Non Existence Theorems for Systems of Quasilinear Partial Differential Equations

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1 Introduction

In this paper we shall investigate the non existence of ground states of quasi-linear systems of the form

$$\begin{cases} -div\left(A\left(|Du|\right)Du\right) = f\left(|x|, u, v\right), \\ -div\left(B\left(|Dv|\right)Dv\right) = g\left(|x|, u, v\right), \end{cases} \tag{1}$$

in $I\!R^N$, $N \geq 3$. Here D denotes the gradient operator, A, B are positive scalar functions and f, g are given nonlinearities that will be specified later.

We recall that a ground state of (1) is a positive radially symmetric solution of (1) such that

$$\lim_{|x|\to\infty} u\left(|x|\right) = \lim_{|x|\to\infty} v\left(|x|\right) = 0. \tag{2}$$

The corresponding scalar problem, i.e.

$$-\,div\left(A\left(|Du|\right)Du\right) = f\left(|x|, u\right) \qquad \text{in } I\!R^N \tag{3}$$

has been studied extensively in a series of pioneering papers by Ni and Serrin ([15], [16] and [17]).

In this paper we shall consider some extensions of their results to the system in (1). As a consequence we shall obtain some results which to our knowledge are new even in the scalar case.

Besides their intrinsic interest, non-existence results are a useful tool for proving related existence theorems for the corresponding Dirichlet problem

$$\begin{cases} -div\left(A\left(|Du|\right)Du\right) = f\left(|x|, u, v\right) & \\ -div\left(B\left(|Dv|\right)Dv\right) = g\left(|x|, u, v\right) & \text{in } B_{r_0}(0) \subset I\!R^N, \\ u\left(x\right) = v\left(x\right) = 0 & \text{for } |x| = r_0. \end{cases} \tag{4}$$

This aspect is by now well understood in the case when the operator on the left hand side of (4) are linear, that is, in the semilinear case. See for example

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the paper by Gidas and Spruck, [9], or [22]. In view of new applications it is natural to investigate the possibility of extending the blow-up method (see [9] and [22]) to the quasilinear situation. Other strategies are available for the study of (4) (see for example [1], [2], [5], [6] [10], [18], [23], [24] and the references therein). However, it seems that for not necessarily variational systems containing strong nonlinearities (i.e. 'superlinear') methods based on non existence results are more powerful. Some results concerning this last aspect are contained in [3]. Our study in this paper is based on a property shared by positive radial supersolutions of the scalar problem:

$$\begin{cases} -div\,(A\,(|Du|)\,Du) \geq 0 \quad \text{in } I\!R^N \backslash B_{r_0}(0)\,, \\ \frac{\partial}{\partial n} u\,(x) \leq 0 \quad \text{for } |x| = r_0, \end{cases} \tag{5}$$

where n denotes the normal at $x \in \partial B_{r_0}$ $(n = \dfrac{x}{|x|})$.

The paper is organized as follows. In section 2 we state the lemma that allows us to control the behavior at $+\infty$ of positive radial supersolutions of (5). In section 3 we will use this control to obtain non-existence results for quasilinear elliptic systems. Whenever the system has a variational structure it is possible to obtain a refined result by using a variational identity from [20], [25]. This is done in the fourth section.

The sections end with some examples. In the examples quasilinear elliptic operators such as the p-Laplacian and the mean curvature operator in non-parametric form will appear.

2 A fundamental Lemma

In this section we shall prove a lemma concerning positive radial supersolutions of (5). This result is our fundamental tool in the rest of the paper.

Let us specify the assumption.

(A1) Let $A : [0, \infty) \to [0, \infty)$ be such that

$$\exists A_M \in I\!R : \quad 0 < A\,(t) \leq A_M \quad \text{for all } t \in [0, \infty)\,. \tag{6}$$

Lemma 2.1 *Let A satisfy* **(A1)** *and let $r_0 > 0$. If $N \geq 3$ and $u \in C^1(r_0, \infty)$, with $r \to A(|u'(r)|)\, u'(r) \in C^1(r_0, \infty)$, is a positive radial supersolution satisfying (5),*
then

1. $u'(r) \leq 0 \quad$ *for* $\quad r \geq r_0$
2. *the function $M(\cdot)$ defined by*

$$M(r) = A(|u'(r)|)\ r\ u'(r) + A_M\ (N-2)\ (u(r) - u_\infty) \tag{7}$$

where $u_\infty = \inf_{r \geq r_0} u(r) = \lim_{r \to \infty} u(r)$, is nonnegative and non increasing on (r_0, ∞).

Proof. We may suppose that u is not constant. First observe that the positivity of u and $u'(r_0) \leq 0$ imply that $u'(r) < 0$ for $r > r_0$. Let us put

$$-div\,(A(|Du|)\ Du) = f,$$

which can be written in radial coordinates as

$$-\left(r^{N-1} A(|u'(r)|)\ u'(r)\right)' = r^{N-1} f(r) \quad \text{for } r > r_0$$

i.e.

$$-\left(r^{N-2}\ r\ A(|u'(r)|)\ u'(r)\right)' = r^{N-1} f(r)$$

or

$$-(N-2)\quad A(|u'(r)|)\ u'(r) - \frac{d}{dr}\left[r\ A(|u'(r)|)\ u'(r)\right] = rf(r)\,. \tag{8}$$

Hence by (8), (6) and $u'(r) < 0$ it follows that

$$-M'(r) = -\frac{d}{dr}\left[rA(|u'(r)|)\, u'(r) + A_M\ (N-2)\ (u(r) - u_\infty)\right] \geq r\ f(r) \tag{9}$$

which shows that M is non increasing. It remains to prove that M is nonnegative.

By contradiction let us suppose that there exists $r_1 > r_0$ such that $M(r_1) < 0$. Integrating (9) on (r_1, r) we obtain

$$r\ A(|u'(r)|)\ u'(r) + A_M\ (N-2)\ (u(r) - u_\infty) \leq M(r_1). \qquad (10)$$

The positivity of u and the fact that u is decreasing gives

$$A_M\ u'(r) \leq A(|u'(r)|)\ u'(r) + \frac{1}{r} A_M\ (N-2)(u(r) - u_\infty) \leq \frac{1}{r} M(r_1). \qquad (11)$$

Integrating (11) on (s, t) we obtain for $r_1 < s < t$ that

$$u(t) - u(s) \leq \frac{M(r_1)}{A_M} \log\left(\frac{t}{s}\right). \qquad (12)$$

By letting t go to infinity we obtain a contradiction. □

Remark: In the above lemma one of the crucial assumptions is that $A(\cdot)$ is uniformly bounded on $(0, \infty)$. This assumption can be weakened somewhat if we assume that $A \in C^1(I\!R^+)$ holds and that A is bounded near 0. This is due to the fact that, if u is a positive supersolution of (5), then $u'(r) < 0$ for large r. Hence, the function $\tilde{u}(r) = u(r) - u_\infty$, where $u_\infty = \lim_{r\to\infty} u(r)$, is a positive supersolution of (5) such that $\lim_{r\to\infty} \tilde{u}(r) = 0$. By using an argument of Ni and Serrin (see page 10 of [16]) for $\tilde{u}(r)$, it follows that $\tilde{u}'(r) = u'(r) \to 0$ as r goes to infinity. Then, if $A(t) \leq M$ for small t, we can apply the same kind of proof as in Lemma 2.1. For easy reference we state this result.

Lemma 2.2 *Let $A \in C([0,\infty)) \cap C^1((0,\infty))$ satisfy*

$$0 < A(t) \leq A_M \quad \text{for sufficiently small } t > 0 \qquad (13)$$

If $N \geq 3$ and $u \in C^1(r_0, \infty)$, with $r \to A(|u'(r)|)\, u'(r) \in C^1(r_0, \infty)$, is a positive radial supersolution satisfying (5),
then

1. *$u'(r) \leq 0 \quad$ for $\quad r \geq r_0$,*
2. *there is $\tilde{r}_0 \geq r_0$ such that the function $M(\cdot)$ defined by (7) is nonnegative and non increasing on $(\tilde{r}_0, \infty)$.*

The following lemma is essentially contained in [15].

