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SEMILINEAR ELLIPTIC PROBLEMS ON DOMAINS WITH CORNERS

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1. INTRODUCTION

In this note we will study the relation between the boundary of the domain and the existence of a positive solution for a semilinear elliptic problem. Consider

$$(1.1)\qquad \begin{cases} -\Delta u = \lambda\, f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$, f is continuous and $\lambda > 0$. For f with $f(0) < 0$ we will establish a cone condition for the boundary which is necessary and sufficient for existence of a positive solution. This result is used to obtain a sign–changing stable solution.

Definition of solution.

Since we will not assume more regularity for f than continuity, we have to specify the notion of solution. A function u is called a solution if it satisfies

(1.2) a) $u \in C(\overline{\Omega})$,

b) $\int_\Omega (u\, \Delta\varphi + f(u)\, \varphi)\, dx = 0 \qquad$ for all $\varphi \in C_0^\infty(\Omega)$,

c) $u = 0 \qquad$ on $\partial\Omega$.

The function u is called positive if:

d) $u \geq 0$ in Ω.

A necessary condition for a)/ c) is the regularity of every boundary point. See [9, Theorem 2.14]. (Regular in the sense of Perron: there exists a barrier function at every boundary point.) Therefore we will assume that $\partial\Omega$ is regular.

The first result.

We will restrict ourselves to functions f which satisfy $f(u) \leq 0$ for $u > \rho > 0$. Then, by the maximum principle, there is no solution with $\max u > \rho$. Hence we assume

$$f = 0 \qquad \text{on } [\rho,\infty). \tag{1.3}$$

If $f(0) \geq 0$ this assumption directly guarantees the existence of a positive solution u_λ for all $\lambda > 0$; see [2,7]. For functions f with $f(0) < 0$, which we will consider, the situation is more complicated. In [5,6] it is shown for $f \in C^1(\mathbb{R})$ that there is a positive solution for λ large when $f(0) < 0$, if the following conditions are satisfied:

$$\int_t^\rho f(s)\,ds > 0 \qquad \text{for all } t \in [0,\rho) \tag{1.4}$$

and

$$\Omega = \bigcup_x \{ B(x,\varepsilon);\, x \in \Omega,\, d(x,\partial\Omega) > \varepsilon \} \quad \text{for some } \varepsilon > 0, \tag{1.5}$$

where $B(x,\varepsilon) = \{ y;\, d(x,y) < \varepsilon \}$.

W. Jäger raised the question of whether or not the uniform interior sphere condition (1.5) is necessary. We will show the following. There exists a critical aperture θ_f (depending on f) such that, if the domain satisfies a uniform interior cone condition with aperture larger than θ_f, then $\exists\lambda_0 > 0$ such that $\forall\lambda > \lambda_0$ there exists a positive solution of (1.1). On the other hand, as soon as the domain is cone–shaped with a smaller aperture at some boundary point, (1.1) has no positive solution. (In our proofs we do need some nice behaviour of $\partial\Omega$ in a neighborhood of a critical boundary point.)

For dimension $n = 2$ the number θ_f satisfies $\pi/2 < \theta_f < \pi$. This shows that there is no positive solution on a square if $f(0) < 0$. Since $\theta_f > \pi/2$ for every such f, it is possible to show (after cutting off an arbitrary f above the maximum of a solution) that there is no positive solution of (1.1) on the square for any $f \in C(\mathbb{R})$ with $f(0) < 0$, but without satisfying (1.3).

The proof.

The main proof will be done by linking specific super– respectively subsolutions on a domain which has a smooth boundary except at one point. Near this point the domain will look like a cone. (For supersolutions in a different sense such linking is also used in [11, 4].)

For the negative result we will use a cone on which we can construct a supersolution which has its minimum inside. If Ω is contained in this cone and has a point which is not to far from the vertex of the same cone we continue as follows. First we show that every solution has to lie below a compound supersolution. Thereupon we will use a sweeping principle (see [18,17]) to show that in fact a solution lies below a supersolution with negative minimum. Hence it is negative somewhere in the intersection of Ω and the cone.

The proof in the other direction uses the existence of a positive subsolution on a top–shaped domain which satisfies a cone condition with a smaller aperture. Using the continuous dependance of the solution on the boundary we obtain a critical aperture for the top–shaped domain. By filling up a domain, which satisfies a cone condition for $\theta > \theta_f$, with small tops instead of balls as in (1.5), we can use the argument from [7] to get a positive subsolution and hence a positive solution.

The second result.

For bounded λ smooth domains close to these 'edgy' domains will not posses a positive solution either. Nevertheless there might exist a stable solution with a positive maximum. Hence such a stable solution will change sign. We will establish an example of such a sign–changing stable solution, on a convex domain. That result answers a question of Matano. Matano himself recently found sign–changing stable solutions on convex domains with even $f(0) = 0$, [14].

We will use the following notion of stability. A solution u of (1.1) is called stable, if $\forall\varepsilon > 0$ $\exists\delta > 0$ such that for $U_0 \in L_\infty(\Omega)$ with $\| U_0 - u \|_\infty < \delta$, the solution U of the related parabolic problem:

$$\begin{cases} U_t - \Delta U = \lambda f(U) & \text{in } \Omega \times \mathbb{R}^+, \\ U = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \end{cases} \tag{1.6}$$

with $\lim_{t\downarrow 0} \| U(t) - U_0 \|_{L_1(\Omega)} = 0$, satisfies $\| U(t) - u \|_\infty < \varepsilon$ for all $t > 0$.

Section 3 contains some lemmata for weak subsolutions. Moreover, we will recollect in that section results about stability from [3,13,16,17] and modify them for our purposes.

2. MAIN RESULTS

We will prove results for domains in $\mathbb{R}^n$ using 'radially symmetric' cones. In dimensions larger than 2 we can find similar results by using other families of cones. At the end of this section we will give an example. In $\mathbb{R}^2$ the first lemma shows that there will not be a positive solution if for example the domain is convex and has a corner with angle less than $\frac{1}{2}\pi$ or close to $\frac{1}{2}\pi$, or if the domain is close to such a domain.

