When is the first eigenfunction for the clamped plate equation of fixed sign?

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1 Introduction

For most 'nice' elliptic boundary value problems there is a general expectation that the first eigenfunction is unique and of fixed sign. And indeed, for second order elliptic differential equations with Dirichlet boundary conditions such a result holds as a consequence of the maximum principle. It is well known that such a maximum principle does not have a direct generalization to higher order elliptic problems. Nevertheless, the hypothesis that the principal eigenfunction for the biharmonic Dirichlet problem is of fixed sign does appear in earlier papers, see for example [32] from 1950. Let us be more precise.

The biharmonic eigenvalue problem with Dirichlet boundary conditions is the following:

$$\begin{cases} \Delta^2\varphi = \lambda\varphi & \text{in } \Omega, \\ \varphi = \frac{\partial}{\partial \boldsymbol{n}}\varphi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$. Closely related to the eigenvalue problem is the biharmonic differential equation with Dirichlet boundary conditions, the so-called clamped plate equation:

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \frac{\partial}{\partial \boldsymbol{n}}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

The famous conjectures for these two problems were as follows; by now both of them have numerous counterexamples.

Conjecture 1 (Szegö, 1950) *If Ω is a 'nice' domain, then the first eigenfunction for (1) is of fixed sign.*

Conjecture 2 (Boggio-Hadamard ±1908) *If Ω is a 'nice' (convex) domain, then (2) is sign-preserving, that is, $f \geq 0$ implies $u \geq 0$.*

Or as Hadamard writes on page 14 of [23]:

> *Malgré l'absence de démonstration rigoureuse, l'exactitude de cette proposition ne paraît pas douteuse pour les aires convexes.*

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Note that a Krein-Rutman type argument shows that a true second conjecture implies the result of the first. In other words, if (2) is sign-preserving for Ω, then the first eigenfunction of (1) on Ω is of fixed sign.

The second problem, (2), forms a model for the clamped plate where f is the load and u the deviation of the plate Ω. Boggio ([4], [5]) and Hadamard ([22], [23]) extensively studied this model.

Both the Boggio-Hadamard conjecture and the Szegö conjecture proved to be wrong. Duffin and others ([12], [15], [28], [7], [9], [25] and [31]), starting in 1949, established convex smooth domains for which the problem in (2) is not sign-preserving. Coffman in [9] proved that the first eigenfunction on a square changes sign. Sign changing first eigenfunctions are also found in [25].

The aim of this present note is to review the relation between the domain and the sign changing of this first eigenfunction. We will do so by considering some families of domains. Isoperimetric questions for the principal eigenfunction of (1) will not be addressed. For those type of results we refer to the papers by Talenti ([36]) and Ashbaugh-Laugesen ([1]). Finally we refer to [27] for eigenfunctions on a number of special domains.

2 Rectangles

The first counterexample to the Boggio-Hadamard conjecture is due to Duffin. In [12] he showed that the Green function changes sign on an infinitely long rectangle. See also [30]. If Ω is a long rectangle a positive function f with small support yields an oscillatary behaviour for the solution u away from that support. A numerical experiment for a rectangle with ratio 3×1 confirms such behaviour. See Figure 1. There $f \geq 0$ is zero except for some area near the short right hand side around $2/3$ from the front side. Note that even this rough approximation also shows a small negative effect in a corner. Such a sign-change near right angles for (2) has been proven by Coffman and Duffin in [7].