Lemma 2.3 *Let A satisfy* **(A1)** *and let $r_0 > 0$. If $N \geq 3$ and $u \in C^1(r_0, \infty)$, with $r \to A(|u'(r)|)\, u'(r) \in C^1(r_0, \infty)$ is a non constant positive radial supersolution satisfying*

$$\begin{cases} -\left(r^{N-1} A(|u'|)\, u'\right)' \geq 0 & \text{for } r > r_0, \\ u'(r_0) \leq 0, \end{cases} \tag{14}$$

then there exists $c > 0$ such that for all r sufficiently large, we have

$$u(r) - u_\infty \geq c\, r^{2-N}. \tag{15}$$

Proof. We may assume that $u'(r_0) < 0$. By integrating (14) on (r_0, r) we obtain

$$-u'(r)\; A(|u'(r)|)\; r^{N-1} \;\geq -u'(r_0)\; A(|u'(r_0)|)\; r_0^{N-1},$$

and hence for some $c_0 > 0$:

$$-u'(r) \;\geq \frac{c_0}{A(|u'(r)|)}\, r^{1-N} \;\geq \frac{c_0}{A_M}\, r^{1-N}.$$

A further integration on (r, t) gives

$$u(r) - u_\infty \geq -u(t) + u(r) \geq \frac{1}{2-N} \frac{c_0}{A_M} \left(t^{2-N} - r^{2-N}\right).$$

By letting $t \to \infty$ we obtain

$$u(r) - u_\infty \geq \frac{c_0}{(N-2)\, A_M}\, r^{2-N}.$$

□

Now we shall illustrate with some examples how we can apply Lemma 2.1.

Example i) The mean curvature operator in non parametric form.

Let $u \in C^2(r_0, \infty)$ be radial and positive, and such that

$$\begin{cases} -div\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) \geq 0 & \text{for } r \geq r_0, \\ u'(r_0) \leq 0. \end{cases}$$

Therefore Lemma 2.2 applies with

$$A(t) = \frac{1}{\sqrt{1+t^2}} \leq 1.$$

The conclusion is that the function

$$M(r) = \frac{r\ u'(r)}{\sqrt{1+(u'(r))^2}} + (N-2)\ (u(r) - u_\infty)$$

is non-negative and non-increasing for $r \geq r_0$. Hence

$$\frac{-r\ u'(r)}{\sqrt{1+(u'(r))^2}} \leq (N-2)\ (u(r) - u_\infty), \tag{16}$$

and thus $\lim_{r\to\infty} r\ u'(r) = 0$. This implies that there exists $\varepsilon > 0$ such that for $r \geq r_1$ sufficiently large we have

$$(u'(r))^2 < \varepsilon. \tag{17}$$

From (16) and (17) it follows that

$$-r\ u'(r) \leq (1+\varepsilon)\ (N-2)\ (u(r) - u_\infty) \quad \text{for } r > r_1,$$

and hence

$$\left(r^{(N-2)(1+\varepsilon)} u(r)\right)' \geq \left(r^{(N-2)(1+\varepsilon)} (u(r) - u_\infty)\right)' \geq 0 \quad \text{for } r > r_1. \tag{18}$$

If we assume that

$$0 < A_{\inf} = \inf_{t\in[0,\infty)} A(t) \quad \text{and} \quad \sup_{t\in[0,\infty)} A(t) = A_{\sup} < \infty, \tag{19}$$

holds, then we have the same conclusion as in (18). That is, if u satisfies

$$\begin{cases} -\left(r^{N-1}A(|u'|)\ u'\right)' \geq 0 \quad \text{for } r > r_0. \\ u'(r_0) \leq 0 \end{cases}$$

then the function $M(\cdot)$, defined on $[r_0, \infty)$ by

$$M(r) = r\ u'(r) + (N-2)\frac{A_{\sup}}{A_{\inf}}(u(r) - u_\infty)$$

satisfies

$$M(r) \geq 0 \quad \text{for } r \in [r_0, \infty).$$

Putting $\theta = (N-2)\frac{A_{\sup}}{A_{\inf}}$ we find

$$\left(r^\theta\ u(r)\right)' \geq \left(r^\theta (u(r) - u_\infty)\right)' \geq 0. \tag{20}$$

An example of such a situation is the following.

Example ii) Let $\varepsilon \in (0,1]$ and consider

$$A_\varepsilon(t) = \varepsilon + \frac{1-\varepsilon}{\sqrt{1+t^2}} \qquad t \geq 0.$$

The corresponding differential inequality associated to A_ε is

$$-\left(\varepsilon\Delta u + (1-\varepsilon)\ div\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)\right) \geq 0 \qquad \text{in } I\!R^N.$$

This regularizing operator is used for example in [11]. It follows from the preceding analysis that (20) holds with $\theta = (N-2)\varepsilon^{-1}$. The particular case when $\varepsilon = 1$ was already known, see for example [12].

Another example is given by the following generalized mean curvature operator. (See [15], [16])

Example iii) Let $u \in C^2(r_0, \infty)$ be radial, positive and satisfying

$$\begin{cases} -div \left(\dfrac{Du}{\left(1+|Du|^2\right)^{\frac{2-m}{m}}} \right) \geq 0 \qquad \text{for } r \geq r_0, \\ u'(r_0) \leq 0. \end{cases}$$

where $m \in (1, 2]$. The case $m = 1$ is treated above. Lemma 2.2 applies with

$$A(t) = \frac{1}{(1+t^2)^{\frac{2-m}{2}}} \qquad t \geq 0.$$

We find for r sufficiently large that

$$\frac{r\ u'(r)}{\left(1+(u'(r))^2\right)^{\frac{2-m}{2}}} + (N-2)(u(r) - u_\infty) \geq 0.$$

Since $A(t) \leq 1$ we find for r sufficiently large that

$$\left(r^{\ (N-2)(1+\varepsilon)} u(r)\right)' \geq \left(r^{\ (N-2)(1+\varepsilon)} (u(r) - u_\infty)\right)' \geq 0. \tag{21}$$

One may also proceed using Lemma 2.1 of [3]. If A is continuously differentiable and

$$1) \qquad (t\ A(t))' > 0 \qquad \forall t > 0,$$

$$2) \qquad 0 < \frac{A(t)}{(t\ A(t))'} \leq C \qquad \forall t > 0,$$

then the function $M(r) = ru'(r) + \theta(u(r) - u_\infty)$, with $\theta = (N-1)C - 1$, satisfies

$$M(r) \geq 0 \quad \text{for } r \geq r_0.$$

We have $C = \frac{1}{m-1}$ and with $\theta = \frac{N-m}{m-1}$ we obtain

$$\left(r^{\frac{N-m}{m-1}} u(r)\right)' \geq \left(r^{\frac{N-m}{m-1}} (u(r) - u_\infty)\right)' \geq 0. \tag{22}$$

For small ε the exponent in (21) is larger than in (22) and hence gives a better estimate. However, since the estimate depends on ε and it is valid only for large r, it is not uniform.

3 Non-existence results

3.1 A system with mean curvature operators

As a first example of a non-existence result we consider the following system.

$$\begin{cases} -\left(r^{N-1}\dfrac{u'(r)}{\sqrt{1+(u'(r))^2}}\right)' = r^{N-1}\ f(v(r)) & \text{for } r>0, \\ \\ -\left(r^{N-1}\dfrac{v'(r)}{\sqrt{1+(v'(r))^2}}\right)' = r^{N-1}\ g(u(r)) & \text{for } r>0. \end{cases} \tag{23}$$

Proposition 3.1 *Let $f, g \in C(I\!R)$ with $f(0) = g(0) = 0$. Suppose that there exists $c > 0$ such that for small u and v we have*

$$f(v) \geq c\,|v|^{p-1}\,v, \tag{24}$$

$$g(u) \geq c\,|u|^{q-1}\,u, \tag{25}$$

where $p, q > 1$. If $N \geq 3$ and

$$\frac{(N-2)(pq-1)}{2} \leq \max\{p+1, q+1\} \tag{26}$$

then problem (23) has no non trivial ground states.

Proof. Let (u, v) be a non trivial ground state of (23). By integrating the first equation in (23) on (r_0, r), for r large, and using $u'(r) < 0$ respectively $v'(r) < 0$ jointly with (24), (25) we obtain

$$\frac{-r\ u'(r)}{\sqrt{1 + (u'(r))^2}} \geq c\ r^2\ v(r)^p, \tag{27}$$

$$\frac{-r\ v'(r)}{\sqrt{1 + (v'(r))^2}} \geq c\ r^2\ u(r)^q, \tag{28}$$

and hence by Lemma fundamentallemma

$$(N-2)\,u(r) \geq c\ r^2\ v(r)^p, \tag{29}$$

$$(N-2)\,v(r) \geq c\ r^2\ u(r)^q. \tag{30}$$

This implies that there exist $c_1, c_2 > 0$ such that for large r

$$u(r) \leq c_1\ r^{-2\frac{p+1}{pq-1}}, \tag{31}$$

and

$$v(r) \leq c_2\ r^{-2\frac{q+1}{pq-1}}. \tag{32}$$

By Lemma 14 we also deduce that

$$c_3\ r^{2-N} \leq u(r), \tag{33}$$

and

$$c_4\ r^{2-N} \leq v(r), \tag{34}$$

for some $c_3, c_4 > 0$ and r sufficiently large. Combining (31)-(33) and (32)-(34) we obtain a contradiction for $r \to \infty$ when the strict inequality holds in (26). Next consider the case that

$$\frac{(N-2)(pq-1)}{2} = \max\{p+1, q+1\}.$$

Without loss of generality we may assume that $p \geq q$. In this case we have $pq = \frac{2p+N}{N-2}$. From the first equation of (23) and the assumption that $f(v) \geq c\, v^p$ for small v we find that for large r that

$$-\left(r^{N-1}\frac{u'(r)}{\sqrt{1+(u'(r))^2}}\right)' \geq c\; r^{N-1}\; u^{pq}\; r^{2p}.$$

