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ON THE EXISTENCE OF A MAXIMAL WEAK SOLUTION FOR A SEMILINEAR ELLIPTIC EQUATION

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0. Introduction. Consider the semilinear problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$. It is well known, see e.g. [16], that for $f \in C^1(\overline{\Omega} \times \mathbb{R})$ and hence solutions in $C^{2+\vartheta}(\Omega)$, there is a solution in between a sub- and a supersolution. (The supersolution has to lie above the subsolution) There the super- and subsolutions are assumed to be in $C^2(\Omega)$. Similar results for sub- and supersolutions in $W^{2,p}(\Omega)$ are shown in [5, 6].

A first place where a weaker supersolution is used is [13]. Deuel and Hess established existence of a solution between weaker sub- and supersolutions in [10].

Amann showed in [3, 4] for the classical case ($u \in C^2(\Omega) \cap C^\vartheta(\overline{\Omega})$) in fact the existence of a minimal and a maximal solution between a sub- and a supersolution in $C^{2+\vartheta}(\overline{\Omega})$.

The classical proofs can be extended to functions f which are Lipschitz. In this note we will show that the result is still true even if f is not Lipschitz. In section 1 we will use super (sub) solutions in $C(\overline{\Omega})$. In section 2 we will use super (sub) solutions in $W^{1,2}(\Omega)$ and allow general bounded domains. Neither definition of super (sub) solution is included in the other even for regular domains, though a $C_0(\overline{\Omega})$-solution is necessarily a $W_0^{1,2}(\Omega)$-solution. Thus neither of our two main results is included in the other.

1. A maximal solution in $C(\overline{\Omega})$. In this section we consider (0.1) for functions f in $C(\overline{\Omega} \times \mathbb{R})$. Moreover we assume that every boundary point is regular with respect to the Laplacian. A boundary point is regular if there exists a barrier-function at that point. For a definition see [12, p. 25]. In this section we are interested in solutions u in $C(\overline{\Omega})$.

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Definition 1.1. *The function u is called a C-solution of (0.1) if:*

1) $u \in C(\overline{\Omega})$,
2) $\int_\Omega (u. -\Delta\varphi - f(x,u)\varphi)\,dx = 0$ *for all* $\varphi \in C_0^\infty(\Omega)$,
3) $u = 0$ *on* $\partial\Omega$.

In this section the following definition of weak super (sub) solution will be used.

Definition 1.2. *The function u is called a super (sub) solution if:*

1) $u \in C(\overline{\Omega})$,
2) $\int_\Omega (u. -\Delta\varphi - f(x,u)\varphi)\,dx \geq (\leq)\, 0$ *for all* $\varphi \in \mathcal{D}^+(\Omega) = \{\varphi \in C_0^\infty(\Omega);\ \varphi \geq 0\}$,
3) $u \geq (\leq)\, 0$ *on* $\partial\Omega$.

This definition is previously used in [7].

Theorem 1.3. *Assume*

$$f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \text{ is continuous,} \tag{1.1}$$

$$\Omega \text{ is a bounded domain of } \mathbb{R}^n \text{ and every boundary point is regular.} \tag{1.2}$$

Let v_1 respectively v_2 be a sub- respectively a supersolution of (0.1), satisfying $v_1 \leq v_2$ in $\overline{\Omega}$. Then there exists a minimal C-solution u_1 and a maximal C-solution u_2 such that, for every C-solution u with $v_1 \leq u \leq v_2$ we have

$$v_1 \leq u_1 \leq u \leq u_2 \leq v_2 \quad \text{in } \overline{\Omega}. \tag{1.3}$$

In the proof we will approximate a subsolution by smooth functions. We cannot hope that these approximations themselves are subsolutions. However, a less strong result is sufficient.