Lemma 2.1: *Set* $\lambda = 1$ *and suppose* f *satisfies* (1.3) (1.4) *and* $f(0) < 0$.

a. *There is* $t_1 \in (0,1)$ *such that if*

(2.1) $$(t_1,0,\dots,0) \in \Omega \subset \{ (x_1,\dots,x_n) \in \mathbb{R}^n ;\ (x_2^2+\dots+x_n^2)^{\frac{1}{2}} < (n-1)^{\frac{1}{2}}\, x_1 \}$$

there will be no positive solution of (1.1).

b. *There is* $c > 1$ *such that if*

(2.2) $$\{ (t,0,\dots,0);\ 0 < t \le 1 \} \subset \Omega \quad \text{and}$$
$$\Omega \subset \{ (x_1,\dots,x_n) \in \mathbb{R}^n ;\ (x_2^2+\dots+x_n^2)^{\frac{1}{2}} < c\,(n-1)^{\frac{1}{2}}\, x_1 \}$$

there will be no positive solution of (1.1).

Remark: By rescaling one finds that Lemma 2.1 b. holds for all $\lambda > 1$. Lemma 2.1 a. holds for $\lambda > 0$ if one replaces t_1 by $t_\lambda = \lambda^{-\frac{1}{2}} t_1$.

In the proofs we will use a weak version of sub- and supersolutions. For a definition see section 3.

Proof: To simplify notations we set $x_r = (x_2,\dots,x_n) \in \mathbb{R}^{n-1}$.

i) *Estimating solutions from above.*

Set $f^m = \max \{ f(s) ;\ 0 \le s \le \rho \}$ and $K = (2\rho f^m)^{\frac{1}{2}}$. Define $U \in C^1(\mathbb{R})$ by

(2.3) $$\begin{cases} U(t) = K\,t - \frac{1}{2} f^m\, t^2 & \text{for } t \le K\, {f^m}^{-1}, \\ U(t) = \rho & \text{for } t > K\, {f^m}^{-1}. \end{cases}$$

Let u be a solution of (1.1) and suppose (2.1) or (2.2) is satisfied. The maximum principle shows:

(2.4) $$u(x_1,x_r) \le U(x_1) < K\,x_1 \quad \text{for } (x_1,x_r) \in \Omega.$$

ii) *Restriction to a subdomain of* Ω.

Take $\varepsilon \in (0,K)$ such that

(2.5) $$f(s) < \tfrac{1}{2} f(0) \qquad \text{for } |s| < \varepsilon$$

and set

(2.6) $$\Omega^v = \Omega \cap \{ (x_1,x_r) \in \mathbb{R}^n;\ x_1 < K^{-1}\varepsilon \}.$$

From (2.4) it follows that

(2.7) $$-\Delta u = f(u) < \tfrac{1}{2} f(0) \qquad \text{for } x \in \Omega^v.$$

iii) *Defining a superfunction on the subdomain* Ω^v.

Define

(2.8) $$k = K\,\varepsilon^{-1} \exp(-\,8\, f(0)^{-1}\, K^2 \varepsilon^{-1}),$$

(2.9) $$v(x_1,x_r) = -\tfrac{1}{8} f(0) \left((x_1+ \tfrac{1}{2}k^{-1})^2 - (n-1)^{-1} x_r^2 \right) \ln(k x_1 + \tfrac{1}{2}).$$

Then we find for

(2.10) $$|x_r| < (n-1)^{\frac{1}{2}} (x_1 + \tfrac{1}{2}k^{-1})$$

that

(2.11) $$-\,\Delta v(x_1,x_r) = \tfrac{1}{8} f(0) \left(3 + (n-1)^{-1} x_r^2 (x_1+ \tfrac{1}{2}k^{-1})^{-2} \right) > \tfrac{1}{2} f(0).$$

The function v has a negative minimum for x satisfying (2.10) in

(2.12) $$\bar{x} = (k^{-1}(e^{-\frac{1}{2}} - \tfrac{1}{2}) , 0 ,\dots, 0).$$

Define

(2.13) $$t_1 = k^{-1}(e^{-\frac{1}{2}} - \tfrac{1}{2})$$

(2.14) $$c = 1 + K(2\varepsilon k)^{-1}.$$

Note that $t_1 \in (0,1)$ and hence $\bar{x} \in \Omega$ if Ω satisfies (2.2).

By this choice of c one finds that if Ω satisfies (2.2) then every $x \in \Omega^v$ satisfies (2.10).

iv) *Contradicting positivity of* u *by a lowest supersolution.*

Let α be the smallest number such that

(2.15) $$u \le v + \alpha \qquad \text{in } \overline{\Omega^v}.$$

If $\alpha \le 0$ then

(2.16) $$u(\bar{x}) \le v(\bar{x}) < 0.$$

If $\alpha > 0$, let $x^* \in \overline{\Omega^v}$ be such that

(2.17) $$u(x^*) = v(x^*) + \alpha.$$

By (2.15), (2.7) and (2.11) $v+\alpha-u$ is nonnegative and superharmonic in Ω^v:

(2.18) $$-\Delta(v+\alpha-u) = -\Delta v + \Delta u > \tfrac12 f(0) - f(u) \ge 0$$

and hence the minimum principle shows $x^* \in \partial\Omega^v$.

First we will prove $x^* \in \partial\Omega$ by showing that $u < v$ on $\partial\Omega^v\backslash\partial\Omega$.

