma'_1(1) = \gamma'_2(0) = (1,0)$. This mapping is no longer regular but its singularity is explicit. As a result we find that the transformed $\Omega$ (let's call it $\Omega^*$) has a $C^{k,\gamma}$ boundary and moreover, if $\ell$ denotes the parametrization by arclength of $\partial\Omega^*$ with $\ell(0) = (0,0)$, we find

$$
\begin{aligned}
\varphi(\ell(t)) - \varphi(0,0) &= -\left(-t + \mathcal{O}\left(|t|^{k+\gamma}\right)\right)^2 &&\text{for } t \leq 0, \\
\psi(\ell(t)) - \psi(0,0) &= \left(t + \mathcal{O}\left(t^{k+\gamma}\right)\right)^2 &&\text{for } t \geq 0.
\end{aligned}
\tag{3.4}
$$

Writing $T_\Omega$ for these combined transformations we find that the determinant satisfies in the new coordinates

$$
\det(\nabla T_\Omega) = \sqrt{x^2 + y^2}\,(4 + \mathcal{O}(1)).
$$

In a similar way we transform the rectangle $R$ to a $C^{k,\gamma}$-domain (even $C^\infty$) $R^*$. For the rectangle the 'boundary' conditions one starts with are $(\varphi_R, \psi_R) = Id$. After stretching we obtain a formula as in (3.4) for $\varphi_R$, $\psi_R$ near $(0,0)$.

We continue as in the proof of Proposition 1. The Kellogg-Warschawski extension of the Riemann Mapping Theorem yields the existence of $f_{\Omega^*} \in C^{k,\gamma}\left(\bar{\mathbf{D}}; \bar{\Omega}^*\right)$ that is conformal inside $\mathbf{D}$. See (2.9) for $\mathbf{D}$. Similar results hold for $f_{R^*}$. In order to show that the function $\alpha$ determined by $\tilde{h} : \mathbf{T} \to \mathbf{T}$ is $C^{k,\gamma}$ and still satisfies $0 < \alpha' < \infty$ we use (3.4) both for $(\varphi, \psi)$ and $(\varphi_R, \psi_R)$. From $\alpha \in C^{k,\gamma}$ it follows that $\hat{\Phi} \in C^{k,\gamma}$. Finally we have

$$
\Phi = T_R \circ f_{R^*} \circ \hat{\Phi} \circ f_{\Omega^*}^{inv} \circ T_\Omega^{inv}.
$$

Since the singularities in the determinants of $T_R$ and $T_\Omega^{inv}$ cancel, we find that

$$0 < \det\left(\nabla\Phi\right) < \infty \quad \text{on } \bar{\Omega}.$$

$\square$

**Theorem 5** *Let $\Omega$ be a simply connected domain in $I\!\!R^2$, with Lipschitz boundary consisting of finitely many $C^2$-curves and finitely many corners. Let $\Sigma$ be convex and let $\Phi \in C^3(\bar{\Omega}; I\!\!R^2)$ satisfy (2.10) and (2.11). Again set $(\varphi, \psi) = \Phi_{|\partial\Omega}$.*
*Then problem (2.4) possesses exactly one solution $u, v \in C^{1,\gamma}(\bar{\Omega}) \cap C^{2,\gamma}(\Omega)$ and (2.12), (2.13) hold.*

**Remark 5.1** The boundary is $C^{1,\gamma}$ except for finitely points, and Lipschitz. The assumption implies that $\Omega$ is similar to a convex domain in the sense of Kadlec [11]. Due to a result of Kadlec [11] solving (2.4) on a domain with $C^2$-smooth 'convex' corners one still has $C^{1,\gamma}(\bar{\Omega})$-solutions. To apply his result we have to assume that $\Phi \in C^3(\bar{\Omega}; I\!\!R^2)$. Assuming $\Phi \in C^2(\bar{\Omega}; I\!\!R^2)$ gives $u, v \in C^{0,\gamma}(\bar{\Omega}) \cap C^{2,\gamma}(\Omega)$. Near a 'concave' corner the derivatives of a solution will become unbounded in general.