Let J be the mollifier defined in [12, p. 147].

$$J(x) = \begin{cases} \exp((|x|-1)^{-1}) & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1, \end{cases} \tag{1.4}$$

and set

$$J_\epsilon(x) = \Big(\int_{\mathbb{R}^n} J(\frac{y}{\epsilon})\,dy\Big)^{-1} J(\frac{x}{\epsilon}). \tag{1.5}$$

For $v \in C(\overline{\Omega})$ define $J_\epsilon * v \in C_0^\infty(\mathbb{R}^n)$ by

$$(J_\epsilon * v)(x) = \int_\Omega J_\epsilon(x-y)v(y)\,dy. \tag{1.6}$$

Moreover, for sake of convenience, define for $\delta > 0$

$$\Omega(\delta) = \{x \in \Omega;\ d(x, \partial\Omega) > \delta\}. \tag{1.7}$$

Lemma 1.4. *Let v be a subsolution of (0.1) and let $\delta > 0$. Then for all $\epsilon < \delta$ we have*

$$-\Delta(J_\epsilon * v) - (J_\epsilon * f(\cdot, v)) \leq 0 \quad \text{in } \Omega(\delta). \tag{1.8}$$

Moreover $\lim_{\epsilon\downarrow 0} J_\epsilon * f(\cdot, v) = f(\cdot, v)$ *uniformly in* $\Omega(\delta)$.

Proof: Since $f(\cdot, v) \in C(\overline{\Omega})$ the second statement follows immediately.

To prove (1.8) let $\varphi \in \mathcal{D}^+(\Omega(\delta))$. Hence $J_\epsilon * \varphi \in \mathcal{D}^+(\Omega)$ if $\epsilon < \delta$. Since v is a subsolution we find

$$\begin{aligned} 0 &\geq \int_\Omega (v. - \Delta(J_\epsilon * \varphi) - f(\cdot, v)(J_\epsilon * \varphi))\,dx \\ &= \int_\Omega (v(J_\epsilon * -\Delta\varphi) - f(\cdot, v)(J_\epsilon * \varphi))\,dx \\ &= \int_{\mathbb{R}^n} ((J_\epsilon * v). - \Delta\varphi - (J_\epsilon * f(\cdot, v))\varphi)\,dx \\ &= \int_{\Omega(\delta)} (-\Delta(J_\epsilon * v) - (J_\epsilon * f(\cdot, v)))\varphi\,dx. \end{aligned} \tag{1.9}$$

Since $-\Delta(J_\epsilon * v) - (J_\epsilon * f(\cdot, v))$ is continuous and since (1.9) is valid for all $\varphi \in \mathcal{D}^+(\Omega(\delta))$ we find (1.8).

Proof of Theorem 1.3: The existence of a solution is shown in [8] using similar arguments as in [2]. For the sake of completeness we repeat the proof.

Step 1: *Existence of a solution.* First we will modify f. Define

$$f^*(x, u) = \begin{cases} f(x, v_1(x)) & \text{if } u < v_1(x), \\ f(x, u) & \text{if } v_1(x) \leq u \leq v_2(x), \\ f(x, v_2(x)) & \text{if } v_2(x) < u, \text{ and } x \in \overline{\Omega}. \end{cases} \tag{1.10}$$

Since $f \in C(\overline{\Omega} \times \mathbb{R})$ and $v_1, v_2 \in C(\overline{\Omega})$ we find that $f^* \in C(\overline{\Omega} \times \mathbb{R})$ and even that f^* is bounded. By Schauder's Theorem we will show the existence of a C-solution of (0.1) with f replaced by f^*. Let $K : C(\overline{\Omega}) \to C(\overline{\Omega})$ denote the solution operator of

$$\begin{cases} -\Delta u = \phi & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

that is $u = K\phi$. K is a compact linear operator in $C(\overline{\Omega})$, where $C(\overline{\Omega})$ is equipped with the maximum norm. Let $F : C(\overline{\Omega}) \to C(\overline{\Omega})$ denote the Nemytskii operator for f^*, that is

$$F(u)(x) = f^*(x, u(x)) \qquad \text{for } u \in C(\overline{\Omega}),\ x \in \overline{\Omega}. \tag{1.12}$$

Then F is continuous and bounded: there is $M > 0$ such that

$$\|F(u)\|_\infty \leq M \qquad \text{for all } u \in C(\overline{\Omega}). \tag{1.13}$$

By the Schauder Fixed Point Theorem there is $u \in C(\overline{\Omega})$ with

$$u = KF(u). \tag{1.14}$$

This function u is a C-solution of (0.1) with f replaced by f^*.