Integrating on (s,r) and using (31) shows, for some $c^* > 0$, that

$$\frac{-r^{N-1}u'(r)}{\sqrt{1+(u'(r))^2}} \geq c\int_s^r \xi^{N-1}\; u(\xi)^{pq}\;\; \xi^{2p}\; d\xi \geq$$

$$\geq c^*\int_s^r \xi^{N-1}\; \xi^{(2-N)pq}\; \xi^{2p}\; d\xi = c^*\int_s^r \xi^{-1}\; d\xi = c^*\;\log\frac{r}{s}.$$

Hence, by Lemma 2.1:

$$(N-2)\, r^{N-2}u(r) \geq \frac{-r^{N-1}u'}{\sqrt{1+(u')^2}} \geq c\;\log r,$$

which gives a contradiction with

$$r^{N-2}u(r) \leq c\; r^{-2\frac{p+1}{pq-1}}r^{N-2} = c$$

following from (31). □

Remark: The same result holds if we replace the mean curvature operators by

$$T_A = -div\,(A(|Du|)\,Du)\,,$$
$$T_B = -div\,(B(|Dv|)\,Dv)\,,$$

with A and B satisfying the assumption **(A1)**.

3.2 Uniformly bounded elliptic operators

In this section we shall consider some non-existence results for ground states of systems of the form

$$\begin{cases} -div\left(A\left(|Du|\right)Du\right) = f\left(|x|, u, v\right), \\ -div\left(B\left(|Dv|\right)Dv\right) = g\left(|x|, u, v\right), \end{cases} \text{ in } I\!R^N \tag{35}$$

Let $N \geq 3$ and suppose that:

(A2) $A, B : [0, \infty) \to (0, \infty)$ are C^1 functions that satisfy

$$0 < A_{\inf} = \inf_{t\in[0,\infty)} A(t) \leq A(t) \leq \sup_{t\in[0,\infty)} A(t) = A_{\sup} < \infty,$$

$$0 < B_{\inf} = \inf_{t\in[0,\infty)} B(t) \leq B(t) \leq \sup_{t\in[0,\infty)} B(t) = B_{\sup} < \infty.$$

Define

$$\alpha_A = (N-2)\frac{A_{\sup}}{A_{\inf}}, \quad \alpha_B = (N-2)\frac{B_{\sup}}{B_{\inf}}.$$

The following result generalizes Proposition 3.1:

Theorem 3.2 *Suppose that condition* **(A2)** *is satisfied. Let* $f, g \in C\left(I\!R^+ \times I\!R^2\right)$ *with* $f(r,0,0) = g(r,0,0) = 0\ \forall r \geq 0$*. Moreover suppose that for large* r *and small* u, v *we have*

$$f(r,u,v) \geq a(r)\ |v|^{q-1} v, \tag{36}$$

$$g(r,u,v) \geq b(r)\ |u|^{p-1} u, \tag{37}$$

for some $p, q > 1$*, where* $a, b : [0,\infty) \to [0,\infty)$ *are continuous.*
If, either

$$(i) \qquad \int_{r_0}^{\infty} \vartheta^{1+\alpha_A}\ a(\vartheta) \left(\int_{\vartheta}^{\infty} b(s)\ s^{1-q\alpha_A}\ ds\right)^p d\vartheta = +\infty, \tag{38}$$

or

$$(ii) \qquad \int_{r_0}^{\infty} \vartheta^{1+\alpha_B}\ b(\vartheta) \left(\int_{\vartheta}^{\infty} a(s)\ s^{1-p\alpha_B}\ ds\right)^q d\vartheta = +\infty, \tag{39}$$

then the system (35) has no non trivial ground state.

Proof. The proof is by contradiction. Let us prove the assertion in the case when (38) holds. The other case is similar. Let (u, v) be a non trivial positive and radial ground state. Then (u, v) satisfies for $r > 0$:

$$\begin{cases} -\left(r^{N-1}\ A\left(\left|u'\left(r\right)\right|\right)\ u'\left(r\right)\right)' = r^{N-1} f\left(r, u, v\right), \\ -\left(r^{N-1}\ B\left(\left|v'\left(r\right)\right|\right)\ v'\left(r\right)\right)' = r^{N-1} g\left(r, u, v\right). \end{cases} \tag{40}$$

Since (u, v) is positive and $u(r), v(r) \to 0$ as $r \to \infty$ it follows that there exists r_0 sufficiently large such that $u'(r_0) \leq 0$, and hence $u'(r) \leq 0$ for $r \geq r_0$. A similar result holds for v. We may assume that for some large r_0 we have

$$\begin{cases} u'(r) \leq 0 & \text{for } r \geq r_0, \\ v'(r) \leq 0 & \text{for } r \geq r_0. \end{cases} \tag{41}$$

Let us define

$$\begin{cases} \tilde{M}_A(r) = r\ A(|u'(r)|)\, u'(r) + A_{\sup}(N-2)\, u(r), \\ \tilde{M}_B(r) = r\ B(|v'(r)|)\, v'(r) + B_{\sup}(N-2)\, v(r). \end{cases} \tag{42}$$

Similar as (9) one shows that

$$\begin{cases} -\frac{d}{dr}\tilde{M}_A(r) & \geq & r\ a(r)\ v^p, \\ -\frac{d}{dr}\tilde{M}_B(r) & \geq & r\ b(r)\ u^q, \end{cases} \tag{43}$$

on (r_0, ∞). By applying Lemma (2.1) it also follows that the two functions $\tilde{M}_A(\cdot)$ and $\tilde{M}_B(\cdot)$ are nonnegative on (r_0, ∞). Integrating both inequalities in (43) on (r, t) with $r > r_0$ it follows that

$$\begin{cases} \tilde{M}_A(r) & \geq & -\tilde{M}_A(t) + \tilde{M}_A(r) & \geq & \int_r^t s\ a(s)\ v(s)^p\, ds, \\ \tilde{M}_B(r) & \geq & -\tilde{M}_B(t) + \tilde{M}_B(r) & \geq & \int_r^t s\ b(s)\ u(s)^q\, ds. \end{cases} \tag{44}$$

and hence using the fact that $A_{\inf}, B_{\inf} > 0$ and $u', v' \leq 0$ for $r \geq r_0$ we obtain, with $C_A = A_{\sup}(N-2)\, u(r_0) > 0$, that

$$C_A \geq A_{\sup}(N-2)\, u(r) \geq \tilde{M}_A(r) \geq \int_r^t s\ a(s)\ v(s)^p\, ds \geq$$

$$\geq \left(r^{\alpha_B}\ v(r)\right)^p \int_r^t s\ a\left(s\right)\ s^{1-\ p\alpha_B} ds, \tag{45}$$

and similarly, with $C_B = B_{\sup}\left(N-2\right)\ v\left(r_0\right) > 0$, that

$$C_B \geq B_{\sup}\left(N-2\right) v\left(r\right) \geq \left(r^{\alpha_A}\ u(r)\right)^q \int_r^t s\ b\left(s\right)\ s^{1-q\alpha_A} ds. \tag{46}$$

In the last two estimates we have used the fact that on (r_0, ∞)

$$\begin{cases} \left(r^{\alpha_A}\ u\left(r\right)\right)' \geq 0, \\ \left(r^{\alpha_B}\ v\left(r\right)\right)' \geq 0. \end{cases} \tag{47}$$

Indeed, the first estimate of (47) follows from (44) and

$$\left(r^{\alpha_A}\ u\left(r\right)\right)' = \frac{1}{A_{\inf}} r^{\alpha_A - 1}\left(A_{\inf} r\ u'(r) + \left(N-2\right) A_{\sup} u(r)\right) \geq \frac{1}{A_{\inf}} r^{\alpha_A - 1} \tilde{M}_A\left(r\right) \tag{48}$$

for $r > r_0$. By (45) and (46) we find that for $r > r_0$

$$\begin{cases} \displaystyle\int_r^\infty a\left(s\right) s^{1-p\alpha_B} ds < \infty, \\ \displaystyle\int_r^\infty b\left(s\right) s^{1-q\alpha_A} ds < \infty. \end{cases} \tag{49}$$

If one of the integrals in (49) equals $+\infty$ the proof is complete. If not we proceed as follows. For some $c_a, c_b > 0$ it follows from (45)-(46) that

$$\begin{cases} \left(u\left(r\right)\right)^q \geq c_a \left(r^{\alpha_B}\ v\left(r\right)\right)^{pq} \left(\displaystyle\int_r^\infty a\left(s\right) s^{1-p\alpha_B} ds\right)^q, \\ \left(v\left(r\right)\right)^p \geq c_b \left(r^{\alpha_A}\ u\left(r\right)\right)^{pq} \left(\displaystyle\int_r^\infty b\left(s\right) s^{1-q\alpha_A} ds\right)^p. \end{cases} \tag{50}$$