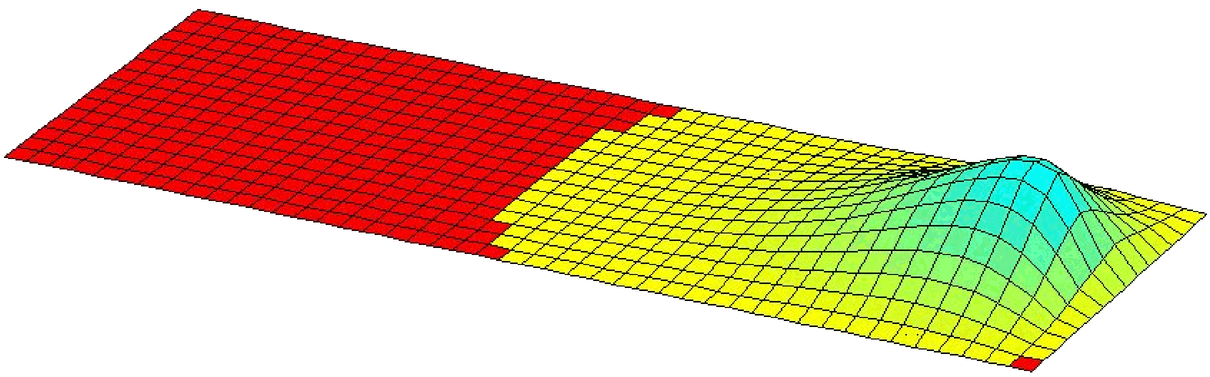

Figure 1: the red area corresponds with $u < 0$

Before Coffman proved in 1984 by analytical means ([9]) that the first eigenfunction on a square changes sign, numerical results in 1972 and 1982 ([2] and [21]) predicted so. Recently, in 1996, these numerical results on the square have been revisited by Wieners ([37]). He could prove that the sign-changing of the numerically approximated first eigenfunction is rigorous, that is, the sign changing effect is to large to be explained by numerical errors.

For the problem in (2) Boggio and Hadamard expected that (assuming smoothness?) convexity was a sufficient condition for the sign-preserving property. Although convexity is not sufficient for (2) one could ask if convexity and smoothness would be sufficient for the Szegö hypothesis. Since the first eigenfunction on a square, convex but nonsmooth, is not of one sign ([9]) there is not much hope for a positive answer. See [25].

The stronger Boggio-Hadamard conjecture is not true on smooth domain sufficiently close to a square. Coffman and Grover in [8, Theorem 8.1] were able to prove a much more general result.

Theorem 3 (Coffman-Grover, 1980) *Let $\{\Omega_i; i \in \mathbb{N}\}$ be a family of domains with $\Omega_i \subset \Omega_{i+1}$ for all $i \in \mathbb{N}$, and $\cup_{i\in\mathbb{N}}\Omega_i = \Omega$. If (2) is not sign preserving on Ω, then it is not sign preserving if Ω is replaced by Ω_i with i large enough.*

A similar result for the first eigenfunction can be found in the paper by Kozlov, Kondrat'ev en Maz'ya ([25, Theorem 1]). They construct a sequence of smooth convex domains that exhaust an appropriately chosen cone. Since the corresponding first eigenfunctions are proven to converge to the sign-changing first eigenfunction on the cone the conclusion follows.

3 Ellipses

Boggio obtained an explicit formula for all polyharmonic Green functions for the Dirichlet problem on a ball in any dimension. For the biharmonic problem on $\Omega = \left\{x \in \mathbb{R}^2; |x| < 1\right\}$ this formula becomes

$$G(x,y) = \frac{1}{8\pi}|x-y|^2 \int_1^{\theta(x,y)} \frac{v^2-1}{v} dv \quad \text{with } \theta(x,y) = \sqrt{1+\frac{\left(1-|x|^2\right)\left(1-|y|^2\right)}{|x-y|^2}}.$$

Since this Green function is positive it proves the sign-preserving property on the disk. From [17] it follows that for domains close, in a smooth sense, to the disk the Green function remains positive. However, for more eccentric ellipses positivity breaks down. Garabedian showed that on an ellipse with ratio of the axes larger then 2, a positive f exist for which u changes sign. Setting

$$E_\ell = \left\{x \in \mathbb{R}^2; (x_1/\ell)^2 + x_2^2 < 1\right\} \tag{3}$$

we summarize:

Theorem 4 (Garabedian 1951 *1.*, Grunau-Sweers 1996 *2.*)

1. *There are large ℓ such that (2) is not positivity preserving for $\Omega = E_\ell$.*
2. *There is $\ell_* > 1$ such that if $\ell \in \left[\ell_*^{-1}, \ell_*\right]$, then (2) is positivity preserving for $\Omega = E_\ell$.*

There are numerical approximations (mentioned in [24]) for the first ℓ where sign change appears. Although it seems likely that there exists a number ℓ_0 such that (2) is positivity preserving for $\Omega = E_\ell$ if and only if $\ell \in \left[\ell_0^{-1}, \ell_0\right]$ the author is not aware of a corresponding theorem.