Finally we will show that $v_1 \leq u \leq v_2$ in $\overline{\Omega}$. This implies u is a C-solution of (0.1) for the original f. Suppose u is a solution of (0.1) with f^* and set $\Omega^+ = \{x \in \Omega;\ v_2(x) < u(x)\}$. We will show that Ω^+ is empty. Suppose not. Since u and v_2 are continuous, Ω^+ is open. Moreover we have:

$$\int_{\Omega^+} (u - v_2)(-\Delta\varphi)\,dx \leq \int_{\Omega^+} (f^*(x, u(x)) - f(x, v_2(x)))\varphi\,dx = 0. \tag{1.15}$$

for every $\varphi \in \mathcal{D}^+(\Omega^+)$. Then $u - v_2$ in $C(\overline{\Omega^+})$ is subharmonic and nonnegative in Ω^+. Since subharmonic function on Ω^+ achieve its maximum on the boundary $\partial\Omega^+$ (see [12]), $u = v_2$ in Ω^+, a contradiction. Similarly, one proves $v_1 \leq u$ in $\overline{\Omega}$.

Step 2: *The Zorn Lemma.* Let P denote the collection of all C-solutions u of (0.1) with $v_1 \leq u \leq v_2$ in $\overline{\Omega}$. Let $\{u_i\}_{i\in I}$ be a totally ordered subset of P.

First we will show that $\{u_i\}_{i\in I}$ is an equicontinuous family. Since $F(u_i) \in L^p(\Omega)$ (for all $p \in (1,\infty)$) we find that $u_i \in W^{2,p}_{\text{loc}}(\Omega)$ (see [12, Th. 9.9]). By the Sobolev Imbedding Theorem (see [1, Th. 5.4]) we find that $u_i \in C^1(\Omega)$. Since both u_i and $F(u_i)$ are bounded uniformly with respect to i, it is even true that $\{u_i\}_{i\in I}$ is equicontinuous on every compact subset of Ω.

To show the equicontinuity near the boundary $\partial\Omega$, we use a weak version of the maximum principle. Set

$$M = \max\left\{|f(x,u)|;\ \min v_1 < u < \max v_2,\ x \in \overline{\Omega}\right\} \tag{1.16}$$

and let $u^* \in C_0(\overline{\Omega}) \cap C^2(\Omega)$ be the function which satisfies

$$\begin{cases} -\Delta u^* = M & \text{in } \Omega, \\ \quad u^* = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.17}$$

Since $\partial\Omega$ is regular, u^* exists, see [12, Th. 2.14]. Moreover, $u^* - u_i$ and $u^* + u_i$ are superharmonic in Ω and zero at the boundary. Hence similarly to step 1, we have for all $i \in I$ that:

$$-u^* \leq u_i \leq u^* \quad \text{in } \overline{\Omega}, \tag{1.18}$$

and hence $\{u_i\}_{i\in I}$ is equicontinuous on $\overline{\Omega}$.

We will show that $u(x) = \sup_{i\in I} u_i(x)$ is a C-solution. From the equicontinuity of the u_i it follows that $u \in C_0(\overline{\Omega})$. Moreover, because of the equicontinuity and the total ordering there is a sequence $\{u_n\}_{n\in\mathbb{N}}$ in this family, with

$$v_1 \leq u_1 \leq u_2 \leq u_3 \leq \cdots \leq v_2 \quad \text{in } \overline{\Omega}, \tag{1.19}$$

and

$$u(x) = \lim_{n\to\infty} u_n(x). \tag{1.20}$$

Then condition 2) of Definition 1.1 is a consequence of the Lebesgue Dominated Convergence Theorem.