From the first inequality of (43) and (50) it follows that

$$-\frac{d}{dr}\tilde{M}_A\left(r\right) \geq c_b\ r\ a\left(r\right)\left(r^{\alpha_A}\ u\left(r\right)\right)^{pq}\left(\int_r^\infty b\left(s\right) s^{1-q\alpha_A} ds\right)^p. \tag{51}$$

Integrating (51) on (r,t) yields

$$\tilde{M}_A(r) \geq -\tilde{M}_A(t) + \tilde{M}_A(r) \geq$$

$$\geq c_b \int_r^t s\ a(s) \left(s^{\alpha_A}\ u(s)\right)^{pq} \left(\int_s^\infty b(\xi)\, \xi^{1-q\alpha_A}\ d\xi\right)^p ds \geq$$

$$\geq c_b \left(r^{\alpha_A}\ u(r)\right)^{pq} \int_r^t s\ a(s) \left(\int_s^\infty b(\xi)\, \xi^{1-q\alpha_A}\ d\xi\right)^p ds, \qquad (52)$$

and then, using (48) again,

$$\left(r^{\alpha_A}\ u(r)\right)' \geq \frac{1}{A_{\inf}}\ r^{\alpha_A - 1}\ \tilde{M}_A(r) \geq$$

$$\geq \frac{c_b}{A_{\inf}}\ r^{\alpha_A-1} \left(r^{\alpha_A}\ u(r)\right)^{pq} \int_r^t s\ a(s) \left(\int_s^\infty b(\xi)\, \xi^{1-q\alpha_A}\ d\xi\right)^p ds. \qquad (53)$$

Setting $\phi(r) = r^{\alpha_A} u(r)$ we find from (53) that

$$\frac{1}{1-pq} \frac{d}{dr} (\phi(r))^{1-pq} \geq c_a'\ r^{\alpha_A-1} \int_r^t s\ a(s) \left(\int_s^\infty b(\xi)\, \xi^{1-q\alpha_A}\ d\xi\right)^p ds, \qquad (54)$$

with $c_a' = \dfrac{c_b}{A_{\inf}}$. An integration of (54) on $(r,\tilde{r})$ gives

$$\frac{-1}{1-pq} (\phi(r))^{1-pq} \geq \frac{1}{1-pq} (\phi(\tilde{r}))^{1-pq} - \frac{1}{1-pq} (\phi(r))^{1-pq} \geq$$

$$\geq c_b' \int_r^{\tilde{r}} \varsigma^{\alpha_A-1} \int_\varsigma^t s\ a(s) \left(\int_s^\infty b(\xi)\, \xi^{1-q\alpha_A} d\xi\right)^p ds\ d\varsigma =$$

$$= \ c_b'' \left[\varsigma^{\alpha_A} \int_\varsigma^t s\ a(s) \left(\int_s^\infty b(\xi)\, \xi^{1-q\alpha_A} d\xi\right)^p ds\right]\Bigg|_{\varsigma=r}^{\varsigma=\tilde{r}} +$$

$$+\ c_b'' \int_r^{\tilde{r}} \varsigma^{\alpha_A+1} a(\varsigma) \left(\int_\varsigma^\infty b(\xi)\, \xi^{1-q\alpha_A} d\xi\right)^p d\varsigma \qquad =$$

$$
\begin{aligned}
= & + \; c_b'' \, \tilde{r}^{\alpha_A} \int_{\tilde{r}}^{t} s \; a(s) \left(\int_s^\infty b(\xi)\, \xi^{1-q\alpha_A} d\xi \right)^p ds \; + \\
& - \; c_b'' \, r^{\alpha_A} \int_{r}^{t} s \; a(s) \left(\int_s^\infty b(\xi)\, \xi^{1-q\alpha_A} d\xi \right)^p ds \; + \\
& + \; c_b'' \int_r^{\tilde{r}} \varsigma^{\alpha_A+1} a(\varsigma) \left(\int_\varsigma^\infty b(\xi)\, \xi^{1-q\alpha_A} d\xi \right)^p d\varsigma \quad ,
\end{aligned}
\tag{55}
$$

with $c_b'' = c_b'/\alpha_A$. Now we first let t go to infinity and then $\tilde{r}$. Since the second term is bounded by (45) and (52), and the third term goes to infinity by assumption, one finds that

$$\tfrac{-1}{1-pq} \left(\phi(r)\right)^{1-pq} \to \infty \quad \text{when } r \to \infty$$

which shows that

$$\lim_{r\to\infty} \phi(r) = 0.$$

But this contradicts the fact that $\phi'(r) \geq 0$ for $r > r_0$. □

3.3 Possibly unbounded elliptic operators

Next we shall consider a general non-existence result for a different class of quasilinear operators. Our theorem can be applied for example to the case when the left hand side of (1) involves two generalized Laplacians. The explicit statement can be found in the last theorem of this section.

(A1') Let $A, B : I\!R^+ \to (0,\infty)$ be C^1 and suppose that for some $\delta_A, \delta_B \geq 0$, $c_A, c_B > 0$ and $m_A, m_B > \frac{1}{N-1}$ we have

$$
\begin{aligned}
&(i)_A \qquad A(t) \leq m_A \tfrac{\partial}{\partial t} (t \; A(t)) \leq c_A \; t^{\delta_A} \quad t > 0, \\
&(i)_B \qquad B(t) \leq m_B \tfrac{\partial}{\partial t} (t \; B(t)) \leq c_B \; t^{\delta_B} \quad t > 0.
\end{aligned}
\tag{56}
$$

(A2') Let $f, g : I\!R^+ \times I\!R^2 \to I\!R$ be C^0 with $f(x,0,0) = g(x,0,0) = 0$ and suppose that for small u, v and large $x \in I\!R^N$ there are continuous $a_{ij} : [0,\infty) \to [0,\infty)$, with $i, j \in \{1,2\}$, such that we have

$$
\begin{aligned}
&(ii)_A \qquad f(x,u,v) \geq a_{11}(|x|) \; |u|^{p_{11}-1} u + a_{12}(|x|) \; |v|^{p_{12}-1} v, \\
&(ii)_B \qquad g(x,u,v) \geq a_{21}(|x|) \; |u|^{p_{21}-1} u + a_{22}(|x|) \; |v|^{p_{22}-1} v.
\end{aligned}
\tag{57}
$$

Theorem 3.3 *Suppose that the above conditions (56) and (57) are satisfied. Moreover assume that*

1. $p_{12}p_{21} > (1+\delta_A)(1+\delta_B)$ *and*

$$\int_{r_0}^{\infty} \xi^{(1+\delta_B)(1+\theta_B)}\ a_{21}(\xi)\ \left(\int_{\xi}^{\infty} s^{1+\delta_A-\theta_B p_{12}} a_{12}(s)\quad ds\right)^{\frac{p_{21}}{1+\delta_A}} d\xi = +\infty, \tag{58}$$

or

2. $p_{12}p_{21} > (1+\delta_A)(1+\delta_B)$ *and*

$$\int_{r_0}^{\infty} \xi^{(1+\delta_A)(1+\theta_A)}\ a_{12}(\xi)\ \left(\int_{\xi}^{\infty} s^{1+\delta_B-\theta_A p_{21}} a_{21}(s)\quad ds\right)^{\frac{p_{12}}{1+\delta_B}} d\xi = +\infty, \tag{59}$$

or

3. $p_{11} > 1+\delta_A$ *and*

$$\int_{r_0}^{\infty} \xi^{(1+\delta_A)(1+\theta_A)-\theta_A p_{11}}\ a_{11}(\xi)\ d\xi = +\infty, \tag{60}$$

or

4. $p_{22} > 1+\delta_B$ *and*

$$\int_{r_0}^{\infty} \xi^{(1+\delta_B)(1+\theta_B)-\theta_B p_{22}}\ a_{22}(\xi)\ d\xi = +\infty, \tag{61}$$

with $\theta_A = m_A(N-1)-1$ *and* $\theta_B = m_B(N-1)-1$.
Then the problem

$$\left\{\begin{array}{rcl} -div\left(A\left(|Du|\right)Du\right) & = & f\left(|x|,u,v\right), \\ -div\left(B\left(|Dv|\right)Dv\right) & = & g\left(|x|,u,v\right), \end{array}\right. \quad in\ I\!R^N \tag{62}$$

has no non trivial ground state.

Proof. We start by showing that there is no non trivial ground state when (58) holds.