Similar to (2.4) the maximum principle yields that on Ω^v:

(2.19) $$u(x_1,x_r) \le U\left((c^2(n-1)+1)^{-1}(c(n-1)^{\frac12}x_1 - \theta.x_r)\right) <$$

$$< K\,(c^2(n-1)+1)^{-1}(c(n-1)^{\frac12}x_1 - \theta.x_r) \quad \text{for all } \theta \in \mathbb{R}^{n-1} \text{ with } |\theta| = 1.$$

Then, if $x \in \partial\Omega^v\backslash\partial\Omega$, which means that $|x_r| < c\,(n-1)^{\frac12} K^{-1}\varepsilon$ and $x_1 = K^{-1}\varepsilon$, we use respectively (2.9), (2.8), (2.14) and (2.19):

(2.20) $$v(K^{-1}\varepsilon,x_r) =$$

$$= -\tfrac18 f(0)\,\left((K^{-1}\varepsilon + \tfrac12 k^{-1})^2 - (n-1)^{-1}x_r^2 \right).\ln(k\,K^{-1}\varepsilon + \tfrac12) >$$

$$> -\tfrac18 f(0)\,\left((K^{-1}\varepsilon + \tfrac12 k^{-1})^2 - (n-1)^{-1}x_r^2 \right).-8\,f(0)^{-1}\,K^2\,\varepsilon^{-1} =$$

$$= \quad K^2\,\varepsilon^{-1}\left(c^2K^{-2}\varepsilon^2 - (n-1)^{-1}x_r^2\right) \qquad \ge$$

$$\ge \quad c\,K\left(c\,K^{-1}\varepsilon - (n-1)^{-\frac12}|x_r|\right) \qquad >$$

$$> \quad K\,(c^2(n-1)+1)^{-1}\left(c\,(n-1)^{\frac12}\,K^{-1}\varepsilon - |x_r|\right) \qquad >$$

$$> \quad u(K^{-1}\varepsilon,x_r)\,.$$

Finally $x^* \in \partial\Omega$ yields $u(x^*) = 0 = v(x^*) + \alpha$ and hence for $\bar{x}$ from (2.12):

(2.21) $$u(\bar{x}) \le v(\bar{x}) + \alpha < v(x^*) + \alpha = 0.$$ □

The first part of Lemma 2.1 can be used to construct sign–changing stable solutions on smooth domains. As an example:

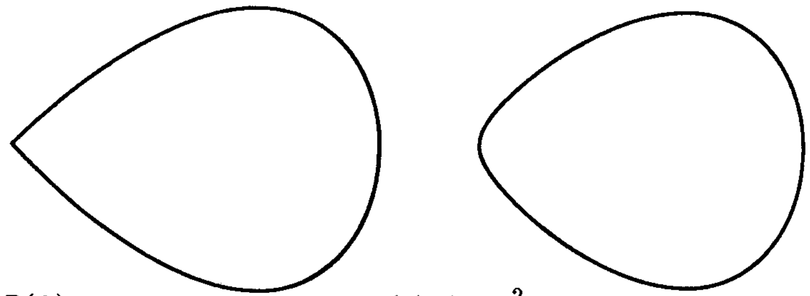
Corollary 2.2: *Set* $f(u) = (u^2-1)(10-u)$ *and*

(2.22) $$D(\varepsilon) = \{ (x_1,x_2) \in \mathbb{R}^2;\ x_1 > 0,\ x_2^2 < x_1^2(1-x_1) - \varepsilon \}$$

a. *Then there is* $\lambda_1 > 0$ *such that for all* $\lambda > \lambda_1$ *and* $\varepsilon \in (0,{}^1/_{10})$ *there is a stable solution* $u_{\lambda,\varepsilon}$ *of* (1.1) *on* $D(\varepsilon)$ *with* $\max u_{\lambda,\varepsilon} \in (1,10)$.

b. *For all* $\lambda > \lambda_1$ *there is* $\varepsilon(\lambda) > 0$ *such that, for* $\varepsilon \in (0,\varepsilon(\lambda)]$, $u_{\lambda,\varepsilon}$ *changes sign in* $D(\varepsilon)$.

Note that $D(\varepsilon)$ has a C^∞– boundary for $\varepsilon \in (0,{}^1/_{10})$.



D(0) $\qquad$ $D({}^1/_{100})$ in $\mathbb{R}^2$.

FIG. *1*

Proof: For $\varepsilon \in (0,{}^1/_{10})$ we find that

(2.23) $$({}^2/_3,0) \in D({}^1/_{10}) \subset D(\varepsilon) \subset D(0) \subset \{ (x_1,x_2);\ |x_2| < x_1 \}.$$

Hence there is $\delta > 0$ with $B(({}^2/_3,0),\delta) \subset D(\varepsilon)$ for all $\varepsilon \in [0,{}^1/_{10}]$. By Lemma 3.1 there exist $\mu > 0$ and $v \in C^2\left(\overline{B(0,1)}\right)$, with v radially symmetric, which satisfy

(2.24) $$\begin{cases} -\Delta v = \mu\, f(v) & \text{in } B(0,1), \\ 1 < v(0) < 10\,, & \\ v'(r) < 0 & \text{for } 0 < r \le 1, \\ v(1) = -1\,. & \end{cases}$$

Extend v by –1 outside of B(0,1).

Now we define $\lambda_1 = \mu\,\delta^{-2}$ and

(2.25) $$V(x_1,x_2) = v\left((\lambda/\mu)^{\frac12}(x_1-\tfrac23)\,,\ (\lambda/\mu)^{\frac12}x_2 \right)$$

which is, see Corollary 3.5, a subsolution of (1.1) for all $\lambda > \lambda_1$ and $\varepsilon < {}^1/_{10}$ on $D(\varepsilon)$, satisfying $V = -1$ on $\partial D(\varepsilon)$. The constant function $W = 10$ is a supersolution for all $\lambda > 0$. By Lemma 3.6 there exists a stable solution $u_{\lambda,\varepsilon}$ in $[V,W]$ of (1.1) on $D(\varepsilon)$. This proves the first part. We fix $\lambda > \lambda_1$ and we let t_1 be as in Lemma 2.1. Take $\varepsilon(\lambda)$ so small that $\lambda^{-\frac12}t_1 \in D(\varepsilon(\lambda))$. The second part of the corollary is a consequence of the remark following Lemma 2.1. □

In the next lemma we will show that for a top-shaped domain, with an aperture corresponding with a very wide cone, there does exist a positive solution of (1.1).