Step 3:

Lemma 1.5. *The maximum of two subsolutions is a subsolution. (for subsolutions as in Definition 1.2)*

For the proof of Theorem 1.3 it is sufficient to show that the maximum of two C-solutions is a subsolution. Since the result is interesting in itself we will prove this slightly stronger lemma.

Suppose both v_0 and v_s are subsolutions. We will show that v^* defined by

$$v^*(x) = \max(v_0(x), v_s(x)) \tag{1.21}$$

is also a subsolution of (0.1).

The conditions 1) and 3) from Definition 1.2 are immediately satisfied. We will show condition 2) by the Kato-inequality (see [14]):

$$-\int_\Omega |w|\Delta\varphi\,dx \le -\int_\Omega \operatorname{sign}(w)\Delta w\,\varphi\,dx \quad \text{for } w\in C^2(\Omega),\ \varphi\in\mathcal{D}^+(\Omega). \tag{1.22}$$

For $v_0, v_s \in C^2(\Omega)$ the result directly follows:

$$\begin{aligned}
-\int_\Omega v^*\Delta\varphi\,dx &= -\frac{1}{2}\int_\Omega (v_0+v_s+|v_0-v_s|)\Delta\varphi\,dx \\
&\le -\frac{1}{2}\int_\Omega (\Delta v_0+\Delta v_s+\operatorname{sign}(v_0-v_s)(\Delta v_0-\Delta v_s))\varphi\,dx \\
&\le \int_\Omega (\chi_{[v_0>v_s]}F(v_0)+\chi_{[v_0<v_s]}F(v_s)+\frac{1}{2}\chi_{[v_0=v_s]}(F(v_0)+F(v_s)))\varphi\,dx \\
&= \int_\Omega F(v^*)\varphi\,dx, \quad \text{for } \varphi\in\mathcal{D}^+(\Omega).
\end{aligned} \tag{1.23}$$

For $v_0, v_s \in C(\overline{\Omega})$ we will use the mollifier J_ϵ defined in (1.5) and define $v_{0,\epsilon} = J_\epsilon * v_0$, $v_{s,\epsilon} = J_\epsilon * v_s$.

Fix $\varphi \in \mathcal{D}^+(\Omega)$ and set $\delta = d(\partial\Omega, \operatorname{supp}(\varphi))$. If $\epsilon < \delta$ we may use Lemma 1.4 to show that:

$$\begin{cases} -\Delta v_{0,\epsilon} \le J_\epsilon * F(v_0) & \text{in } \operatorname{supp}(\varphi), \\ -\Delta v_{s,\epsilon} \le J_\epsilon * F(v_s) & \text{in } \operatorname{supp}(\varphi). \end{cases} \tag{1.24}$$

Similarly to (1.23) we find

$$\begin{aligned}
-\int_\Omega \max(v_{0,\epsilon}, v_{s,\epsilon})\Delta\varphi\,dx \le \int_\Omega (&\chi_{[v_{0,\epsilon}>v_{s,\epsilon}]}J_\epsilon * F(v_0)+\chi_{[v_{0,\epsilon}<v_{s,\epsilon}]}J_\epsilon * F(v_s) \\
&+\frac{1}{2}\chi_{[v_{0,\epsilon}=v_{s,\epsilon}]}(J_\epsilon * F(v_0)+J_\epsilon F(v_s)))\varphi\,dx
\end{aligned} \tag{1.25}$$

Since v_0, v_s are continuous $\max(v_{0,\epsilon}, v_{s,\epsilon}) \to \max(v_0, v_s)$ and $J_\epsilon * F(v_i) \to F(v_i)$ $(i = 0,1)$ uniformly on $\operatorname{supp}(\varphi)$ for $\epsilon \downarrow 0$. Moreover, the first term in the right hand side of (1.25) can be estimated as follows.