First we set

$$\begin{aligned} E_A(t) &= \tfrac{\partial}{\partial t}\left(t\ A(t)\right), \\ E_B(t) &= \tfrac{\partial}{\partial t}\left(t\ B(t)\right). \end{aligned} \tag{63}$$

System (62) can be rewritten as

$$\begin{cases} -E_A(|u'|)\,u'' + \dfrac{1-N}{r}u' A(|u'|) = f(r,u,v) & \text{for } r>0, \\ -E_B(|v'|)\,v'' + \dfrac{1-N}{r}v' B(|v'|) = g(r,u,v) & \text{for } r>0, \end{cases} \tag{64}$$

and by the estimates in (57) we obtain for large r

$$\begin{cases} -rE_A(|u'|)\,u'' + (1-N)\,u' A(|u'|) \geq a_{12}(r)\ \ r\ v^{p_{12}}, \\ -rE_B(|v'|)\,v'' + (1-N)\,v' B(|v'|) \geq a_{21}(r)\ \ r\ u^{p_{21}}, \end{cases} \tag{65}$$

and hence

$$\begin{cases} -ru'' + (1-N)\,u'\dfrac{A(|u'|)}{E_A(|u'|)} \geq \dfrac{a_{12}(r)\ \ r}{E_A(|u'|)}v^{p_{12}}, \\ -rv'' + (1-N)\,v'\dfrac{B(|v'|)}{E_B(|v'|)} \geq \dfrac{a_{21}(r)\ \ r}{E_B(|v'|)}u^{p_{21}}. \end{cases} \tag{66}$$

By (56) and $u'<0$ we find

$$-\,ru'' + (1-N)\,m_A\ u' \geq \frac{a_{12}(r)\,r}{E_A(|u'|)}v^{p_{12}} \geq \frac{c_A}{m_A}\,|u'|^{-\delta_A}\ \ a_{12}(r)\ \ r\ v^{p_{12}}, \tag{67}$$

and similarly with $v'<0$ that

$$-\,rv'' + (1-N)\,m_B\ v' \geq \frac{c_B}{m_B}\,|v'|^{-\delta_B}\ \ a_{21}(r)\ \ r\ u^{p_{21}}. \tag{68}$$

We put

$$\begin{cases} \tilde{M}_A(r) = r\ u' + \left(m_A(N-1)-1\right)u, \\ \tilde{M}_B(r) = r\ v' + \left(m_B(N-1)-1\right)v. \end{cases} \tag{69}$$

From (67) and (68) it follows that for r large enough we have

$$\begin{cases} \frac{d}{dr}\tilde{M}_A(r) \leq 0, \\ \frac{d}{dr}\tilde{M}_B(r) \leq 0. \end{cases} \tag{70}$$

With θ_A, θ_B as above this is equivalent to

$$\begin{cases} \left(r^{\theta_A}\ u(r)\right)' \geq 0 \quad \text{for} \quad r > r_1, \\ \left(r^{\theta_B}\ v(r)\right)' \geq 0 \quad \text{for} \quad r > r_1, \end{cases} \tag{71}$$

for some $r_1 > 0$. Hence by Lemma 2.1 of [3] we find that

$$\begin{cases} \tilde{M}_A(r) \geq 0, \\ \tilde{M}_B(r) \geq 0. \end{cases} \tag{72}$$

Since $u' \leq 0$, $v' \leq 0$ we deduce from (72) that

$$\begin{cases} \theta_A\ u(r) \geq -r\ u'(r) = r\ |u'(r)| \quad \text{for} \quad r > r_1, \\ \theta_B\ v(r) \geq -r\ v'(r) = r\ |v'(r)| \quad \text{for} \quad r > r_1. \end{cases} \tag{73}$$

By integrating (68) on (s, r) with $s \geq r_1$ and using (73) we find that for some positive c_i we have

$$c_0\ v(s) \geq \tilde{M}_B(s) \geq -\tilde{M}_B(r) + \tilde{M}_B(s) \geq$$

$$\geq c_1 \int_s^r |v'(\xi)|^{-\delta_B}\ a_{21}(\xi)\ \xi\ u(\xi)^{p_{21}}\, d\xi \geq$$

$$\geq c_2 \int_s^r \xi^{\delta_B}\ v(\xi)^{-\delta_B}\ a_{21}(\xi)\ \xi\ u(\xi)^{p_{21}}\, d\xi \geq$$

$$\geq c_2\ v(s)^{-\delta_B} \int_s^r \xi^{1+\delta_B-\theta_A\, p_{21}}\ a_{21}(\xi)\ \left(\xi^{\theta_A}\ u(\xi)\right)^{p_{21}} d\xi \geq$$

$$\geq c_2 \; v\,(s)^{-\delta_B} \; \left(s^{\theta_A} \; u\,(s)\right)^{p_{21}} \int\limits_s^r \xi^{1+\delta_B-\theta_A\, p_{21}} \; a_{21}\,(\xi) \; d\xi. \tag{74}$$

Hence there exists $c > 0$ such that

$$v\,(s)^{1+\delta_B} \geq c \; \left(s^{\theta_A} \; u\,(s)\right)^{p_{21}} \int\limits_s^\infty \xi^{1+\delta_B-\theta_A\, p_{21}} \; a_{21}\,(\xi) \; d\xi. \tag{75}$$

Using (67) and (73) it follows that there exist positive constants c_i such, that for all $r > r_1$

$$-\tfrac{d}{dr}\tilde{M}_A\,(r) \geq c_1 \; r^{\delta_A} \; u\,(r)^{-\delta_A} \; a_{12}\,(r) \; r \; v\,(r)^{p_{12}} \geq$$

$$\geq c_2 \; r^{\delta_A} \; u\,(r)^{-\delta_A} \; a_{12}\,(r) \; r \; \left(\left(r^{\theta_A} \; u\,(r)\right)^{p_{21}} \int\limits_r^\infty \xi^{1+\delta_B-\theta_A\, p_{21}} \; a_{21}\,(\xi) \; d\xi\right)^{\frac{p_{12}}{1+\delta_B}}. \tag{76}$$

We continue as in the uniform case. Integrating (76) on (s,t) for $s \geq r_1$ yields

$$c_0 \; u\,(s) \geq \tilde{M}_A\,(s) \geq -\tilde{M}_A\,(t) + \tilde{M}_A\,(s) \geq$$

$$\geq c_2 \int\limits_s^t r^{1+\delta_A(1+\theta_A)} \, a_{12}\,(r) \; \left(r^{\theta_A} \; u\,(r)\right)^{\frac{p_{12}p_{21}}{1+\delta_B}-\delta_A} \left(\int\limits_r^\infty \xi^{1+\delta_B-\theta_A\, p_{21}} \; a_{21}\,(\xi) \; d\xi\right)^{\frac{p_{12}}{1+\delta_B}} dr \geq$$

$$\geq c_2 \; \left(s^{\theta_A} \; u\,(s)\right)^{\frac{p_{12}p_{21}}{1+\delta_B}-\delta_A} \int\limits_s^t r^{1+\delta_A(1+\theta_A)} \, a_{12}\,(r) \; \left(\int\limits_r^\infty \xi^{1+\delta_B-\theta_A\, p_{21}} \; a_{21}\,(\xi) \; d\xi\right)^{\frac{p_{12}}{1+\delta_B}} dr. \tag{77}$$

And again for some $c > 0$ we have

$$\left(r^{\theta_A} \; u\,(r)\right)' = r^{\theta_A-1}\tilde{M}_A\,(r) \geq$$

$$\geq c\, r^{\theta_A-1} \left(r^{\theta_A} \; u\,(r)\right)^{\frac{p_{12}p_{21}}{1+\delta_B}-\delta_A} \int\limits_r^t \eta^{1+\delta_A(1+\theta_A)} \, a_{12}(\eta) \left(\int\limits_\eta^\infty \xi^{1+\delta_B-\theta_A\, p_{21}} \; a_{21}(\xi) \; d\xi\right)^{\frac{p_{12}}{1+\delta_B}} d\eta. \tag{78}$$

Hence

$$\left(\left(r^{\theta_A}\ u(r)\right)^{1+\delta_A-\frac{p_{12}p_{21}}{1+\delta_B}}\right)' \geq$$

$$\geq c\, r^{\theta_A-1}\int_r^t \eta^{1+\delta_A(1+\theta_A)}\ a_{12}(\eta)\ \left(\int_\eta^\infty \xi^{1+\delta_B-\theta_A p_{21}}\ a_{21}(\xi)\ d\xi\right)^{\frac{p_{12}}{1+\delta_B}} d\eta. \quad (79)$$