Define for $c > 0$ and $x_r = (x_2, x_3, \dots, x_n) \in \mathbb{R}^{n-1}$:

(2.26) $$A(c) = \{(x_1,x_r);\ (n-1)^{-\frac12}|x_r| < c x_1 + ((n-1)^{-1}+c^2)^{\frac12},\quad x_1 < -c((n-1)^{-1}+c^2)^{-\frac12}\},$$

(2.27) $$S(c) = A(c) \cup B(0,1),$$

where $B(0,1)$ is the unit ball.

Let $\tilde{x}$ denote the vertex of $S(c)$: $\left(-c^{-1}((n-1)^{-1}+c^2)^{\frac12},0,\dots,0\right)$.

Then $\partial S(c)\backslash\{\tilde{x}\}$ is $C^{1,1}$.

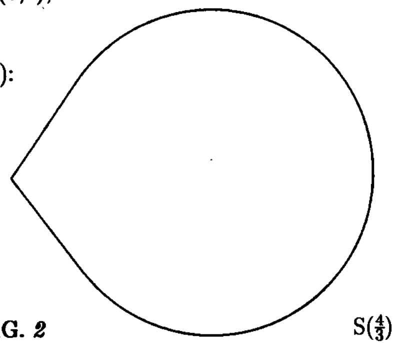


FIG. 2 $S(\frac43)$ for n=2

Lemma 2.3: *Let* f *satisfy* (1.3), (1.4) *and suppose* $f(0) < 0$. *Then there are* $c > 1$, λ *and* u, *with* $u \geq 0$, *which satisfy* (1.1) *on* S(c).

Proof: There is a radially symmetric subsolution (λ, U) of (1.1) on $B(0,1)$, which satisfies $U'(r) < 0$ for $r \in (0,1]$. Indeed, there is $f^* \in C^1$ with $f^* \leq f$, which still satisfies (1.3) and (1.4). Thereupon Lemma 3.1 yields the existence of a positive radially symmetric solution on $B(0,1)$ with f replaced by f^*, which is a subsolution of the original problem. We will show that for some $c \in (1,\infty)$ there exists a positive solution on S(c) with the same λ.

Fix the negative number $f_m = \min\{ f(s)\ ;\ 0 \leq s \leq \rho \}$ and define for $c > 1$:

(2.28) $$V_c(x_1,x_r) = -\tfrac12\lambda f_m (c^2-1)^{-1}\left((c x_1 + ((n-1)^{-1}+c^2)^{\frac12})^2 - (n-1)^{-1}x_r^2 \right),$$

which is positive on S(c). Moreover, a direct computation shows:

(2.29) $$-\Delta V_c = \lambda f_m \leq \lambda f(V_c) \qquad \text{if } 0 \leq V_c \leq \rho.$$

Define

(2.30) $$\begin{cases} W_c(x) = V_c(x) & \text{for } x \in \overline{A(c)} \setminus B(0,1),\\ W_c(x) = \max(V_c(x), U(x)) & \text{for } x \in A(c) \cap B(0,1),\\ W_c(x) = U(x) & \text{for } x \in \overline{B(0,1)} \setminus A(c). \end{cases}$$

Let $\alpha > 0$ be such that $U(x) > \alpha(1-|x|^2)$ for $x \in B(0,1)$. Then for c defined by

(2.31) $$c = \left(1 - \tfrac12\lambda f_m (n-1)^{-1} \alpha^{-1} \right)^{\frac12} > 1$$

we find that

(2.32) $$\begin{aligned} V_c(x_1,x_r) &= -\tfrac12\lambda f_m (c^2-1)^{-1} (n-1)^{-1} \left((1 + c^2(n-1))^{-1} - x_r^2 \right) = \\ &= \alpha\left((1+c^2(n-1))^{-1} - x_r^2 \right) = \alpha(1 - x_1^2 - x_r^2) < \\ &< U(x_1,x_r) \qquad \text{for } (x_1,x_r) \in \partial A(c) \cap B(0,1). \end{aligned}$$

For $x \in A(c) \cap \partial B(0,1)$ it follows that

(2.33) $$V_c(x) > 0 = U(x).$$

Hence $W_c \in C(\overline{S(c)})$ and Corollary 3.5 shows that W_c is a subsolution. By the construction W_c is positive in S(c). Applying the results in [6] shows the existence of a solution $u \in [W_c,\rho] \subset C(\overline{S(c)})$. Hence u is positive. □

Before we are able to state the main result we need the following. See also Definition 1.2.2.1 of [9].

Definition 2.4: *A domain* Ω *has the uniform interior cone property with constant* c *if* $\Omega = \cup \{ \varepsilon S_i\ ;\ i \in I \}$ *for some* $\varepsilon > 0$, *where every* S_i *is the image of* S(c) *under an orthonormal transformation.*

(S(c) is defined in (2.27) ; T is an orthonormal transformation if $|T(x)-T(y)| = |x-y|$ for all $x, y \in \mathbb{R}^n$, where $|\cdot|$ is the Euclidean norm).

Proposition 2.5: *Let* f *satisfy* (1.3), (1.4) *and* $f(0) < 0$. *Fix the dimension* $n > 1$. *Then there is* $c_0 \in (1,\infty)$ *for which the following holds. Let* Ω *a be bounded domain in* $\mathbb{R}^n$.

1) *If* Ω *has the uniform interior cone property with constant* $c > c_0$, *then* λ_0 *exists such that for all* $\lambda > \lambda_0$ *there is a positive solution* u_λ *of* (1.1).

2) *If* $\partial\Omega$ *contains a point* y *such that for some orthonormal transformation* T, *some* $\varepsilon > 0$ *and for some* $c < c_0$, *the following holds:*

$$\text{(2.34)}\qquad \begin{cases} T(\varepsilon\Omega) \subset S(c), \\ T(\varepsilon y) = \tilde{x} \ \ (\textit{the vertex of } S(c)\,), \end{cases}$$

Then there is no positive solution (λ,u) *of* (1.1).