$$\begin{aligned}
&\int_\Omega |\chi_{[v_{0,\epsilon}>v_{s,\epsilon}]}(J_\epsilon * F(v_0)-F(v^*))\varphi|\,dx \\
&\le \int_\Omega \chi_{[v_{0,\epsilon}>v_{s,\epsilon}]}|J_\epsilon * F(v_0)-F(v_0)|\varphi\,dx+\int_\Omega \chi_{[v_{0,\epsilon}>v_{s,\epsilon}]}|F(v_0)-F(v^*)|\varphi\,dx \\
&\le \|J_\epsilon * F(v_0)-F(v_0)\|_{L^\infty(\operatorname{supp}\varphi)}\int_\Omega \varphi\,dx+\|F(v_s)\|_\infty\int_\Omega \chi_{[v_{0,\epsilon}>v_{s,\epsilon}]}\chi_{[v_0<v_s]}\varphi\,dx.
\end{aligned} \tag{1.26}$$

By using the continuity of $F(v_0)$ on Ω for the first term and the Lebesgue Dominated Convergence Theorem for the second term we see that the right hand side in (1.26) goes to zero for $\epsilon \downarrow 0$. The two remaining terms in (1.25) can be estimated similarly. Hence as required

$$-\int_\Omega v^*\Delta\varphi\,dx \le \int_\Omega f(x, v^*)\varphi\,dx. \tag{1.27}$$

Step 4: *Completion of the proof.* By the first step we have the existence of a C-solution $u \in [v_1, v_2]$. Step 2 shows that there is a maximal C-solution in the sense of partially ordering. From the third step we know that if there are two C-solution u_a and u_b, then $\max(u_a, u_b)$ is a subsolution. Hence as a consequence there is a C-solution $u_c \in [\max(u_a, u_b), v_2]$. By this result and step two, there is a unique maximal C-solution u_2 in the sense of partially ordering and hence a maximal solution in the sense of Theorem 1.3. Similarly one shows the existence of the minimal C-solution u_1.

2. A maximal solution in $W^{1,2}(\Omega)$. In this section we prove a variant of Theorem 1.3 for general bounded domains Ω in $\mathbb{R}^n$ but for solutions in $W^{1,2}(\Omega)$. Thus we consider the equation (0.1), where now by a solution we mean an element of $W_0^{1,2}(\Omega)$, with $f(\cdot, u) \in L^1(\Omega)$, satisfying the equation in the weak sense as in Definition 1.1. To make the distinction we will call this a W-solution. For sub- and supersolution we use a slightly different definition than in section 1.

Definition 2.1. *The function u is called a super (sub) solution if*

1) $u \in W^{1,2}(\Omega)$,
2) $\int_\Omega (u. - \Delta\varphi - f(x,u)\varphi)\, dx \geq (\leq)\, 0$ *for all* $\varphi \in \mathcal{D}^+(\Omega) = \{\varphi \in C_0^\infty(\Omega); \varphi \geq 0\}$,
3) $u \geq (\leq)\, 0$ *on* $\partial\Omega$ *in the sense of Kinderlehrer and Stampacchia [15, p. 35].*

In [15] the function u in $W^{1,2}(\Omega)$, is called nonnegative on $E \subset \overline{\Omega}$, if there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $W^{1,\infty}(\Omega)$ with $u_n \to u$ in $W^{1,2}(\Omega)$, such that:

$$u_n(x) \geq 0 \quad \text{for } x \in E. \tag{2.1}$$

Theorem 2.2. *Assume that Ω is a bounded domain in $\mathbb{R}^n$ and that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory condition. Let v_1, respectively v_2, be a subsolution, respectively a supersolution of (0.1), with*

$$v_1(x) \leq v_2(x) \quad \text{in } \Omega, \tag{2.2}$$

and

$$\sup\{|f(x,v)| \mid v_1(x) \leq v \leq v_2(x)\} \in L^p(\Omega) \text{ with } p > \frac{2n}{n+2} \ (p > 1 \text{ if } n = 1). \tag{2.3}$$

Then there exists a minimal W-solution u_1 and a maximal W-solution u_2 such that, for every W-solution u with $v_1 \leq u \leq v_2$ we have

$$v_1 \leq u_1 \leq u \leq u_2 \leq v_2 \quad \text{in } \Omega. \tag{2.4}$$

Remark: A function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisties the Caratheodory condition if $u \to f(x,u)$ is continuous for almost all x in Ω, and $x \to f(x,u)$ is measurable for all u in $\mathbb{R}$.