**Remark 5.2** A similar remark as in Remark 3.3 can be made. If $\partial\Omega$ is non smooth (having finitely many corners) the condition on for example $\Gamma_1$ has to be replaced by

$$\begin{cases} \varphi_{\tau^-} \geq 0, & \psi_{\tau^-} \geq 0 \quad \text{on } \gamma_1\left(0, 1\right], \\ \varphi_{\tau^+} \geq 0, & \psi_{\tau^+} \geq 0 \quad \text{on } \gamma_1\left[0, 1\right), \end{cases} \tag{3.5}$$

where $\Gamma_1$ is parameterized by $\gamma_1 : [0, 1] \to I\!\!R^2$, and $\tau^\pm$ are the upper/lower counterclockwise tangential direction.

**Remark 5.3** The conditions in (2.14-2.15) do not imply convexity of $\Sigma$. For example (see Figure 2.) the boundary of the square $[-1, 1]^2$ is mapped in a non regular way by the following boundary conditions $\varphi$ and $\psi$:

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} -yx^2 + 2x \\ yx^2 + 2x \end{pmatrix} \quad \text{for } (x, y) \in \partial\left([-1, 1]^2\right).$$

A straightforward computation shows that the conditions in (2.14-2.15) in the sense of (3.5) however are satisfied.



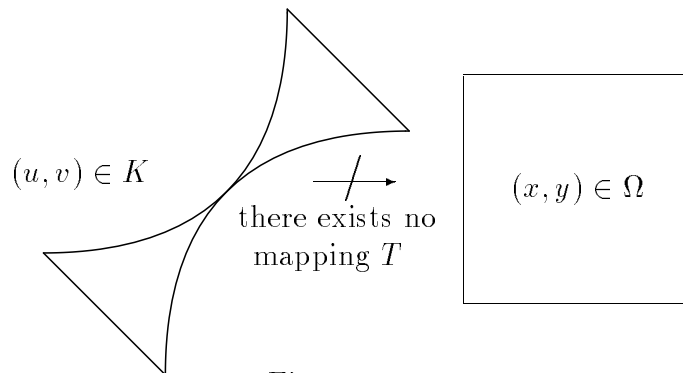$(u, v) \in K$ there exists no mapping $T$ $(x, y) \in \Omega$

Figure 2.

**Remark 5.4** An approach that takes care of the corners in cases II, III and IV is the following. Define an explicit intermediate mapping $\Psi$ on $\Omega$ that stretches the concave corners (and possible some of the convex corners) and apply the mapping defined by the differential equations on $\Psi(\Omega)$. The singularities of the mapping $F \circ \Psi : \Omega \to \Sigma$ will only come from the explicitly known singularities of $\Psi$.

An intermediate mapping $\Psi$ from $\Omega$ onto a square is used by Hagmeijer in [8]. His motivation is based upon the availability of such a mapping within grid adaptation problems and considering adaptation as a modification of an existing mapping instead of constructing a new mapping. Getting rid of possible singularities in solving the differential equations gives a second motivation. Although the approach in [8] uses different boundary conditions, no singularities appear in the solution of the differential equation. See [2].

**The proof of Theorem 5:** We will only mention the parts that differ from the proof of Theorem 2.

i. *Existence:* By Theorem 6.24 of [6] there are solutions $u, v \in C^0(\bar{\Omega}) \cap C^{1,\gamma}(\Omega)$. Denoting the corner points of $\Omega$ by $K$ it can be used to find $u, v \in C^{1,\gamma}(\bar{\Omega} \backslash K)$. See also [5]. Since the corners are convex a result of Kadlec [11] implies that $u - \Phi_1$, $\frac{\partial}{\partial x_1}(u - \Phi_1)$ and $\frac{\partial}{\partial x_2}(u - \Phi_1)$ are in $W^{2,2}(\Omega)$. Since $W^{3,2}(\Omega)$ is imbedded in $C^{1,\gamma}(\bar{\Omega})$ for $C^{0,1}$-domains (see page 144 of [1]) we find $u, v \in C^{1,\gamma}(\bar{\Omega})$.