Remark : For convex domains Ω in $\mathbb{R}^2$ this proposition can be formulated as follows. Let c denote the largest constant such that Ω satisfies the uniform interior cone property with constant c. If $c > c_0$ then there is a positive solution (λ_0,u) of (1.1) on Ω (and hence for all $\lambda \geq \lambda_0$). If $c < c_0$ then there is no positive solution (λ,u) of (1.1) on Ω.

Proof: i) Let Ω_1 and Ω_2 be two bounded domains in $\mathbb{R}^n$. Suppose there exists a positive solution (λ_1,u_1) of (1.1) on Ω_1, and suppose there is $\varepsilon > 0$ and a family $\{ T_i \ ; i \in I \}$ of orthonormal transformations in $\mathbb{R}^n$ such that $\Omega_2 = \cup \{ T_i(\varepsilon\Omega_1) \ ; i \in I \}$. Since Ω_1 and Ω_2 are open one can assume without loss of generality that I is countable. By Corollary 3.5:

$$\text{(2.35)}\qquad v_k(x) = \sup \{ u(\varepsilon^{-1}T_i^{-1}(x)) \ ; i \in \{i_1,\dots,i_k\} \subset I \}$$

is a subsolution on $\cup \{ T_i(\varepsilon\Omega_1) \ ; i \in \{i_1,\dots,i_k\} \}$ for $\lambda = \lambda_1\varepsilon^{-2}$. Using the dominated convergence theorem one finds that

$$\text{(2.36)}\qquad v(x) = \lim_{k\to\infty} v_k(x)$$

satisfies condition ii) in Definition 3.2. Since $\{v_k\}$ is an equicontinuous family, v also satisfies the conditions i) and iii) in Definition 3.2/3.3. Hence v is a subsolution on Ω_2 for $\lambda = \lambda_1\varepsilon^{-2}$. By $v > 0$ in Ω_2, $\max v = \max u$ and again the supersolution $w = \rho$, one gets the existence of a positive solution (λ_2,u_2) of (1.1) on Ω_2 with $\lambda_2 = \lambda_1\varepsilon^{-2}$ and $v \leq u \leq w$.

ii) Suppose (λ_0,u_0) is a positive solution of (1.1) on Ω and Ω is convex. Then $\Omega = \cup \{ x + \theta(\Omega - x) \ ; x \in \Omega \}$ for all $\theta \in (0,1)$. Part i) shows that there will be a positive solution of (1.1) on Ω for all $\lambda \geq \lambda_0$.

iii) Define

$$\text{(2.37)}\qquad J = \{ c \in (0,\infty) \ ; \exists \text{ a positive solution } (\lambda,u) \text{ of (1.1) on } S(c) \}.$$

By Lemma 2.1 there is $c_1 > 1$ such that $c_1 \notin J$. Lemma 2.3 shows that there is $c_2 < \infty$ such that $c_2 \in J$. Part i) of this proof shows that if $c \in J$, then $[c,\infty) \subset J$. Hence $c_0 = \inf \{ c \in J \} \in (1,\infty)$ is well defined. With part ii) this shows that $\forall\, c > c_0 \ \exists\, \lambda_c > 0$ with $\forall\, \lambda \geq \lambda_c$ there is a positive solution (λ,u) of (1.1) on S(c). If Ω satisfies the uniform cone property with a constant $c > c_0$ then from part i) there is a positive solution (λ,u) of (1.1) on Ω for all $\lambda \geq \lambda_c\,\varepsilon^{-2}$. This shows Proposition 2.5.1).

iv) We still have to prove the second statement. Since $\partial S(c)\backslash\{\tilde{x}\}$ is C^1 there is $\theta > 0$ and a family of orthonormal transformations $\{ T_i \ ; i \in I \}$ such that

$$\text{(2.38)}\qquad S(c) = \cup \{ T_i(\theta\varepsilon\Omega) \ ; i \in I \}.$$

If there is a positive solution of (1.1) on Ω, then part i) gives the existence of a positive solution of (1.1) on S(c), which is contradicted by part iii). □

The result of the last proposition may be used for f with $f(0) < 0$ which does not have a falling positive zero ρ, if the domain has a corner with an angle less than or equal to $\frac{1}{2}\pi$. For example for the square one can show the following.

Corollary 2.6: *Set* $\Omega = (0,1)^2$, *let* $\lambda > 0$ *and* $f \in C(\mathbb{R})$ *be such that* $f(0) < 0$. *Then there is no positive solution of* (1.1) *on* Ω.

Proof: Suppose u is a solution of (1.1). After modifying f on $(\max u,\infty)$ such that $f(u) = 0$ for $u \geq \max u + 1$ one may apply Proposition 2.5.2) to find that u is negative somewhere in Ω. □

Similarly to Corollary 2.6 one can show that for all f with $f(0) < 0$ there are no positive solutions of (1.1) on the hypercube, $\Omega = (0,1)^n$.

For domains in dimensions higher than two there is no longer a unique critical cone. For example in $\mathbb{R}^3$ one may use the following superfunctions to prove nonpositivity:

$$\text{(2.39)}\qquad v(x_1,x_2,x_3) = -\tfrac{1}{8}\, f(0) \left((x_1 + \tfrac{1}{2}k^{-1})^2 - \theta x_2^2 - (1-\theta)x_3^2 \right) \ln(kx_1 + \tfrac{1}{2}).$$

With every $\theta \in (0,1)$ one can find a critical cone. Replace S(c) in Definition 2.4 by

$$\text{(2.40)}\qquad S(\theta,c) = \{ (x_1,x_2,x_3) \ ; (x_1,(2\theta)^{\frac{1}{2}}x_2,(2-2\theta)^{\frac{1}{2}}x_3) \in S(c) \}$$

and one can prove the equivalent of Proposition 2.5 for every $\theta \in (0,1)$. Moreover, we may use the result in two dimensions for domains in higher dimensions. For example:

Corollary 2.7: *Let* Ω *denote a half ball,* $\Omega = \{ x \in \mathbb{R}^n;\ |x|<1,\ x_1>0 \}$, *for any dimension* $n > 1$, *let* $\lambda > 0$ *and* $f \in C(\mathbb{R})$ *be such that* $f(0) < 0$. *Then there is no positive solution of* (1.1) *on* Ω.