Proof of Theorem 2.2: The proof is very similar to the proof of Theorem 1.3.

Step 1. *The existence.* That there is a solution between a subsolution and a supersolution follows from Deuel and Hess [10]. There is only one place where their proof needs to be changed. To show that $u \leq v_2$ on Ω, we need to know that, if $u \in W_0^{1,2}(\Omega)$ and $v_2 \geq 0$ on $\partial\Omega$ in the sense of [15] (see (2.1)), then $(u - v_2)^+ \in W_0^{1,2}(\Omega)$. To see this, note that it follows from [15] that there exist functions w_n in $W^{1,\infty}(\Omega)$ such that $w_n \geq 0$ on $\partial\Omega$ and $w_n \to v_2$ in $W^{1,2}(\Omega)$. By replacing w_n by $w_n + \frac{1}{n}$, we can assume $w_n > 0$ near $\partial\Omega$. Moreover, since

$u \in W_0^{1,2}(\Omega)$ there exist smooth functions u_n of compact support in Ω such that $u_n \to u$ in $W^{1,2}(\Omega)$ as $n \to \infty$. Then $(u_n - w_n)^+$ is of compact support in Ω. Hence for ϵ small enough $J_\epsilon * (u_n - w_n)^+ \in C_0^\infty(\Omega)$. Since $J_\epsilon * (u_n - w_n)^+ \to (u_n - w_n)^+$ in $W^{1,2}(\Omega)$ for $\epsilon \downarrow 0$ and $(u_n - w_n)^+ \to (u - v_2)^+$ in $W^{1,2}(\Omega)$ as $n \to \infty$, $(u - v_2)^+ \in W_0^{1,2}(\Omega)$. Since $L^p(\Omega) \subset W^{-1,2}(\Omega)$, our assumptions imply those in [10].

Step 2. *Zorn's Lemma.* The proof of [10] shows that the W-solutions u of (0.1) with $v_1 \le u \le v_2$ on Ω are bounded in $W_0^{1,2}(\Omega)$. Thus if $\{u_i\}_{i \in I}$ is an ordered family of W-solutions with $v_1 \le u_i \le v_2$ there is a sequence $\{u_n\}_{n \in \mathbb{N}}$ in this family, with

$$v_1 \le u_1 \le u_2 \le u_3 \le \cdots \le v_2 \quad \text{in } \overline{\Omega}, \tag{2.5}$$

and

$$u_n \to u \text{ weakly in } W^{1,2}(\Omega) \text{ for } n \to \infty. \tag{2.6}$$

Since $u_n, u \in W_0^{1,2}(\Omega)$, the Sobolev Imbedding Theorem ([1, Th. 6.2, part IV]) shows that there is a subsequence converging strongly to u in $L^2(\Omega)$. We will omit the change of notation necessitated by passing to subsequences. Since a further subsequence must converge pointwise almost everywhere (see [1, Corol. 1.2.11]) and since $\{u_n\}_{n \in \mathbb{N}}$ is increasing,

$$v_1 \le u_n \le u \le v_2 \quad \text{in } \Omega \text{ a.e., for all } n. \tag{2.7}$$

Moreover, since $u_n \to u$ weakly in $W_0^{1,2}(\Omega)$ and pointwise, the Dominated Convergence Theorem shows that u is a W-solution of (0.1). A Zorn's lemma argument implies that the set of solutions between v_1 and v_2 has a maximal element in the sense of the ordering.