What about Szegö's conjecture on eccentric ellipses? As far as the author knows there are no rigorous results for the sign of the eigenfunction on eccentric ellipses. For ellipses with small eccentricity one may show that the first eigenfunction remains positive (see for example [18]). Only numerical results seem to support the conjecture that on rather eccentric ellipses the first eigenfunction changes sign. See Figure 4 and 5 in the Appendix. So a conjecture would be the following.

Conjecture 5 *There exists a number $\ell_e \in (1, \infty)$ such that the first eigenfunction is positive for $\Omega = E_\ell$ if and only if $\ell \in \left[\ell_e^{-1}, \ell_e\right]$.*

Even for the weaker result that there exists a number ℓ such that the first eigenfunction changes sign on E_ℓ is open. If ℓ_0 and ℓ_e exist, then by results in [20] it would follow that $\ell_0 < \ell_e$.

Note from Figure 5 that the size of the minimum (the negative part) is near 10^{-5}; the eigenfunction is normalized by $\max \phi = 1$.

4 Elongated disks

By an elongated disk we mean two half disks joined by a rectangle:

$$D_\ell = \bigcup_{-\ell \leq y_1 \leq \ell} B_1 (y_1, 0), \tag{4}$$

where $B_r(y) = \left\{x \in \mathbb{R}^2; |x - y| < r\right\}$. In contrast to eccentric ellipses the author wasn't able to find numerical evidence for a sign-changing first eigenfunction. Let me put it into a conjecture.

Conjecture 6 *Let D_ℓ be as in (4). For every $\ell \geq 0$ the first eigenfunction of (1) on $\Omega = D_\ell$ is of fixed sign.*

For a numerical approximation of the first eigenfunction on an elongated disk see Figure 2.

In [20] it has been shown, roughly spoken, that for an appropriate family of domain perturbations $\ell \to \Omega_\ell$ the statement *'Green function is positive'* breaks down strictly before *'first eigenfunction is positive'* does. For convex domains sign changing seems to be appearing near boundary points where the curvature becomes big. And in fact, next to the eigenvalue, another quantity that remains bounded for D_ℓ and not for E_ℓ is the curvature. A rather bold conjecture in such a direction would be the following.

Conjecture 7 *There is a number $C > 0$ such that the following holds. Suppose that Ω is a convex domain in $\mathbb{R}^2$. Let $\lambda_{1,\Omega}$ be the first eigenvalue of (1) and denote by κ_Ω the maximal curvature of $\partial\Omega$. If $\lambda_{1,\Omega} \geq C\,\kappa_\Omega^4$, then the first eigenfunction is of fixed sign.*

For the family of ellipses E_ℓ in (3) one finds $\kappa_{E_\ell} = \ell^2$. For the elongated disks above $\kappa_{D_\ell} = 1$ for all ℓ and also the first eigenvalue remains bounded. Indeed, one finds that $\lambda_1(D_\ell) \to \lambda_1[-1, 1]$ for $\ell \to \infty$. Here $\lambda_1[-1, 1]$ is the first eigenvalue on the one-dimensional interval. The corresponding eigenfunction is