iv. *The Jacobian is positive on the boundary of $\Omega$:* For all boundary points on the smooth part of $\partial\Omega$ one shows the result as before. Let $x^*$ be a boundary point where $\partial\Omega$ has an angle and let $\tau^-, \tau^+$ denote the upper and lower tangential direction of the boundary at $x^*$. Since $\tau^- \neq \tau^+$ and since $\Phi_{1,\tau\pm} = u_{\tau\pm}$ respectively $\Phi_{2,\tau\pm} = v_{\tau\pm}$ we have $\mathcal{J}_F = \mathcal{J}_\Phi$ at $x^*$.

v. *The global degrees related with the gradients are zero:* Similarly as before one makes the homotopy for the smooth part. At a corner point $\nabla u = \nabla \Phi_1$.  $\square$

# 4 Appendix, some auxiliary results

**Theorem A** (see Theorem 8 of [4]) *Let $A \subset \mathbb{R}^n$ be convex. Then for every $y \in \partial A$ there is a closed half space $S$, with $y \in \partial S$ and $A \subset S$.*

**Theorem B** (see Theorem 11 of [4]) *Let $B \subset \mathbb{R}^n$ be bounded. Then $\text{co}(\bar{B})$, the convex hull of the closure of $B$, is the intersection of all the closed half-spaces that contain $B$.*

**Theorem C** (Corollary of Schulz's [18] version of a theorem by Carleman-Hartman-Wintner, see [2]) *Let $\phi \in W^{2,p}(\Omega)$, with $p > 2$, satisfy*

$$\left( a_{11} \left( \frac{\partial}{\partial x_1} \right)^2 + a_{12} \frac{\partial^2}{\partial x_1 \partial x_2} + a_{22} \left( \frac{\partial}{\partial x_2} \right)^2 + b_1 \frac{\partial}{\partial x_1} + b_2 \frac{\partial}{\partial x_2} \right) \phi = 0 \quad in \ \Omega,$$

*with $a_{ij}, b_i \in C^{0,1}(\Omega)$ and such that for some $c > 0$ we have $\sum a_{ij} \xi_i \xi_j \geq c |\xi|^2$ on $\Omega$ for all $\xi \in I\!R^2$. If $x^* \in \Omega$ is such that $\nabla \phi(x^*) = 0$, then there exists $r > 0$ such that $\overline{B_r(x^*)} \subset \Omega$ and either*

$$\nabla \phi \equiv 0 \quad on \ \overline{B_r(x^*)}$$

*or*

$$\begin{cases} \nabla \phi \neq 0 \quad on \ \overline{B_r(x^*)} \backslash \{x^*\}, \\ \deg(\nabla \phi, B_r(x^*)) < 0. \end{cases}$$

$B_r(x^*) = \left\{ x \in I\!R^2; \|x - x^*\| < r \right\}$.

**Remark:** There is no equivalent of Theorem C in dimensions $n \geq 3$. As a consequence one cannot generalize the proofs in this paper in order to obtain a version of Theorem 1 in higher dimensions.

Before stating the last theorem we will recall a definition.

**Definition** *A set $\Omega$ in $I\!R^n$ is called a Jordan domain if there exists $h : \overline{B_1(0)} \subset I\!R^n \to I\!R^n$ that is continuous, injective and such that $\Omega = h(B_1(0))$.*

**Remark:** A Jordan domain $\Omega$ is open, connected and $\partial \Omega = h(\partial B_1(0))$. For $n = 2$ it is known that the inside of a closed Jordan curve is a Jordan domain (see page 81 of [7]).

**Theorem D** *Let $\Omega, \Sigma \subset I\!R^n$ both be Jordan domains and let $F \in C\left( \bar{\Omega}; I\!R^n \right)$. Suppose that $F : \partial \Omega \to \partial \Sigma$ is bijective and that $F(\Omega) \subset \Sigma$. Then*

    i. *$F : \Omega \to \Sigma$ is surjective.*

    ii. *If moreover $F : \Omega \to \Sigma$ is a locally injective, then $F : \Omega \to \Sigma$ is injective.*

**Proof of Theorem D** *i.*: Let $h_\Omega$ (resp. $h_\Sigma$) be an injective continuous map from $\overline{B_1(0)}$ to $\bar{\Omega}$ (resp. $\bar{\Sigma}$) such as in the definition of a Jordan domain. We consider the continuous map

$$\check{F} := h_\Sigma^{inv} \circ F \circ h_\Omega : \overline{B_1(0)} \to \overline{B_1(0)}.$$

The map $\tilde{F} : \partial B_1 (0) \to \partial B_1 (0)$ is a bijection and $\tilde{F} (B_1 (0)) \subset B_1 (0)$. It remains to show that $\tilde{F} (B_1 (0)) \supset B_1 (0)$. To prove the last inclusion it is sufficient that for every $z \in B_1 (0)$ the Brouwer degree $\deg \left( \tilde{F} (\cdot) - z, B_1 (0) \right)$ is well defined and not equal zero.