Proof: Suppose u is a positive solution of (1.1). We start by modifying f on $(\max u, \infty)$ such that $f(u) = 0$ for $u \geq \max u + 1$. Next we define a positive subsolution on the half ball in $\mathbb{R}^2$:

$$v(x_1,x_2) = \max \{ u(x_1,x_2,x_3,\ldots,x_n)\ ;\ x_3^2 + \ldots + x_n^2 \leq 1 - x_1^2 - x_2^2 \} \tag{2.41}$$

Since $w = \rho = \max u + 1$ is a supersolution above v, there exists a positive solution in $[v,w]$ on the half ball in $\mathbb{R}^2$, which is contradicted by Proposition 2.5. □

3. SOME PRELIMINARIES

Lemma 3.1: *Let* $f \in C^1(\mathbb{R})$ *satisfy*

$$f(\rho) = 0 \quad \textit{for some } \rho > 0 \tag{3.1}$$

and

$$\int_u^\rho f(s)\,ds > 0 \quad \textit{for all } u \in [0,\rho). \tag{3.2}$$

Then for all $\varepsilon > 0$ *there is* $\mu > 0$ *and* $v \in C^2[0,1]$ *such that:*

$$\begin{cases} -\left(v'' + \frac{n-1}{r} v' \right) = \mu f(v), \\ v(0) \in (\rho-\varepsilon,\rho), \\ v'(0) = v(1) = 0, \\ v'(r) < 0 \quad \text{for } r \in (0,1]. \end{cases} \tag{3.3}$$

Proof: For a proof see also [5].
Change f for negative numbers such that

$$f(s) > |f(-s-2)| \quad \text{for } s \leq -1 \tag{3.4}$$

and

$$\int_u^\rho f(s)\,ds > 0 \qquad \text{for all } u < \rho. \tag{3.5}$$

Moreover assume

$$f(s) < 0 \qquad \text{for } s > \rho. \tag{3.6}$$

Take a minimizing sequence $\{u_k\}$, for fixed μ, of

$$I(u,\mu) = \tfrac{1}{2} \int_B |\nabla u|^2 dx - \mu \int_B \int_{-1}^u f(s)\,ds\,dx\ , \tag{3.7}$$

for $u+1 \in W_0^{1,2}(B)$, where B denotes the unit ball in $\mathbb{R}^n$. Since $I(|u_k+1|-1,\mu) < I(u_k,\mu)$ and since $I(.,\mu)$ is sequentially weakly lower semicontinuous and coercive, $I(.,\mu)$ possesses a minimizer $u_\mu \geq -1$ in $W^{1,2}(B)$ with $u_\mu = -1$ on ∂B. Regularity theory, see [8], shows that $u_\mu \in C^2(\overline{B})$. By [7] one finds that u_μ is radially symmetric and $u_\mu'(r) < 0$ for all $r \in (0,1)$. Hence u_μ satisfies the first and fourth condition (except for r=1) in (3.3). By the strong maximum principle one finds $u_\mu(0) < \rho$.

Suppose $u_\mu(0) \leq \rho-\varepsilon$ for all $\mu > 0$. Then define

$$\begin{cases} w_\delta(r) = \rho & \text{for } r < 1-\delta, \\ w_\delta(r) = \delta^{-1}(1-r)(1+\rho) - 1 & \text{for } 1-\delta < r < 1. \end{cases} \tag{3.8}$$

Since $I(u_\mu,\mu) > I(w_\delta,\mu)$ for μ large and δ small if $u_\mu \leq \rho-\varepsilon$, this yields a contradiction. Hence for some μ_1 one finds that $\rho-\varepsilon < u_{\mu_1}(0) < \rho$. Since u_{μ_1} is strictly decreasing for $r > 0$, there is a unique $r_1 \in (0,1)$ with $u_{\mu_1}(r_1) = 0$. Then v and μ defined by

$$v(r) = u_{\mu_1}(r_1 r)\ ,\quad \mu = \mu_1 r_1^2 \tag{3.9}$$

satisfy (3.3). □

Definition 3.2: *Let* Ω *be an open bounded domain in* $\mathbb{R}^n$, *and let* $f \in C(\mathbb{R})$. *We call a function* u *a superfunction* (*subfunction*) *of*

$$-\Delta u = f(u) \qquad \textit{in } \Omega, \tag{3.10}$$

if i) $u \in C(\overline{\Omega})$,

ii) $\int_\Omega (u(-\Delta\varphi) - f(u)\varphi)\,dx \geq (\leq)\ 0$ *for all* $\varphi \in \mathscr{D}^+(\Omega)$,

where $\mathscr{D}^+(\Omega)$ *consists of all nonnegative functions in* $C_0^\infty(\Omega)$.

Definition 3.3: *Let* Ω *be an open bounded domain in* $\mathbb{R}^n$, *let* $f \in C(\mathbb{R})$ *and* $g \in C(\partial\Omega)$. *We call a function* u *a supersolution* (*subsolution*) *of*

$$\begin{cases} -\Delta u = f(u) & \textit{in } \Omega, \\ u = g & \textit{on } \partial\Omega, \end{cases} \tag{3.11}$$

if u *satisfies* i), ii) *and*

iii) $u \geq (\leq)\ g$ *on* $\partial\Omega$.

Lemma 3.4: *Let* u_1 *and* u_2 *be subfunctions of* (3.10) *on an open bounded domain* Ω, *with* f *only continuous. Then* u^* *defined by*

$$u^*(x) = \max(u_1(x), u_2(x)) \quad \text{for } x \in \bar{\Omega}, \tag{3.12}$$

is a subfunction of (3.10) *on* Ω.

Remark : A related result can be found in [4]. However, there the definition of weak subfunction is different.