$$v(y) = \frac{\cos(t\,y)\cosh(t) - \cosh(t\,y)\cos(t)}{\cosh(t) - \cos(t)}$$

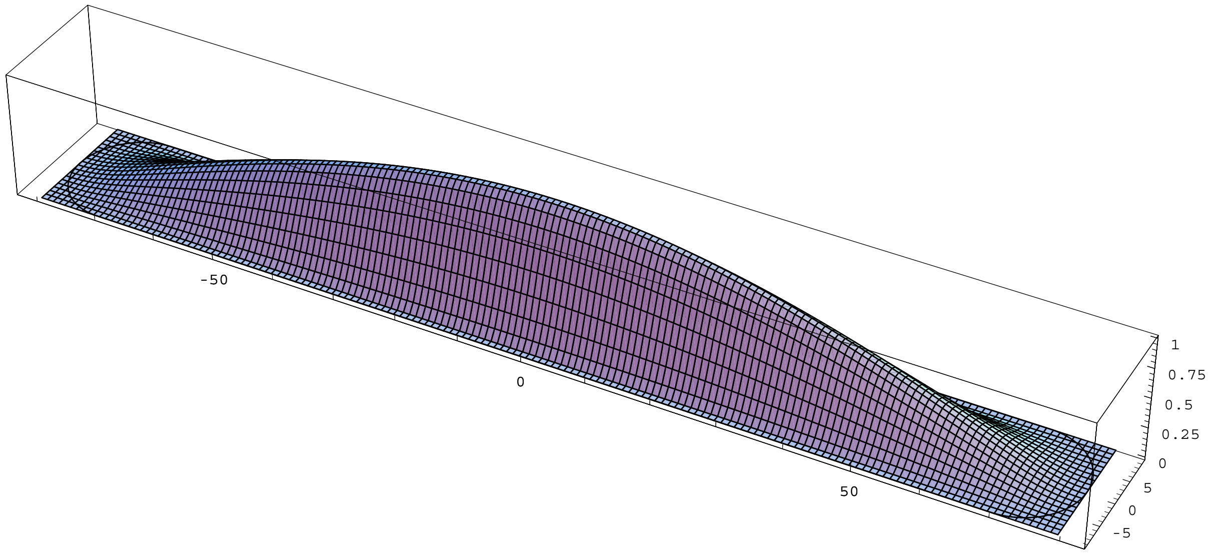

Figure 2: A positive first eigenfunction on an elongated disk

with $t = (\lambda_1 [-1,1])^{\frac{1}{4}}$ the first positive number such that $\tan t + \tanh t = 0$ (that is $t \approx 2.365$). Indeed this limit follows by testing the Rayleigh quotient for $\Omega = D_\ell$

$$\lambda_1 = \inf \left\{ \frac{\int_\Omega (\Delta\phi)^2 \, dx}{\int_\Omega \phi^2 dx}; \phi \in H_0^2(\Omega) \right\}$$

with $\phi(x,y) = \eta(x/\ell)\, v(y)$. Here η is some nonnegative smooth function with support in $[-1,1]$.

5 Annuli

A main assumption that appears over and over again is the convexity of the domain. Domains which are far from convex are domains with holes. The standard examples are the annuli:

$$A_\varepsilon = \left\{ (x,y)\,; \varepsilon^2 < x^2 + y^2 < 1 \right\} \text{ with } 0 < \varepsilon < 1, \tag{5}$$

Already Hadamard knew that the Green function on the annulus (*couronne circulaire* in [23]) changes sign. The sign-changing Green function for the annulus was revisited in [29] and [14]. For general domains with a small hole Coffman and Grover in [8, Proposition 8.1] proved that no sign preserving property can hold for (2).

The eigenvalue problem for domains with holes has first been studied by Duffin and Shaffer ([13]). With Coffman ([6]) they could show that for the annuli with a small hole, the first eigenfunction changes sign. They used an explicit formula and explicit values of the Bessel functions involved and obtained even a critical number for the ratio of the inner and outer radius. The proof has been further simplified in [10].