Integrating (79) on (s_1, s_2) gives

$$\left(s_2{}^{\theta_A}\ u(s_2)\right)^{1+\delta_A-\frac{p_{12}p_{21}}{1+\delta_B}} - \left(s_1{}^{\theta_A}\ u(s_1)\right)^{1+\delta_A-\frac{p_{12}p_{21}}{1+\delta_B}} \geq$$

$$\geq\ c\int_{s_1}^{s_2} r^{\theta_A-1}\int_r^t \eta^{1+\delta_A(1+\theta_A)}\ a_{12}(\eta)\ \left(\int_\eta^\infty \xi^{1+\delta_B-\theta_A p_{21}}\ a_{21}(\xi)\ d\xi\right)^{\frac{p_{12}}{1+\delta_B}} d\eta\, dr =$$

$$=\ c'\ \left[r^{\theta_A}\int_r^t \eta^{1+\delta_A(1+\theta_A)}\ a_{12}(\eta)\ \left(\int_\eta^\infty \xi^{1+\delta_B-\theta_A p_{21}}\ a_{21}(\xi)\ d\xi\right)^{\frac{p_{12}}{1+\delta_B}} d\eta\right]_{r=s_1}^{r=s_2} +$$

$$+\ c'\int_{s_1}^{s_2} r^{\theta_A}\ r^{1+\delta_A(1+\theta_A)}\ a_{12}(r)\ \left(\int_r^\infty \xi^{1+\delta_B-\theta_A p_{21}}\ a_{21}(\xi)\ d\xi\right)^{\frac{p_{12}}{1+\delta_B}} dr\ =$$

$$=\ c'\ s_2^{\theta_A}\int_{s_2}^t \eta^{1+\delta_A(1+\theta_A)}\ a_{12}(\eta)\ \left(\int_\eta^\infty \xi^{1+\delta_B-\theta_A p_{21}}\ a_{21}(\xi)\ d\xi\right)^{\frac{p_{12}}{1+\delta_B}} d\eta\ \ +$$

$$-\ c'\ s_1^{\theta_A}\int_{s_1}^t \eta^{1+\delta_A(1+\theta_A)}\ a_{12}(\eta)\ \left(\int_\eta^\infty \xi^{1+\delta_B-\theta_A p_{21}}\ a_{21}(\xi)\ d\xi\right)^{\frac{p_{12}}{1+\delta_B}} d\eta\ \ +$$

$$+\ c'\int_{s_1}^{s_2} r^{(1+\delta_A)(1+\theta_A)}\ a_{12}(r)\ \left(\int_r^\infty \xi^{1+\delta_B-\theta_A p_{21}}\ a_{21}(\xi)\ d\xi\right)^{\frac{p_{12}}{1+\delta_B}} dr. \quad (80)$$

Now we let t go to infinity. The second term is bounded independently of s_2. Next we let s_2 go to infinity and with (58) we obtain a contradiction. Similarly a contradiction is obtained in the case that (59) holds. This concludes the proof in the cases when (58) or (59) hold.

In order to obtain a contradiction in the cases when (60) or (61) hold, we proceed as before until (67). This estimate is replaced by

$$-\tilde{M}_A'(r) = -ru''(r) + (1-N)\, m_A\; u'(r) \geq$$

$$\geq \frac{a_{11}(r)\;\; r}{E_A(|u'|)} u(r)^{p_{11}} \geq \frac{c_A}{m_A} |u'(r)|^{-\delta_A}\;\; a_{11}(r)\;\; r\;\; u(r)^{p_{11}}. \tag{81}$$

By (73) we find that there exists $c > 0$ such that

$$\frac{c_A}{m_A} |u'(r)|^{-\delta_A}\;\; a_{11}(r)\;\; r\;\; u(r)^{p_{11}} \geq c\; r^{\delta_A}\;\; u(r)^{-\delta_A}\;\; a_{11}(r)\;\; r\;\; u(r)^{p_{11}},$$

and as in (74), since $p_{11} > \delta_A + 1 > \delta_A$, we obtain that

$$\tilde{M}_A(s) \geq -\tilde{M}_A(t) + \tilde{M}_A(s) \geq$$

$$\geq c \int_s^t\;\; r^{1+\delta_A}\;\; u(r)^{p_{11}-\delta_A}\;\; a_{11}(r)\;\; dr \geq$$

$$\geq c\;\; \left(s^{\theta_A}\;\; u(s)\right)^{p_{11}-\delta_A} \int_s^t\;\; r^{1+\delta_A-\theta_A(p_{11}-\delta_A)}\;\; a_{11}(r)\;\; dr. \tag{82}$$

The last inequality is equivalent to

$$\left(s^{\theta_A}\;\; u(s)\right)' \geq c\;\; \left(s^{\theta_A}\;\; u(s)\right)^{p_{11}-\delta_A}\;\; s^{\theta_A-1} \int_s^t\;\; r^{1+\delta_A-\theta_A(p_{11}-\delta_A)}\;\; a_{11}(r)\;\; dr$$

or

$$\frac{1}{1+\delta_A-p_{11}} \left(\left(s^{\theta_A}\;\; u(s)\right)^{1+\delta_A-p_{11}}\right)' \geq c\;\;\; s^{\theta_A-1} \int_s^t\;\; r^{1+\delta_A-\theta_A(p_{11}-\delta_A)}\;\; a_{11}(r)\;\; dr. \tag{83}$$

From (83) and $p_{11} > 1 + \delta_A$ we find for some $c' > 0$ and for $s_2 > s_1$ that

$$\frac{1}{p_{11}-(1+\delta_A)} \left(s_1^{\theta_A}\;\; u(s_1)\right)^{1+\delta_A-p_{11}} \geq$$

$$\begin{aligned}&\geq c \int_{s_1}^{s_2} s^{\theta_A - 1} \int_s^t \xi^{1+\delta_A - \theta_A (p_{11} - \delta_A)} a_{11}(\xi)\ d\xi\ ds = \\ &= c' \left[s^{\theta_A} \int_s^t \xi^{1+\delta_A - \theta_A (p_{11} - \delta_A)} a_{11}(\xi)\ d\xi\ ds \right] \Bigg|_{s_1}^{s_2} + \\ &\quad + c' \int_{s_1}^{s_2} s^{(1+\delta_A)(1+\theta_A) - \theta_A p_{11}} a_{11}(s)\ ds.\end{aligned} \tag{84}$$

A contradiction follows if (60) holds. A similar argument is used when (61) holds. This concludes the proof of the theorem. □

Example iv) Consider (1) with

$$f(x,u,v) = |x|^a |u|^{p_{11}-1} u + |x|^b |v|^{p_{12}-1} v,$$
$$g(x,u,v) = |x|^c |u|^{p_{21}-1} u + |x|^d |v|^{p_{22}-1} v.$$

If $p_{12} p_{21} > (1+\delta_A)(1+\delta_B)$ and

$$N - 1 \leq (m_B)^{-1} \frac{(1+c+p_{21})(1+\delta_A) + (1+b+p_{12}) p_{21}}{p_{12} p_{21} - (1+\delta_A)(1+\delta_B)},$$

or $p_{12} p_{21} > (1+\delta_A)(1+\delta_B)$ and

$$N - 1 \leq (m_A)^{-1} \frac{(1+b+p_{12})(1+\delta_B) + (1+c+p_{21}) p_{12}}{p_{12} p_{21} - (1+\delta_A)(1+\delta_B)},$$

or $p_{11} > 1 + \delta_A$ and

$$N - 1 \leq (m_A)^{-1} \frac{1+a+p_{11}}{p_{11} - (1+\delta_A)},$$

or $p_{22} > 1 + \delta_B$ and

$$N - 1 \leq (m_B)^{-1} \frac{1+d+p_{22}}{p_{22} - (1+\delta_B)},$$

then (1) has no non trivial ground state.

Example v) (a generalization of Corollary 2.2 of [12]). Consider

$$\begin{cases} -div\left(\dfrac{Du}{\left(1+|Du|^2\right)^{\frac{2-m}{2}}}\right) = |x|^{\alpha}\ \ |v|^{q-1}\, v \\ -div\left(\dfrac{Dv}{\left(1+|Dv|^2\right)^{\frac{2-n}{2}}}\right) = |x|^{\beta}\ \ |u|^{p-1}\, u \end{cases} \quad \text{in } I\!R^N \tag{85}$$

with $m, n \in (1, 2]$. If $N > m, n$ and $p, q > 1$ and

$$N - 1 \leq (n-1)\, \frac{1 + \beta + (1 + \alpha + q)\, p}{pq - 1}$$

or

$$N - 1 \leq (m-1)\, \frac{1 + \alpha + (1 + \beta + p)\, q}{pq - 1},$$

then (85) has no non trivial ground state.

Remark: The technique used above allow us to obtain non-existence theorems for ground states of systems containing an arbitrary (finite) number of equations. However, for systems containing more than two equations the conditions will become rather involved. Some examples in this direction are studied in [14].

An application of Theorem 3.3 is the following. Consider a system that involves two degenerate generalized Laplacians, namely

$$\begin{cases} -div\left(|Du|^{p-2}\, Du\right) &= f\left(|x|, u, v\right) \\ -div\left(|Dv|^{q-2}\, Dv\right) &= g\left(|x|, u, v\right) \end{cases} \quad \text{in } I\!R^N, \tag{86}$$

where p, q are such that $2 < p, q < N$ and $f, g : I\!R^+ \times I\!R^2 \to I\!R$ are given functions specified next.

The result is an analogy of Theorem 3.2 in the degenerate case (i.e. $p, q \geq 2$). It generalizes Theorem 3.1 of [3].