Corollary 3.5: *Let* v_i *be a subfunction of* (3.10) *on* $\Omega = \Omega_i$, *where* i *is* 1 *or* 2. *Define* v *by*

$$\begin{cases} v(x) = v_1(x) & \text{for } x \in \bar{\Omega}_1 \setminus \Omega_2, \\ v(x) = \max(v_1(x), v_2(x)) & \text{for } x \in \Omega_1 \cap \Omega_2, \\ v(x) = v_2(x) & \text{for } x \in \Omega_2 \setminus \bar{\Omega}_1. \end{cases} \tag{3.13}$$

If

$$\begin{cases} v_1 < v_2 & \text{on } \partial\Omega_1 \cap \Omega_2, \\ v_2 < v_1 & \text{on } \partial\Omega_2 \cap \Omega_1, \end{cases} \tag{3.14}$$

then v *is a subfunction of* (3.10) *on* $\Omega = \Omega_1 \cup \Omega_2$.

Remark 1. Let v_i be a subsolution of (3.11) on Ω_i with $g = g_i$, where i=1 or 2. If v_1, v_2 and v are like in Corollary 3.5, then v is a subsolution on $\Omega_1 \cup \Omega_2$ for every g with $g \geq g_1$ on $\partial\Omega_1 \setminus \Omega_2$ and $g \geq g_2$ on $\partial\Omega_2 \setminus \Omega_1$.

Remark 2. Let $\{ u_i ; i=1,...,k \}$ be a family of subfunctions on Ω. Then one finds that the maximum of these subfunctions is again a subfunction.

Remark 3. Similar results hold for superfunctions and supersolutions if one replaces maximum by minimum and reverses the inequality signs.

Proof of the Corollary : By construction i) in Definition 3.2 is immediate; ii) remains to be proved. Let $\varphi \in \mathscr{D}^+(\Omega_1 \cup \Omega_2)$. Because of (3.14) and the continuity of v_1 and v_2, it is possible to find φ_1, φ_2 and φ_3 in $\mathscr{D}^+(\Omega_1 \cup \Omega_2)$ such that $\varphi = \varphi_1 + \varphi_2 + \varphi_3$, $v = v_i$ on support(φ_i) for i=1, 2, and support(φ_3) $\subset \Omega_1 \cap \Omega_2$. Hence it is sufficient to prove ii) of Definition 3.2 for all $\varphi \in \mathscr{D}^+(\Omega_1 \cap \Omega_2)$. This follows from Lemma 3.4. □

Proof of Lemma 3.4.:

Condition i) from Definition 3.10 is immediately satisfied.

We will show condition ii) by the Kato–inequality (see [12, Lemma A]):

$$-\int_\Omega |w| \Delta\varphi \, dx \leq -\int_\Omega \text{sign}(w) \, \Delta w \, \varphi \, dx \quad \text{for } w \in C^2(\Omega), \ \varphi \in \mathscr{D}^+(\Omega). \tag{3.15}$$

For $u_1, u_2 \in C^2(\Omega)$ the result directly follows:

$$\begin{aligned} -\int_\Omega u^* \Delta\varphi \, dx &= -\tfrac{1}{2}\int_\Omega (u_1 + u_2 + |u_1 - u_2|) \, \Delta\varphi \, dx \leq \\ &\leq -\tfrac{1}{2}\int_\Omega \big(\Delta u_1 + \Delta u_2 + \text{sign}(u_1 - u_2)(\Delta u_1 - \Delta u_2)\big)\varphi \, dx \leq \\ &\leq \int_\Omega \Big(\chi_{[u_1>u_2]} f(u_1) + \chi_{[u_1<u_2]} f(u_2) + \tfrac{1}{2}\chi_{[u_1=u_2]}(f(u_1)+f(u_2)) \Big) \varphi \, dx = \\ &= \int_\Omega f(u^*) \, \varphi \, dx, \qquad \text{for } \varphi \in \mathscr{D}^+(\Omega). \end{aligned} \tag{3.16}$$

For $u_1, u_2 \in C(\bar{\Omega})$ we will use the mollifier J_ε defined in [9, p 147], that is, with:

$$J(x) = \begin{cases} \exp\left((|x|-1)^{-1}\right) & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1, \end{cases} \tag{3.17}$$

$$J_\varepsilon(x) = \left(\int_{\mathbb{R}^n} J\left(\tfrac{y}{\varepsilon}\right) dy \right)^{-1} J\left(\tfrac{x}{\varepsilon}\right). \tag{3.18}$$

For $v \in C(\bar{\Omega})$ define $J_\varepsilon * v \in C_0^\infty(\mathbb{R}^n)$ by

$$(J_\varepsilon * v)(x) = \int_\Omega J_\varepsilon(x-y) \, v(y) \, dy, \tag{3.19}$$

and set $u_1^\varepsilon = J_\varepsilon * u_1$, $u_2^\varepsilon = J_\varepsilon * u_2$.

Let v be a subsolution and let $\delta > 0$.

Moreover, for sake of convenience, define

$$\Omega(\delta) = \{x \in \Omega; \, d(x, \partial\Omega) > \delta \}. \tag{3.20}$$

Then for all $\varepsilon < \delta$ we have

$$-\Delta (J_\varepsilon * v) - J_\varepsilon * f(v) \leq 0 \qquad \text{in } \Omega(\delta). \tag{3.21}$$

Indeed, we find for $\varphi \in \mathscr{D}^+(\Omega(\delta))$ that:

$$\begin{aligned} 0 \;&\geq\; \int_\Omega \Big(v.-\Delta(J_\varepsilon * \varphi) - f(v)\,(J_\varepsilon * \varphi) \Big) dx = \\ &= \int_{\Omega(\delta)} \Big(-\Delta(J_\varepsilon * v) - J_\varepsilon * f(v) \Big) \varphi \, dx. \end{aligned} \tag{3.22}$$

Hence, if $\varepsilon < \delta$, (3.21) shows that:

$$\begin{cases} -\Delta u_1^\varepsilon \leq J_\varepsilon * f(u_1) & \text{in } \Omega(\delta), \\ -\Delta u_2^\varepsilon \leq J_\varepsilon * f(u_2) & \text{in } \Omega(\delta). \end{cases} \tag{3.23}$$