Theorem 8 (Coffman-Duffin-Shaffer) *There exists $\varepsilon_0 > 0$ such that the following holds.*

1. *If $\varepsilon < \varepsilon_0$, then the first eigenvalue has multiplicity two. There exist two independent eigenfunctions for this first eigenvalue with diametral nodal lines.*

2. *If $\varepsilon = \varepsilon_0$, then the first eigenvalue has multiplicity three. There exists a positive eigenfunction for this eigenvalue and there are two independent eigenfunctions with diametral nodal lines.*

3. *If $\varepsilon > \varepsilon_0$, then the first eigenvalue has multiplicity one and the corresponding eigenfunction is of fixed sign.*

6 Dumb-bells

A sign changing first eigenfunction for a dumb-bell shaped, hence simply connected, domain has been obtained numerically in a recent preprint of Brown e.a. ([3]). The numerics show that reducing the size of the connecting bar forces the first eigenfunction to increase the number of sign changes. One might describe this as a wobbling effect in the bar, which is similar as Duffin's oscillating in long rectangles. As Davies explained ([11]) in the numerical probem a continuous reduction of the size of the connection caused the first odd and the first even eigenfunction to alternate in having the smallest eigenvalue.

Graphics of numerical approximations on the first two eigenfunctions on two dumb-bell shaped domains are found in Figure 6 and 7.

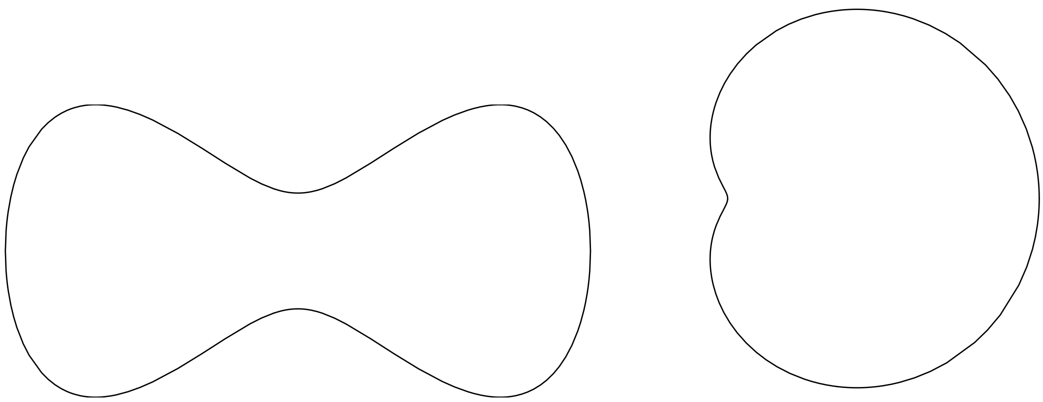

Figure 3: A dumb-bell shaped domain resp. a limaçon for $r = 1 + \frac{4}{5}\cos\varphi$

7 Limaçon

A final question could be if convexity is necessary for either problem. Such a question was answered long ago by Hadamard. He already noticed that the convexity condition is not necessary even for the sign preserving property for (2). The domain he considered was the interior of a nonconvex Limaçon of Pascal ([23]): in polar coordinates $r < 1 + 2a\cos\varphi$ with $a < \frac{1}{2}$.

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Numerics: The eigenfunctions have been approximated and plotted by MATHEMATICA 4.0 on a Pentium III. The steps in the computation are as follows.

- A uniform grid is distributed on a rectangle containing the domain. By a simple test points are either inside or out.
- On the interior points the biharmonic operator is approximated by finite differences. In order to reduce the number of nodes symmetry in the y-direction is used for all examples; for the ellipse and the elongated disk also symmetry in the x-direction. The resulting matrix is a block five diagonal matrix (13 nonzero diagonals).
- The eigenvectors are computed by the MATHEMATICA command `Eigenvectors`. The eigenvectors are ordered by the size of the eigenvalues.
- Finally the eigenvectors are plotted by standard MATHEMATICA commands.