Since $\left\| \tilde{F} (x) \right\| = 1$ for $x \in \partial B_1 (0)$ we find that $\tilde{F} - z \neq 0$ on $\partial B_1 (0)$, hence $\deg \left( \tilde{F} (\cdot) - z, B_1 (0) \right)$ is well defined. By considering the homotopy

$$H (t, \cdot) = \tilde{F} (\cdot) - t \, z \quad \text{for } t \in [0, 1]$$

we obtain that

$$\deg \left( \tilde{F} (\cdot) - z, B_1 (0) \right) = \deg \left( \tilde{F} (\cdot), B_1 (0) \right).$$

Since $\tilde{F} : \partial B_1 (0) \to \partial B_1 (0)$ is a bijection , it follows from the multiplicative property of the degree that

$$\deg \left( \tilde{F} (\cdot), B_1 (0) \right) = \pm 1. \tag{4.1}$$

Indeed, for an open bounded set $A \ni 0$ with a continuous injection $J : \bar{A} \to J \left( \bar{A} \right)$ the multiplicative property (see Theorem 5.1 on page 24 of [3]) shows

$$1 = \deg (Id, A) = \deg \left( J^{inv} \circ J, A \right) = \deg \left( J^{inv}, J (A) \right) \deg (J, A) \tag{4.2}$$

which implies $\deg (J, A) = \pm 1$. Since $J$ defined by

$$J (x) = \begin{cases} \|x\| \; \tilde{F} \left( \dfrac{x}{\|x\|} \right) & \text{for } x \neq 0, \\ \\ 0 & \text{for } x = 0, \end{cases}$$

is a continuous bijection from $\overline{B_1 (0)}$ onto itself, with $J = \tilde{F}$ on $\partial B_1 (0)$, we find (by property d6 on page 17 of [3]) that (4.1) holds.

**D** *ii.*: Now suppose that $F : \Omega \to \Sigma$, and hence $\tilde{F} : B_1 (0) \to B_1 (0)$, is locally injective. Then the local degree of $\tilde{F} (\cdot) - \tilde{F} (x)$ near $x$, for $x \in B_1 (0)$, is well defined by

$$d (x) = \lim_{\varepsilon \downarrow 0} \deg \left( \tilde{F} (\cdot) - \tilde{F} (x), B_\varepsilon (x) \right).$$

By (4.2) one finds that it equals $\pm 1$. We will show that $d (\cdot)$ is locally constant on $B_1 (0)$ and hence constant. Indeed, if $x \in B_1 (0)$ there exists $\delta > 0$ such that $\tilde{F} (\cdot)$ is locally injective on $B_{3\delta} (x^*)$. If $|x - y| < \delta$ we find by homotopy that

$$d (x) = \deg \left( \tilde{F} (\cdot) - \tilde{F} (x), B_{2\delta} (x) \right) = \deg \left( \tilde{F} (\cdot) - \tilde{F} (y), B_{2\delta} (x) \right).$$

Since for all $\varepsilon \in (0, \delta]$ we have that $B_{2\delta} (x)$ contains $B_\varepsilon (y)$ and $\tilde{F} (\cdot) - \tilde{F} (y) \neq 0$ on $B_{2\delta} (x) \backslash B_\varepsilon (y)$, it follows that

$$\deg \left( \tilde{F} (\cdot) - \tilde{F} (y), B_{2\delta} (x) \right) = \deg \left( \tilde{F} (\cdot) - \tilde{F} (y), B_\varepsilon (y) \right) = d (y).$$

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Using (4.1) and the additivity property of the degree we find that $\tilde{F}(\cdot) = z$ has at most one solution for all $z \in B_1(0)$. $\square$

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