Similarly to (3.16) we find

(3.24) $$-\int_\Omega \max(u_1^\varepsilon, u_2^\varepsilon)\,\Delta\varphi\,dx \le$$
$$\le \int_\Omega \Big(\chi_{[u_1^\varepsilon > u_2^\varepsilon]}\, J_\varepsilon * f(u_1) + \chi_{[u_1^\varepsilon < u_2^\varepsilon]}\, J_\varepsilon * f(u_2) +$$
$$+ \tfrac{1}{2}\chi_{[u_1^\varepsilon = u_2^\varepsilon]}(J_\varepsilon * f(u_1) + J_\varepsilon * f(u_2)) \Big)\,\varphi\,dx.$$

Since u_1 and u_2 are continuous $\max(u_1^\varepsilon, u_2^\varepsilon) \to \max(u_1, u_2) = u^*$ and $J_\varepsilon * f(u_i) \to f(u_i)$ (i=0,1) uniformly on supp(φ) for $\varepsilon \downarrow 0$. Moreover, the first term in the right hand side of (3.24) can be estimated as follows.

(3.25) $$\int_\Omega |\chi_{[u_1^\varepsilon > u_2^\varepsilon]}\,(J_\varepsilon * f(u_1) - f(u^*))\,\varphi|\,dx \le$$

$$\le \int_\Omega \chi_{[u_1^\varepsilon > u_2^\varepsilon]}\,|J_\varepsilon * f(u_1) - f(u_1)|\,\varphi\,dx + \int_\Omega \chi_{[u_1^\varepsilon > u_2^\varepsilon]}\,|f(u_1) - f(u^*)|\,\varphi\,dx \le$$

$$\le \|J_\varepsilon * f(u_1) - f(u_1)\|_{L_\infty(\Omega(\delta))} \int_\Omega \varphi\,dx + \|f(u_2)\|_\infty \int_\Omega \chi_{[u_1^\varepsilon > u_2^\varepsilon]}\chi_{[u_1 < u_2]}\,\varphi\,dx.$$

By using the continuity of $f(u_1)$ on Ω for the first term and the Lebesgue Dominated Convergence Theorem for the second term we see that the right hand side in (3.25) goes to zero for $\varepsilon \downarrow 0$. The two remaining terms in (3.24) can be estimated similarly.

Hence

(3.26) $$-\int_\Omega u^*\,\Delta\varphi\,dx \le \int_\Omega f(u^*)\,\varphi\,dx \qquad \text{for all } \varphi \in \mathscr{D}^+(\Omega(\delta)).$$

Since (3.26) is true for every $\delta > 0$, the inequality holds for all $\varphi \in \mathscr{D}^+(\Omega)$. □

Lemma 3.6: *Let* $f \in C^1(\mathbb{R})$, Ω *be bounded and* $\partial\Omega \in C^3$ *and set* g=0. *If* u_1, *respectively* u_2, *with* $u_1 < u_2$ *in* Ω, *are respectively* a *sub– and a supersolution of* (3.11) *with* $u_1 < 0 < u_2$ *on* $\partial\Omega$, *then there exists a stable solution* $u \in [u_1,u_2] \subset C(\overline{\Omega})$ *of* (3.11).

Proof: In order to get sub and supersolutions in $C^2(\overline{\Omega})$, we will use the first two steps in a monotone iteration scheme.

Set $\varepsilon = \min\{ -u_1(x), u_2(x)\ ;\ x \in \partial\Omega \}$ and define

(3.27) $$\omega = \max\{ f'(u)\ ;\ \min u_1(x) \le u \le \max u_2(x) \},$$

and the operator $T_\sigma: C(\overline{\Omega}) \to C(\overline{\Omega})$ by

(3.28) $$T_\sigma(u) = (-\Delta + \omega)^{-1}_{\sigma\varepsilon}(\omega u + \lambda f(u)),$$

where $(-\Delta + \omega)^{-1}_{\sigma\varepsilon}$ is the inverse of $-\Delta + \omega$ with Dirichlet boundary condition $u = \sigma\varepsilon$, $\sigma \in \{-,+\}$. The operators T_σ are order preserving. Moreover, if v is a subsolution of (3.11) with $g = -\varepsilon$, then $T_-(v) \ge v$ and $T_-(v)$ is also a subsolution (see e.g. [16] or [7]). By regularity theory (see [8]) $T_-^2(u_1), T_+^2(u_2) \in C^2(\overline{\Omega})$. Since $T_-^2(u_1) < T_+^2(u_2)$ in $\overline{\Omega}$, are respectively a sub and a supersolution, one can use [16, Th.3.6]. Sattinger showed that the unique solution U_1 of

(3.29) $$\begin{cases} U_t - \Delta U = f(U) & \text{in } \Omega \times \mathbb{R}^+, \\ U = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ U(0) = \Phi & \text{on } \Omega, \end{cases}$$

with $\Phi = T_-^2(u_1)$, satisfies $U_1(x,t) \uparrow v_1(x)$ for $t \to \infty$, and v_1 is a solution of (3.11) with $g = 0$. Similarly the unique solution U_2 of (3.29) with $\Phi = T_+^2(u_2)$ satisfies $U_2(x,t) \downarrow v_2(x)$ for $t \to \infty$, and $v_2 \ge v_1$ is also a solution of (3.11) with $g = 0$. By the maximum principle for elliptic problems, [15, Th.2.6], one finds:

(3.30) $$u_1 \le T_-^2(u_1) < v_1 \le v_2 < T_+^2(u_2) \le u_2 \quad \text{in } \overline{\Omega},$$

By the maximum principle for parabolic problems, [15, Th.3.12], every solution U of (3.30) with $T_-^2(u_1) \le \Phi \le v_1$ converges to v_1 for $t \to \infty$. Hence v_1 is stable from below. Similarly v_2 is stable from above. By [13, Th.4.3] one finds that there is at least one stable solution $u \in [v_1,v_2] \subset [u_1,u_2] \subset C(\overline{\Omega})$. □

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