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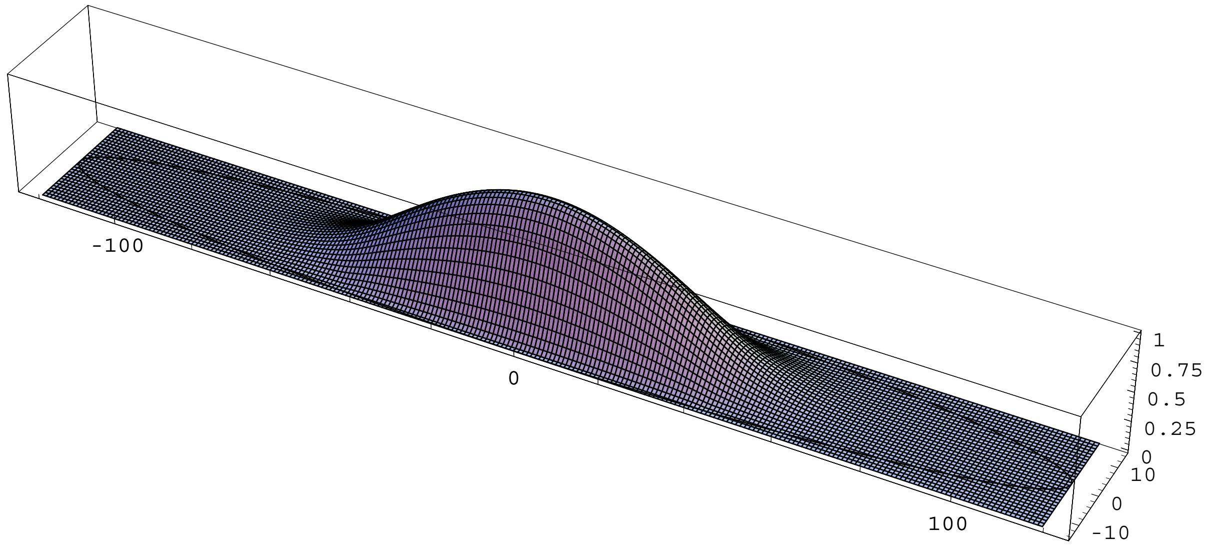

Figure 4: Numerical approximation of the first eigenfunction on the ellipse 120×15.

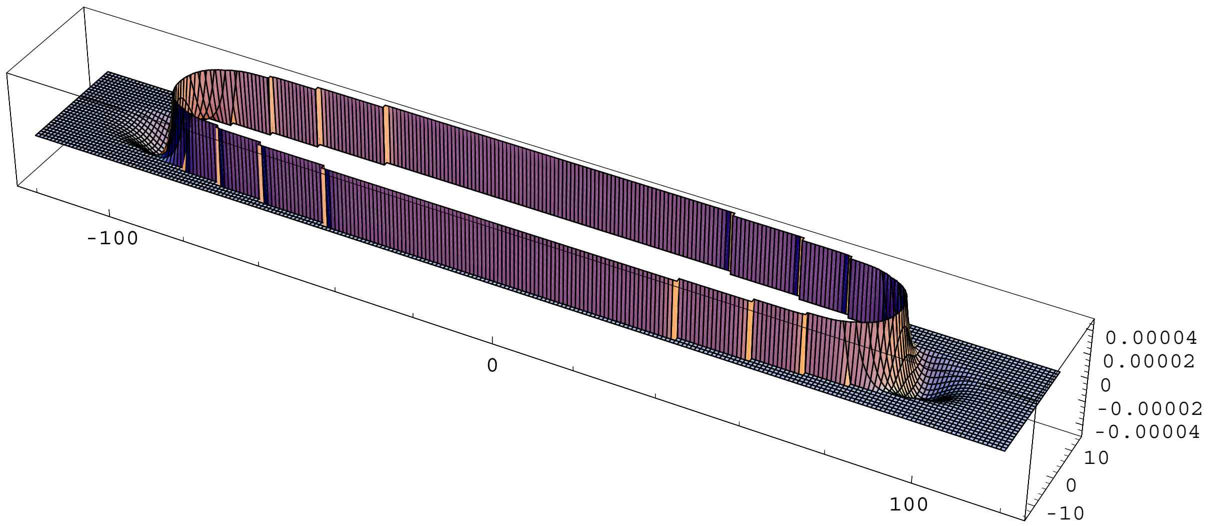

Figure 5: The same eigenfunction blown-up vertically to show the negative part.