Step 3. *The maximum of two solutions is a subsolution.* Let v_3 and v_4 be W-solutions. We will show that $\sup(v_3, v_4) \in W_0^{1,2}(\Omega)$ is a subsolution. If $v \in W_0^{1,2}(\Omega)$ is a W-solution of (0.1) with $v_1 \le v \le v_2$, $f(\cdot, v) \in L^p(\Omega)$ and hence by standard regularity results $v \in W^{2,p}_{\text{loc}}(\Omega)$. Now Kato's proof shows that (1.22) holds for $w \in W^{2,1}_{\text{loc}}(\Omega)$. Hence we can prove that $\sup(v_3, v_4)$ is a subsolution by a similar argument to that in step 3 of the proof of section 1. Indeed we do not need mollifiers.

We can complete the proof of Theorem 2.1 by the same argument as in section 1.

Remark 1. The methods can be generalized to allow f to depend on ∇u, provided that

$$|f(x, u, s)| \le k(x) + K|s| \quad \text{for } v_1(x) \le u \le v_2(x), \text{ where } k \in L^p(\Omega) \ (p \text{ as before}). \tag{2.8}$$

The proof needs only minor modifications except in the analogue of step 2. Here, if u_n is an increasing sequence of solutions with $v_1 \le u_n \le v_2$ for all n, we prove as before that $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $W^{1,2}(\Omega)$. We then deduce from the equation that, if $K \subset \Omega$ is compact, $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $W^{2,p}(K)$ and hence converges strongly in $W^{1,2}_{\text{loc}}(\Omega)$. The rest of the proof of step 2 is much as before. If $v_1, v_2 \in W^{1,\infty}(\Omega)$ we can allow rather better growth rates. The idea here, as in [5], is to choose a smooth map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ with bounded image and

$$\Phi(y) = y \quad \text{if } |y| \le \max\{\|\nabla v_1\|_\infty, \|\nabla v_2\|_\infty\}, \tag{2.9}$$

and apply the above argument to the equation with $f(x, u, \nabla u)$ replaced by $f(x, u, \Phi(\nabla u))$. One then proves an estimate for solutions between v_1 and v_2 which shows that any solution of the new equation with a suitable Φ is a solution of the original equation. This depends on a $W^{1,\infty}$-estimate. For example, if Ω has a C^2-boundary and if

$$|f(x, u, s)| \le K(1 + |s|^2) \quad \text{for } x \in \Omega,\ v_1(x) \le u \le v_2(x), \tag{2.10}$$

one can establish the $W^{1,\infty}$-estimate by using the main estimate in Amann and Crandall [5]. Hence we obtain the result if f has quadratic growth in ∇u. For an arbitrary domain, one can establish our result if

$$|f(x,u,s)| \le K(1+|s|^r) \quad \text{for } x \in \Omega,\ v_1(x) \le u \le v_2(x),\ r < 2. \tag{2.11}$$

The idea here is that, by using the Nirenberg-Gagliardo inequalities (see [11, Th. 10.1]), we can establish $W^{1,\infty}$-estimates on compact subsets of Ω and this suffices to obtain the result. We use a sequence of truncations Φ_n and pass to the limit.

Remark 2. Our result, when f does not depend on ∇u is true for unbounded Ω if we consider solutions and super (sub) solutions which are in $W^{1,2}(\Omega \cap B(0,R))$ for every $R > 0$, where $B(0,R)$ is the ball in $\mathbb{R}^n$ of centre 0 and radius R. We prove the existence by using the existence of solutions on $\Omega \cap B(0,R)$ and passing to the limit as $R \to \infty$. Steps 2 and 3 are established by working on bounded subsets of Ω.

Remark 3. Lastly, one can replace Δu by a more general second order elliptic operator provided that the top order coefficients are in $W^{1,q}_{\text{loc}}(\Omega)$ for $q > n$, and the lower order coefficients are in $L^\infty_{\text{loc}}(\Omega)$. In some cases where f does not depend on ∇u, one can allow a more general linear part by using the technique on page 451 of [9].

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