Theorem 3.4 *Suppose that $N > p, q \geq 2$. Let $f, g \in C\left(I\!R^+ \times I\!R^2\right)$ with $f(r,0,0) = g(r,0,0) = 0\ \forall r \geq 0$. Further assume that there exist continuous $a, b, c, d : I\!R^+ \to I\!R^+$ with $a(r), b(r), c(r), d(r) > 0$ for $r > 0$ and such that for large r and small u, v we have*

$$\begin{cases} f(r,u,v) & \geq & a(r)\,|u|^{\alpha-1} u + b(r)\,|v|^{\beta-1} v, \\ g(r,u,v) & \geq & c(r)\,|u|^{\gamma-1} u + d(r)\,|v|^{\delta-1} v, \end{cases} \tag{87}$$

where $\alpha, \beta, \gamma, \delta$ are positive constants. If one of the following conditions is satisfied:

1. *$\beta\gamma > pq$ and*

$$\int_{r_0}^{\infty} s^{N-1} b(s) \left(\int_s^{\infty} \vartheta^{p-1-\beta\frac{N-q}{q-1}} c(\vartheta)\ d\vartheta \right)^{\frac{\gamma}{p-1}} ds = +\infty, \tag{88}$$

2. *$\beta\gamma > pq$ and*

$$\int_{r_0}^{\infty} s^{N-1} c(s) \left(\int_s^{\infty} \vartheta^{q-1-\gamma\frac{N-p}{p-1}} b(\vartheta)\ d\vartheta \right)^{\frac{\beta}{p-1}} ds = +\infty, \tag{89}$$

3. *$\alpha > p$ and*

$$\int_{r_0}^{\infty} s^{N-1-\alpha\frac{N-p}{p-1}}\ a(s)\ ds = +\infty, \tag{90}$$

4. *$\delta > q$ and*

$$\int_{r_0}^{\infty} s^{N-1-\delta\frac{N-q}{q-1}}\ d(s)\ ds = +\infty, \tag{91}$$

then (86) has no non trivial ground states.

Proof. The demonstration follows straightforwardly from Theorem 3.3 by choosing

$$
\begin{array}{rclcrcl}
\delta_A & = & p-1 & \qquad & \delta_B & = & q-1 \\
\theta_A & = & \dfrac{N-p}{p-1} & & \theta_B & = & \dfrac{N-q}{q-1} \\
m_A & = & \dfrac{1}{p-1} & & m_B & = & \dfrac{1}{q-1} \\
p_{11} & = & \alpha & & p_{21} & = & \gamma \\
p_{12} & = & \beta & & p_{22} & = & \delta
\end{array}
$$

Remark: The above theorem can be applied to the scalar case

$$-div\left(|Du|^{q-2}\, Du\right) = f(x,u) \quad \text{in } I\!R^n,$$

with for some continuous $a : [0,\infty) \to [0,\infty)$

$$f(x,u) \geq a(|x|)\,|u|^{\alpha-1}\, u \quad \text{for } r \text{ large and } u \text{ small.}$$

In this case we have non existence of ground states if

$$\int_{r_0}^{\infty} s^{N-1-\alpha\frac{N-q}{q-1}}\; a(s)\; ds = +\infty. \tag{92}$$

For $a(r) \equiv a > 0$ condition (92) is equivalent with (see [16], [17])

$$\alpha \leq \frac{N}{N-q}\,(q-1)\,.$$

4 Variational systems

In the case that the system under consideration has a variational structure it is possible to obtain refined versions of the preceding results by using the variational identities that were proved in [19], [4], [20] and [25]. In this section we shall consider a general result of this type and discuss some particular case.

We will start by recalling some facts about the scalar case. Consider the Lagrangian density

$$J(u) = \frac{1}{p}|\nabla u|^p - \frac{1}{\alpha+1}|x|^\delta |u|^{\alpha+1},$$

with $1 < p < N$. From the paper [20] one may derive the following identity

$$\frac{1-p}{p}\oint_{\partial\Omega}\left|\frac{\partial u}{\partial n}\right|^p (x\cdot n)\ d\sigma = \left(\frac{N-p}{p} - \frac{N+\delta}{\alpha+1}\right)\int_\Omega |x|^\delta |u|^{\alpha+1}\,dx. \tag{93}$$

Using identity (93) one finds that the boundary value problem

$$\begin{cases} -div\left(|\nabla u|^{p-2}\nabla u\right) = |x|^\delta u\,|u|^{\alpha-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{94}$$

has no positive solution when Ω is a bounded starshaped domain in $I\!R^N$ and

$$\alpha + 1 \geq \frac{p(N+\delta)}{N-p}. \tag{95}$$

See for instance [4]. By variational techniques it can be shown that (94) has a non trivial solution if

$$\alpha + 1 < \frac{p(N+\delta)}{N-p}, \tag{96}$$

which is equivalent to

$$N + \delta < \left(\frac{\frac{\alpha+1}{p}(p+\delta)}{-1+\frac{\alpha+1}{p}}\right). \tag{97}$$

Taking $\Omega = I\!R^N$ and restricting to radially symmetric solutions, the situation reverses, and (94) does not have positive radially symmetric solutions when (96) holds. However, when (95) holds radially symmetric positive solutions are known to exist. See [16]. In the case of systems and more general nonlinearities we can derive a similar non-existence result by imposing natural growth conditions on H.

Our next result deals with the case when the system is the Euler-Lagrange equation $\delta\{J(u)\} = 0$ of the functional

$$J(u) = \sum_{i=1}^{k} \frac{1}{p_i} \int_{I\!R^N} |Du_i|^p \, dx - \int_{I\!R^N} H(|x|, u) \, dx, \tag{98}$$

that is, with a system of the form

$$\left\{ \begin{array}{rcl} -div\left(|Du_1|^{p_1-2} Du_1\right) & = & \frac{\partial}{\partial u_1} H(|x|, u) \\ \vdots & & \\ -div\left(|Du_k|^{p_k-2} Du_k\right) & = & \frac{\partial}{\partial u_k} H(|x|, u) \end{array} \right. \quad \text{in } I\!R^N. \tag{99}$$

Here $k \geq 1$, $N > p_i > 1$ for all $i \in \{1, \ldots, k\}$ and H is a given potential that satisfies the following assumptions.

(H) Let $H : I\!R^+ \times I\!R^k \to I\!R$ be a C^1-function such that $\frac{\partial}{\partial u_i} H(r, 0) = 0 \; \forall r \geq 0, i \in \{1, \ldots, k\}$ and

i) for all $(r, u) \in I\!R^+ \times I\!R^k$:

$$N \; H(r, u) + r \frac{\partial}{\partial r} H(r, u) > \sum_{i=1}^{k} \frac{N - p_i}{p_i} \; u_i \frac{\partial}{\partial u_i} H(r, u), \tag{100}$$

ii) there exist δ_i, α_i and $c_i > 0$ such that for all large r and sufficiently small $u \in I\!R^k$ we have:

$$u_i \; \frac{\partial}{\partial u_i} H(r, u) \geq c_i \; r^{\delta_i} \; \prod_{i=1}^{k} \; |u_i|^{\alpha_i + 1} \quad ,$$

iii) there exists $\delta \in I\!R, c > 0$ such that for all large r and sufficiently small $u \in I\!R^k$ we have:

$$H(r,u) \leq c\ r^\delta \prod_{i=1}^{k} |u_i|^{\alpha_i+1} \ ,$$

iv)

$$\sum_{i=1}^{k} \frac{\alpha_i+1}{p_i} > 1$$

v) subcriticality:

$$N + \delta < \left(\frac{\sum_{i=1}^{k} \frac{\alpha_i+1}{p_i}(p_i+\delta_i)}{-1+\sum_{i=1}^{k} \frac{\alpha_i+1}{p_i}} \right). \qquad (101)$$

Remark 1. For $k = 1$ (i.e. the scalar case) condition (101) reduces to the usual subcriticality condition (96). For the pure power case, that is $H(r,u) = c\ r^\delta\ \prod_{i=1}^{k}\ |u_i|^{\alpha_i+1}$, (100) coincides with (101).

Remark 2. The strict inequality in (100) can be replaced by $\geq$ if one assumes that there is a sequence $\{r_n\}$ with $r_n \to \infty$ for which the strict inequality holds.

Theorem 4.1 *Assume that the above condition* **(H)** *is satisfied. Then the system (99) has no non trivial ground states in* $C^2\left(I\!R^N\right)^k$.

The proof of the theorem is based on the following version of the identity proved in [20].