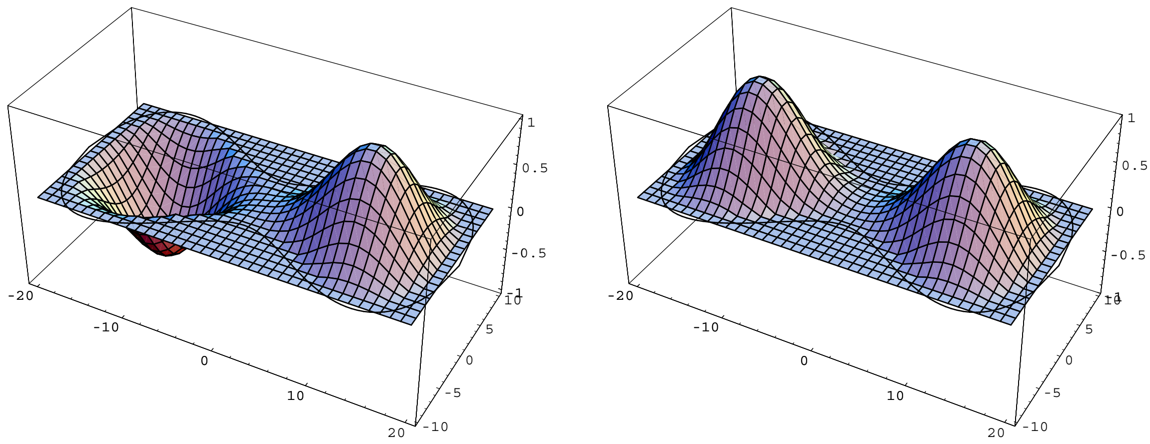

Figure 6: The first (left!) respectively the second eigenfunction on a two-dimensional 'dumb-bell'.

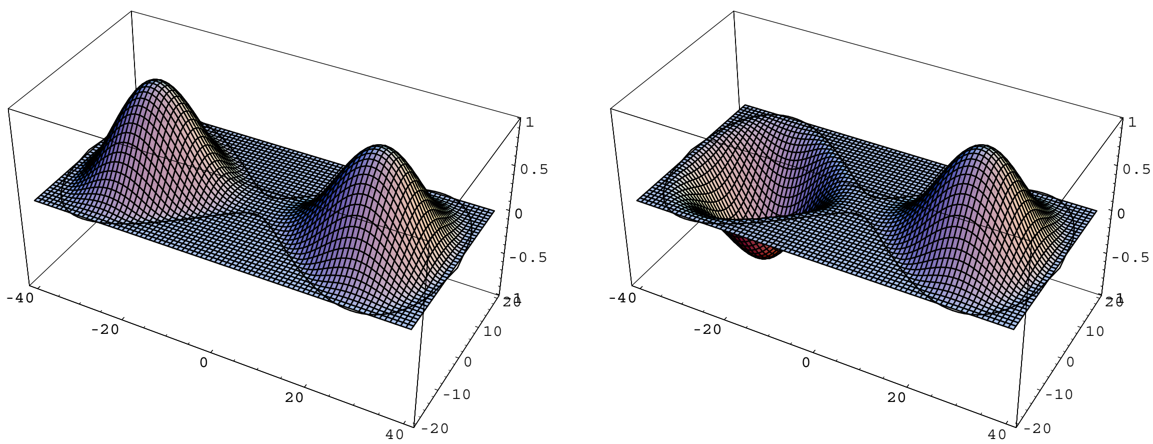

Figure 7: On a more tighter dumb-bell the eigenvalues switched again. The one on the left is the first eigenfunction. In an enlarged picture the approximation of the first and of the second eigenfunction display an extra (small) sign-change in the connecting area.