Proposition 4.2 *Let $u \in C^2\left(I\!R^N\right)^k$ be a radial solution of (99). Then the following identity holds.*
For any $R > 0$ we have

$$\int_0^R \left(N \; H(r,u) + r\,\frac{\partial}{\partial r} H(r,u)\right) r^{N-1}\,dr + \sum_{i=1}^k \left(\frac{p_i - N}{p_i} \int_0^R r^{N-1}\,u_i\;\frac{\partial}{\partial u_i} H(r,u)\;dr\right) =$$

$$= \sum_{i=1}^k \left(\frac{p_i - 1}{p_i} R^N \;\left|u_i'(R)\right|^p + \frac{N - p_i}{p_i}\; R^{N-1}\;\left|u_i'(R)\right|^{p-2}\; u_i'(R)\;\; u_i(R)\right) +$$

$$+ \quad R^N \; H(R, u(R)) \tag{102}$$

In order to prove the theorem we shall also need some asymptotic estimates.

Lemma 4.3 *Let $u \in C^2\left(\mathbb{R}^N\right)^k$ be a ground state of (99). If (H) ii), iii) and iv) hold, then there exists $c > 0$ such that for all sufficiently large r we have*

$$H\left(r, u(r)\right) \le c\; r^{\delta - \mu}$$

with $\mu = \dfrac{\sum_{i=1}^k \frac{\alpha_i + 1}{p_i}\left(p_i + \delta_i\right)}{-1 + \sum_{i=1}^k \frac{\alpha_i+1}{p_i}}.$

Proof. Let $u \in C^2\left(\mathbb{R}^N\right)^k$ be a radial ground state of (99). By ii) it follows that for r sufficiently large we have $u_i'(r) < 0$ for all i, and, for some $c > 0$ and all i,

$$-\left(r^{N-1}\left|u_i'(r)\right|^{p_i-2}\;\; u_i'(r)\right)' \ge c\, r^{\delta_i + N-1}\; u_1(r)^{\alpha_1+1} \dots\; u_i(r)^{\alpha_i} \dots\; u_k(r)^{\alpha_k+1}\,. \tag{103}$$

Integrating on (r, t) we obtain for all i

$$t^{N-1}\left|u_i'(t)\right|^{p_i-1} + r^{N-1}\left|u_i'(r)\right|^{p_i-2} u_i'(r) \ge$$

$$\geq c \int_r^t s^{\delta_i+N-1}\; u_1(s)^{\alpha_1+1}\dots\; u_i(s)^{\alpha_i}\dots\; u_k(s)^{\alpha_k+1}\,ds, \tag{104}$$

and hence by using Lemma 2.2, that is (see (71))

$$\left(r^{\theta_i}\; u_i(r)\right)' \geq 0 \quad \text{for large } r,$$

with $\theta_i = \frac{N-p_i}{p_i-1}$, we obtain

$$t^{N-1}\left|u_i'(t)\right|^{p_i-1} \geq$$

$$\geq c^*\;\left(r^{\theta_1}u_1(r)\right)^{\alpha_1+1}\dots\;\left(r^{\theta_i}u_i(r)\right)^{\alpha_i}\dots\;\left(r^{\theta_k}u_k(r)\right)^{\alpha_k+1}\int_r^t s^{\beta_i}\;ds, \tag{105}$$

where $\beta_i = \delta_i + N - 1 + \theta_i - \sum_{j=1}^k \theta_j(\alpha_j+1)$. Using again the inequality

$$ru_i'(r) + \theta_i\; u_i(r) \geq 0 \qquad \text{for } r \geq r_0, \tag{106}$$

and $u_i'(r) \leq 0$, we deduce from (105) that there exist $c > 0$ such that

$$t^{N-p_i}\; u_i(t)^{p_i-1} \geq c\;\left(r^{\theta_1}u_1(r)\right)^{\alpha_1+1}\dots\;\left(r^{\theta_i}u_i(r)\right)^{\alpha_i}\dots\;\left(r^{\theta_k}u_k(r)\right)^{\alpha_k+1}\int_r^t s^{\beta_i}\;ds. \tag{107}$$

By choosing $t = 2r$ in (107) we obtain

$$2^{N-p_i}\; r^{N-p_i}\; u_i(r)^{p_i-1} \geq$$

$$\geq c\;\left(r^{\theta_1}u_1(r)\right)^{\alpha_1+1}..\;\left(r^{\theta_i}u_i(r)\right)^{\alpha_i}..\;\left(r^{\theta_k}u_k(r)\right)^{\alpha_k+1}\gamma_i\; r^{\beta_i+1}\;, \tag{108}$$

with $\gamma_i = \frac{2^{\beta_i+1}-1}{\beta_i+1}$ if $\beta_i + 1 \neq 0$ and $\gamma_i = \log 2$ if $\beta_i + 1 = 0$. Thus for some C and for every i and sufficiently large r we have

$$C\; u_i(r)^{\alpha_i+1} \geq \left(r^{p_i+\delta_i}\prod_{j=1}^k u_j(r)^{\alpha_j+1}\right)^{\frac{\alpha_i+1}{p_i}}. \tag{109}$$

After combining the k inequalities in (109), we obtain

$$C^k \prod_{i=1}^{k} u_i\left(r\right)^{\alpha_i+1} \geq \prod_{i=1}^{k}\left(r^{p_i+\delta_i} \prod_{j=1}^{k} u_j\left(r\right)^{\alpha_j+1}\right)^{\frac{\alpha_i+1}{p_i}} =$$

$$= r^{\left(\sum_{i=1}^{k}(p_i+\delta_i)\frac{\alpha_i+1}{p_i}\right)}\left(\prod_{j=1}^{k} u_j\left(r\right)^{\alpha_i+1}\right)^{\sum_{i=1}^{k}\frac{\alpha_i+1}{p_i}}, \tag{110}$$

or in other words

$$C^k \geq r^{\left(\sum_{i=1}^{k}\frac{\alpha_i+1}{p_i}(p_i+\delta_i)\right)}\left(\prod_{j=1}^{k} u_j\left(r\right)^{\alpha_j+1}\right)^{\left(\sum_{i=1}^{k}\frac{\alpha_i+1}{p_i}\right)-1}. \tag{111}$$

Now by using condition **(H)** iv) in (111) it follows that

$$C^* \geq r^{\mu} \prod_{j=1}^{k} u_j\left(r\right)^{\alpha_j+1}, \tag{112}$$

with

$$\mu = \frac{\sum_{i=1}^{k}\frac{\alpha_i+1}{p_i}\left(p_i+\delta_i\right)}{-1+\sum_{i=1}^{k}\frac{\alpha_i+1}{p_i}}. \tag{113}$$

From (112) and condition **(H)** iii) we find for large r that

$$H\left(r, u\left(r\right)\right) \leq c\ r^{\delta} \prod_{j=1}^{k} u_j\left(r\right)^{\alpha_j+1} \leq c_1\ r^{\delta-\mu}.$$

This concludes the proof of Lemma 4.3. □

Proof of Theorem 4.1. We argue by contradiction. Let u be a non trivial ground state of (99). Applying identity (102) to this specific situation it follows that the left hand side of (102) is bounded away from zero for R large. On the other hand by using Lemma 2.1 of [3] and the fact that $u_i'\left(R\right) \leq 0\ \forall i$ we know that the first term of the right hand side of (102) is non positive. We conclude the proof by using Lemma 4.3 in order to show that the second term of the right hand side of (102), namely $H(R, u\left(R\right))\ R^N$, converges to zero as R goes to $+\infty$. As a result we obtain a contradiction. □

Remark: The same technique used in the proof of the last theorem allows to study more general systems than (99). It is possible to show that non-existence results can be obtained for systems of the form

$$\begin{cases} -div\left(A_1\left(|Du_1|\right)Du_1\right) = \frac{\partial}{\partial u_1}H\left(|x|,u\right) \\ \vdots \\ -div\left(A_k\left(|Du_k|\right)Du_k\right) = \frac{\partial}{\partial u_k}H\left(|x|,u\right) \end{cases}$$

where each operator A_i satisfies $(\mathrm{i})_{A_i}$ of (56) and where the potential $H\left(|x|,u\right)$ is controlled by a suitable growth condition. This problem will be considered in a forthcoming paper ([14]).

Example vi) See [24]. As an example of a system with a variational structure we consider

$$\begin{cases} -div\left(|Du|^{p-2}Du\right) = |x|^{\delta}\ |u|^{\alpha-1}\ |v|^{\beta+1}\ u, \\ -div\left(|Dv|^{q-2}Dv\right) = |x|^{\delta}\ |u|^{\alpha+1}\ |v|^{\beta-1}\ v. \end{cases} \tag{114}$$

We have, after a possible rescaling, that

$$H\left(r,u\right) = r^{\delta}\ |u|^{\alpha+1}\ |v|^{\beta+1}.$$

Suppose $\frac{(\alpha+1)}{p} + \frac{(\beta+1)}{q} > 1$ and $N > p, q > 2$. If

$$N + \delta < \frac{\frac{(\alpha+1)}{p}(p+\delta) + \frac{(\beta+1)}{q}(q+\delta)}{\frac{(\alpha+1)}{p} + \frac{(\beta+1)}{q} - 1}$$

then there is no non trivial ground state of (114). The proof follows from a straightforward calculation using Theorem 4.1.

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